E_∞-algebras and general linear groups

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Homological stability

All based on joint work with S. Galatius and A. Kupers.

We have developed a general method which is quite powerful for studying homological stability and related questions. I will explain some of the results we have obtained, then say something about the method.

Homological stability is the phenomenon that

$$H_d(GL_n(A), GL_{n-1}(A)) = 0 \text{ for all } d \leq f(n)$$

for some divergent function f.

One can ask this question for homology with $\mathbb{k}$-coefficients: the function f may then depend on $\mathbb{k}$.

Stability with $\mathbb{Z}$-coefficients known when A has "finite stable rank in the sense of Bass" (Maazen–van der Kallen): $f(n) = \frac{n-sr(A)}{2}$ suffices.

The Nesterenko–Suslin theorem

Sometimes one has homological stability in a range of degrees much larger than the slope $\frac{1}{2}$ range of Maazen and van der Kallen.

As a first example, our methods re-prove:

Theorem (Suslin, Nesterenko, Guin). If A is a connected semi-local ring with all residue fields infinite then

$$H_d(GL_n(A), GL_{n-1}(A); \mathbb{Z}) = 0 \text{ for } d < n,$$

and $H_n(GL_n(A), GL_{n-1}(A); \mathbb{Z}) \cong K_n^M(A)$, nth Milnor K-theory.

Milnor K-theory: $K_*^M(A)$ is the graded ring generated by $K_1^M(A) = A^\times$ and subject to the relations $a \cdot b = 0 \in K_2^M(A)$ whenever $a, b \in A^\times$ satisfy $a + b = 1$. (A calculation shows it is graded commutative.)

A degree above the Nesterenko–Suslin theorem

We also study these relative homology groups one degree further up (and rationally). We first show that

$$\bigoplus_{n \geq 1} H_{n+1}(GL_n(A), GL_{n-1}(A); \mathbb{Q})$$

can be made into a $K_*^M(A) \otimes \mathbb{Q}$-module, then analyse how it may be generated efficiently as a $K_*^M(A) \otimes \mathbb{Q}$-module.

Theorem (Galatius–Kupers–R-W). If A is a connected semi-local ring with all residue fields infinite, then there is a map of graded $\mathbb{Q}$-vector spaces

$$\mathrm{Harr}_3(K_*^M(A) \otimes \mathbb{Q}) \longrightarrow \mathbb{Q} \otimes_{K_*^M(A) \otimes \mathbb{Q}} \bigoplus_{n \geq 1} H_{n+1}(GL_n(A), GL_{n-1}(A); \mathbb{Q})$$

which is an isomorphism in gradings ≥ 5.

Here Harr = Harrison homology = André–Quillen homology.
Third Harrison homology measures "relations between relations" in a presentation of the quadratic algebra $K_*^M(A) \otimes \mathbb{Q}$.

Improved homological stability

Under further assumptions on A, our methods (which I have not yet told you) instead give improved homological stability results:

Theorem (Galatius–Kupers–R-W).

(i) If A is a connected semi-local ring with all residue fields infinite and such that $K_2(A) \otimes \mathbb{Q} = 0$ (e.g. $\bar{\mathbb{F}}_q$, $\mathbb{F}_q(t)$, number field, $\bar{\mathbb{Q}}$) then

$$H_d(GL_n(A), GL_{n-1}(A); \mathbb{Q}) = 0 \text{ for } d < \tfrac{4n-1}{3}.$$

(ii) If A is a connected semi-local ring with all residue fields infinite and p is a prime number such that $A^\times \otimes \mathbb{Z}/p = 0$ then

$$H_d(GL_n(A), GL_{n-1}(A); \mathbb{Z}/p) = 0 \text{ for } d < \tfrac{5n}{4}.$$

(iii) If $\mathbb{F}$ is an algebraically closed field then, for all primes p,

$$H_d(GL_n(\mathbb{F}), GL_{n-1}(\mathbb{F}); \mathbb{Z}/p) = 0 \text{ for } d < \tfrac{5n}{3}.$$

The slopes of these stability ranges are all > 1.

Resolving some conjectures

The last part implies that if $\mathbb{F}$ is an algebraically closed field then

$$H_{n+1}(GL_n(\mathbb{F}), GL_{n-1}(\mathbb{F}); \mathbb{Z}/p) = 0$$

for all $n > 1$ and all primes p.

This resolves a conjecture of Mirzaii on certain "higher pre-Bloch groups" $\mathfrak{p}_n(\mathbb{F})$, and a conjecture of Yagunov on a different notion of "higher pre-Bloch groups" $\wp_n(\mathbb{F})$ and $\wp_n(\mathbb{F})_{cl}$.

In a different direction, we can complete an approach of Mirzaii to proving Suslin's "injectivity conjecture":

Theorem (Galatius–Kupers–R-W). If $\mathbb{F}$ is an infinite field and $\Bbbk$ is a field in which $(n-1)!$ is invertible then the stabilisation map

$$H_n(GL_{n-1}(\mathbb{F}); \Bbbk) \longrightarrow H_n(GL_n(\mathbb{F}); \Bbbk)$$

is injective.

Methods

Overview

These results are proved by considering the totality

$$\mathbf{R}^{+} = \coprod_{n \geq 0} BGL_n(A),$$

as a unital E_∞-algebra in the category of $\mathbb{N}$-graded spaces.

Try to construct $\mathbf{R}^{+}$ as a cellular object in this category. Such cell structures can be constrained by calculating or estimating the analogue $H^{E_\infty}_{n,d}(\mathbf{R})$ of cellular homology in this category.

We prove that if A is a connected semi-local ring with all residue fields infinite, then $H^{E_\infty}_{n,d}(\mathbf{R}) = 0$ for $d < 2(n-1)$.

This vanishing range is twice as good as what one might first expect, and opens the door to $H_d(GL_n(A), GL_{n-1}(A); \mathbb{k})$ beyond slope 1.

Going through that door still requires detailed calculations with E_∞-algebras, which I won't say anything about today.

Graded objects

Let C denote sSet, sSet_*, or (because we are eventually interested in taking $\Bbbk$-homology) $\mathrm{sMod}_{\Bbbk}$.

Write $\otimes$ for the cartesian, smash, or tensor product.

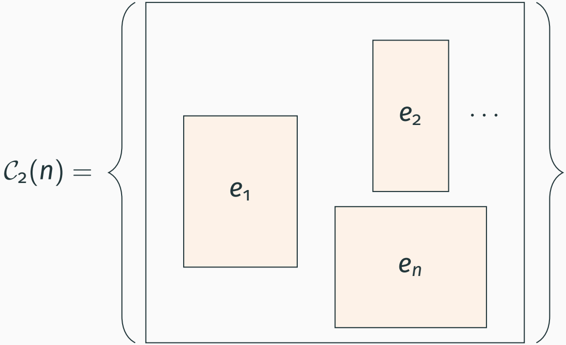
We will consider $\mathbb{N}$-graded objects in C, meaning $\mathsf{C}^{\mathbb{N}} := \mathrm{Fun}(\mathbb{N}, \mathsf{C})$. This is given the Day convolution monoidal structure:

$$(X \otimes Y)(n) = \coprod_{a+b=n} X(a) \otimes Y(b).$$

Define bigraded homology groups as $H_{n,d}(X; \Bbbk) := H_d(X(n); \Bbbk)$.

E_k-algebras

Let $\mathcal{C}_k$ denote the non-unital ($\mathcal{C}_k(0) = \varnothing$) little k-cubes operad.



The categories $\mathsf{C}^{\mathbb{N}}$ are all tensored over Top: can make sense of the monad

$$E_k(X) := \coprod_{n \geq 1} \mathcal{C}_k(n) \odot_{\mathfrak{S}_n} X^{\otimes n}$$

and so of E_k-algebras $\mathbf{X}$ in $\mathsf{C}^{\mathbb{N}}$. Call the category of these $\mathrm{Alg}_{E_k}(\mathsf{C}^{\mathbb{N}})$.

E_k-cells

Let $S^{n,d}$ denote the $\mathbb{N}$-graded space which is S^d in grading n and trivial otherwise, and similarly $D^{n,d}$.

Given an E_k-algebra $\mathbf{X}$ and a map $f : S^{n,d-1} \to X$ can define the cell attachment $\mathbf{X} \cup_f^{E_k} D^{n,d}$ as the pushout in $\mathrm{Alg}_{E_k}(\mathsf{C}^{\mathbb{N}})$ of

$$\mathbf{E_k}(D^{n,d}) \longleftarrow \mathbf{E_k}(S^{n,d-1}) \xrightarrow{f^{ad}} \mathbf{X}.$$

Cellular E_k-algebras are those formed by iterated cell attachments. A *CW-E_k-algebra* is similar but the attaching maps are controlled (e.g. it comes with a skeletal filtration).

(Every object is equivalent to a cellular one, as usual.)

E_k-indecomposables

Have inclusion $\mathsf{C}_*^{\mathbb{N}} \to \mathrm{Alg}_{E_k}(\mathsf{C}_*^{\mathbb{N}})$ by imposing the trivial E_k-action, with left adjoint Q^{E_k}, called the "E_k-indecomposables".

e.g. Have $Q^{E_k}(\mathbf{E_k}(X)) = X$

If $\mathbf{X}$ is a cellular E_k-algebra then it follows that $Q^{E_k}(\mathbf{X})$ is a cellular object in $\mathsf{C}_*^{\mathbb{N}}$ with one (n, d)-cell for each E_k-(n, d)-cell of $\mathbf{X}$.

This construction is not homotopy invariant, so on a general $\mathbf{X}$ one should evaluate the derived functor

$$Q_{\mathbb{L}}^{E_k}(\mathbf{X}) := Q^{E_k}(\text{any cellular } E_k\text{-algebra equivalent to } \mathbf{X}),$$

a.k.a. topological Quillen homology (for the operad $\mathcal{C}_k$).

E_k-homology and minimal cell structures

Define E_k-homology as $H^{E_k}_{n,d}(\mathbf{X}) := H_{n,d}(Q^{E_k}_{\mathbb{L}}(\mathbf{X}))$.

If $\Bbbk$ is a field, the discussion so far shows

$$\dim_{\Bbbk} H^{E_k}_{n,d}(\mathbf{X}; \Bbbk) \leq \begin{array}{c}\text{number of } E_k\text{-}(n,d)\text{-cells in any} \\ E_k\text{-cellular approximation of } \mathbf{X}.\end{array}$$

Just as in classical homotopy theory, homology can be used to detect *minimal* cell structures as long as we work in a stable context.

Theorem. Let $\Bbbk$ be a field and $\mathsf{C} = \mathsf{sMod}_{\Bbbk}$. Then $\mathbf{X} \in \mathsf{Alg}_{E_2}(\mathsf{sMod}_{\Bbbk}^{\mathbb{N}})$ has a cellular approximation $c\mathbf{X} \xrightarrow{\sim} \mathbf{X}$ with precisely $\dim_{\Bbbk} H^{E_2}_{n,d}(\mathbf{X})$ many E_2-(n,d)-cells.

Furthermore $c\mathbf{X}$ can be taken to be “CW”, not just “cellular”.

Computing derived E_k-indecomposables

Crucially to this entire business, $Q^{E_k}_{\mathbb{L}}(\mathbf{X})$ may also be computed another way: by a k-fold bar construction.

(Getzler–Jones, Basterra–Mandell, Fresse, Francis)

Theorem. If $\mathbf{X}$ is an E_k-algebra with unitalisation $\mathbf{X}^+$, then

$$\mathbb{1} \oplus \Sigma^k Q^{E_k}_{\mathbb{L}}(\mathbf{X}) \simeq B^{E_k}(\mathbf{X}^+);$$

the latter is the k-fold bar construction.

Considering the k-fold bar construction as the bar construction of the $(k-1)$-fold bar construction gives a bar spectral sequence

$$E^2_{n,p,q} = Tor_p^{H_{*,*}(B^{E_{k-1}}(\mathbf{X}^+);\mathbb{k})}(\mathbb{k},\mathbb{k})_{n,q} \Rightarrow H_{n,p+q}(B^{E_k}(\mathbf{X}^+);\mathbb{k}).$$

So $H^{E_{k-1}}_{n,d}(\mathbf{X}) = 0$ for $d < \lambda \cdot n - (k-1)$
$\Rightarrow H^{E_k}_{n,d}(\mathbf{X}) = 0$ for $d < \lambda \cdot n - (k-1)$ too.

The general linear groups

The general linear group E_∞-algebra

Let A be a connected commutative ring for which f.g. projective modules are free.

The symmetric monoidal category P_A of f.g. projective A-modules and their isomorphisms has classifying space

$$\mathbf{R}^+ = B\mathsf{P}_A \simeq \coprod_{n \geq 0} BGL_n(A)$$

and is equipped with an action of an E_∞-operad. We consider this as $\mathbb{N}$-graded via the rank functor $r : \mathsf{P}_A \to \mathbb{N}$.

By direct calculation this has

$$B^{E_1}(\mathbf{R}^+)(n) \simeq \Sigma^2 S(A^n)_{hGL_n(A)}$$

where $S(A^n)$ is Charney's *split Tits building*, i.e.

$$[p] \mapsto \{(M_0, \ldots, M_{p+1}) \text{ nonzero submodules of } A^n \text{ s. t. } \bigoplus M_i = A^n\}.$$

Theorem (Charney). If A is Dedekind then $S(A^n)$ is $(n-3)$-connected.

$$\Rightarrow H^{E_1}_{n,d}(\mathbf{R}) = 0 \text{ for } d < n-1.$$

Slope 2

Theorem (Galatius–Kupers–R-W). If A is a connected semi-local ring with all residue fields infinite, then $H^{E_\infty}_{n,d}(\mathbf{R}) = 0$ for $d < 2(n-1)$.

This involves analysing the 2-dimensional version of the split Tits building, and relating it to the square of the ordinary Tits building.

It is related to Rognes' connectivity conjecture, cf. Patzt's talk.

Theorem (Galatius–Kupers–R-W). If A is an infinite field then

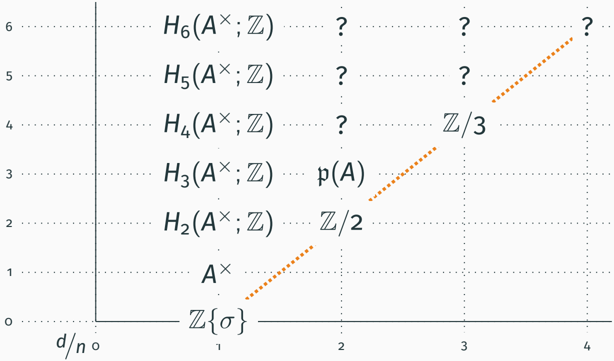
$$H^{E_\infty}_{2n-2}(\mathbf{R}) = \begin{cases} \mathbb{Z} & \text{if } n = 1, \\ \mathbb{Z}/p & \text{if } n = p^k \text{ with } p \text{ prime}, \\ 0 & \text{otherwise}. \end{cases}$$

This involves proving $[St(A^n) \otimes St(A^n)]_{GL_n(A)} \cong \mathbb{Z}$, i.e. classifying equivariant bilinear forms on the classical Steinberg module.

(This implies that $St(A^n)$ is indecomposable, cf. Putman's talk.)

E_∞-homology

Combining the previous results with calculations of Suslin for $GL_2(A)$, we obtain the following chart for $H^{E_\infty}_{n,d}(\mathbf{R})$:



$\mathfrak{p}(A)$ = "pre-Bloch group": generated by $[x] \in A^\times \setminus \{1\}$ subject to

$$[x] - [y] + \left[\frac{y}{x}\right] + \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] = 0$$

whenever x, y, $1-x$, $1-y$, and $x-y \in A^\times$.

Relation to homological stability

For a ring of coefficients $\mathbb{k}$, let

$$\mathbf{R}^+_{\mathbb{k}} \in \mathrm{Alg}_{E_\infty}(\mathrm{sMod}^{\mathbb{N}}_{\mathbb{k}})$$

be the $\mathbb{k}$-linearisation of $\mathbf{R}^+ = \coprod_{n \geq 0} BGL_n(A) \in \mathrm{Alg}_{E_\infty}(\mathrm{sSet}^{\mathbb{N}})$.

For the basepoint $\sigma \in H_0(BGL_1(A); \mathbb{k}) = H_{1,0}(\mathbf{R}^+_{\mathbb{k}})$, stabilisation can be described in terms of the E_∞-structure as

$$- \cdot \sigma : H_d(BGL_{n-1}(A); \mathbb{k}) = H_{n-1,d}(\mathbf{R}^+_{\mathbb{k}}) \longrightarrow H_d(BGL_n(A); \mathbb{k}) = H_{n,d}(\mathbf{R}^+_{\mathbb{k}}).$$

Writing $\mathbf{R}^+_{\mathbb{k}}/\sigma$ for the cofibre in $\mathrm{sMod}^{\mathbb{N}}_{\mathbb{k}}$ of $- \cdot \sigma : \mathbf{R}^+_{\mathbb{k}}[1] \to \mathbf{R}^+_{\mathbb{k}}$, have

$$H_d(GL_n(A), GL_{n-1}(A); \mathbb{k}) = H_{n,d}(\mathbf{R}^+_{\mathbb{k}}/\sigma).$$

Deducing homological stability results

The strategy is to choose a minimal CW-structure on $\mathbf{R}_{\Bbbk}^{+}$ as per the E_∞-homology on the chart, consider its skeletal filtration and study the spectral sequence

$$E^1_{*,*,*} = H_{*,*,*}(gr(\mathbf{R}_{\Bbbk}^{+})/\sigma) \Rightarrow H_{*,*}(\mathbf{R}_{\Bbbk}^{+}/\sigma).$$

Now $gr(\mathbf{R}_{\Bbbk}^{+})$ is the free E_∞-algebra with one generator for each E_∞-cell of $\mathbf{R}_{\Bbbk}^{+}$, and by the chart σ is the only generator of slope < 1. From this and the known homology of free E_∞-algebras, one immediately deduces homological stability of slope $\frac{1}{2}$.

To do better than this, need to analyse how the low-dimensional E_∞-cells are attached to each other, i.e. show that $\mathbf{R}_{\Bbbk}^{+}$ is *better* than the free E_∞-algebra with the same collection of cells.

Literature

Based on work with S. Galatius and A. Kupers:

E_∞-cells and general linear groups of infinite fields.
arXiv:2005.05620.

Cellular E_k-algebras.
arXiv:1805.07184.

For further applications of these ideas see also:

E_2-cells and mapping class groups.
Publ. Math. Inst. Hautes Études Sci. 130 (2019), 1–61.

E_∞-cells and general linear groups of finite fields.
arXiv:1810.11931.