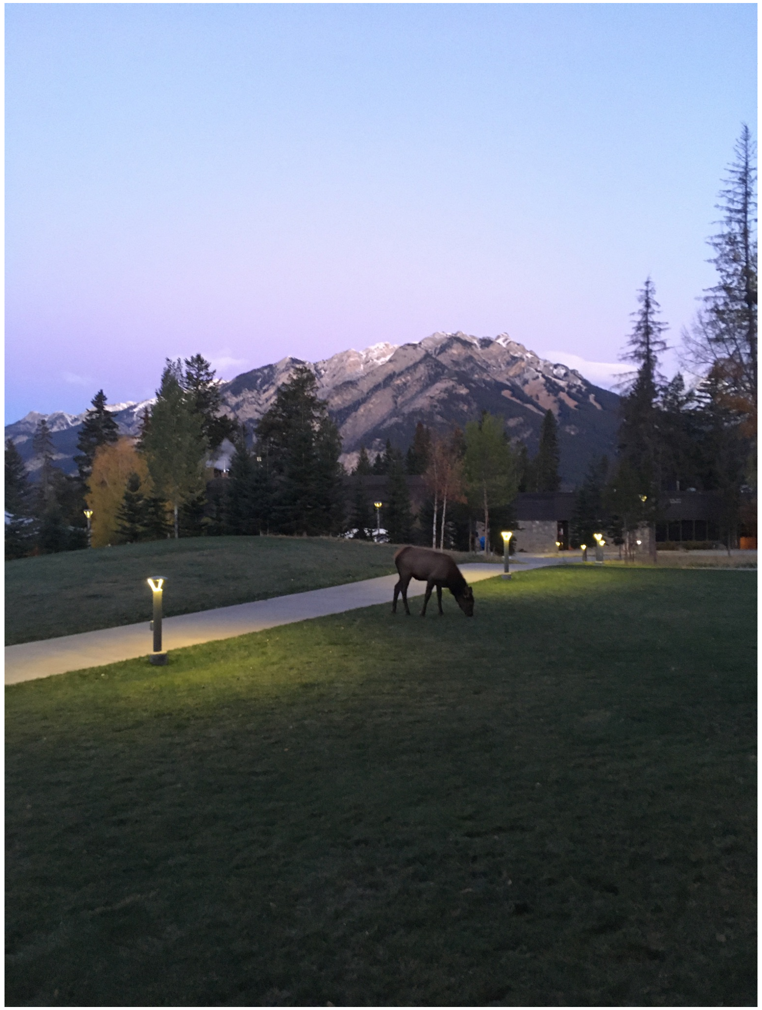
October 11, 2021

Growth of Cohomology in Towers & Endoscopy



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Workshop on Cohomology of Arithmetic Groups

THE PROBLEM

QUESTION:

How fast does $h^i(p^n) := \dim H^i(\Gamma(p^n), \mathbb{C})$ grow as $n \to \infty$?

(VERTICAL QUESTION!)

Let:
- $G = U(N-a, a)$ defined in terms of E/F
- p a prime of F
- $\Gamma(p^n)$, cocompact, full-level congruence subgroups.

SOME FACTS:

→ $X(p^n) := \Gamma(p^n) \backslash G/K$ is a $2 \cdot a \cdot (N-a)$-fold.

→ $\mathrm{Vol}(X(p^n)) \asymp [\Gamma(p^n) : \Gamma(1)] \asymp Nm(p^n)^{N^2-1}$

→ (de George - Wallach '78)

$$\lim_{n \to \infty} \frac{h^i(p^n)}{\mathrm{Vol}(X(p^n))} = \begin{cases} k > 0 & i = a(N-a) \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM (G.-G., '20) Let $\Gamma(p^n) \subset U(N-2,2)$ as before. Assume N is odd & p is large enough. Then

$$h^2(p^n) \ll Nm(p^n)^N.$$

Example: If $G = U(2,1)$, then $X(p^n)$ is a Picard modular surface (real 4-fold), and

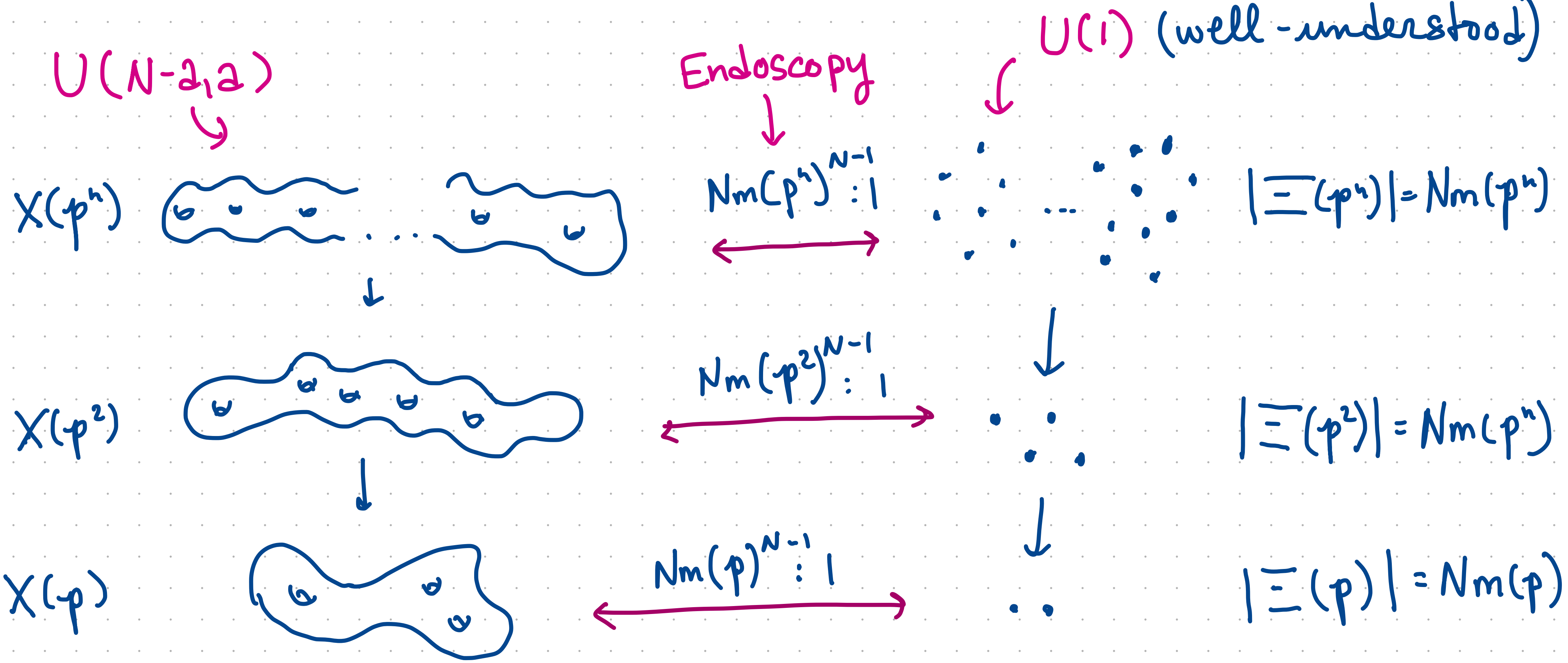
$$h^1(p^n) \leq Nm(p^n)^3.$$

In general, $i=2$ is the smallest nonzero degree for which $\dim H^i(\Gamma(p^n), \mathbb{C}) \neq 0$

WHY THIS HAPPENS: ENDOSCOPY

$\boxed{\text{Specifically, our classes in } H^a \text{ come from "classes in } H^0\text{" associated to } U(1).}$

- → instance of Langlands Functoriality
- → here, describes how cohomology classes "come from smaller groups".



REPRESENTATION THEORY

$U(N-2,2) = G \supset K = U(N-2)\times U(2)$

$\mathfrak{g} \supset \mathfrak{k}$ Lie alg.

MATSUSHIMA'S FORMULA (special case)

$$\boxed{\dim H^2(\Gamma(p^n),\mathbb{C}) = m(\pi_{2,0},p^n) + m(\pi_{0,2},p^n)}$$

- $\pi_{2,0}$ & $\pi_{0,2}$: the two unitary irreducible representations of $U(N-2,2)$ w/
  - → trivial infinitesimal character (harmonic)
  - → $\mathrm{Hom}_K(\Lambda^2(\mathfrak{g}/\mathfrak{k}),\pi) \neq 0$ (2-forms on G/K)
- $m(\pi,p^n)$ = multiplicity of π in $L^2(\Gamma(p^n)\backslash G)$

THE ADELIC PICTURE

→ replace $X(p^n)$ by $G(F)\backslash G(\mathbb{A})/K_\infty K(p^n)$

← compact-open subgroup

→ replace mult. in $L^2(\Gamma(p^n)\backslash G)$ by mult in $L^2(G(F)\backslash G(\mathbb{A}))^{K(p^n)}$

AUTOMORPHIC REPRESENTATIONS:

$$\circlearrowleft G(\mathbb{A})$$

aut. rep.

$$L^2(G(F)\backslash G(\mathbb{A})) = \bigoplus_{\pi} m(\pi)\, \pi$$

with each

$$\pi = \pi_\infty \otimes \pi_f$$

$G(\mathbb{R}) \circlearrowright \pi_\infty$, $\quad \pi_f \circlearrowleft G(\mathbb{A}_f)$

WE WANT TO COUNT:

$$h^a(p^n) \simeq \sum_{\substack{\pi = \pi_{a,0} \otimes \pi_f \\ \text{or } \pi_{0,a} \otimes \pi_f}} m(\pi) \dim \pi_f^{K(p^n)}$$

ENDOSCOPIC CLASSIFICATION OF REPRESENTATIONS (Arthur, Mok, KMSW)

→ attaches to each π a "Galois rep":

$$\rho_\pi : \text{"}Gal(\bar{F}/F)\text{"} \times SL_2(\mathbb{C}) \to GL_N(\mathbb{C})$$

& a local version

$$\rho_{\pi_v} : \text{"}Gal(\bar{F}_v/F_v)\text{"} \times SL_2(\mathbb{C}) \to GL_N(\mathbb{C})$$

→ if ρ_π factors through $GL_{N_1} \times \dots \times GL_{N_r}$ then π comes from $U(N_1) \times \dots \times U(N_r)$

→ $\rho_\pi|_{SL_2(\mathbb{C})}$ is determined locally at any place.

we have $\pi \rightsquigarrow \rho_\pi : \mathrm{Gal}(\bar{F}/F) \times SL_2(\mathbb{C}) \to GL_N(\mathbb{C})$

<u>THM (Arthur)</u> (assume $a=1$)

π_∞ cohomological $\Longrightarrow$ $\rho_{\pi_\infty}|_{SL_2(\mathbb{C})}$ contains the "Lefschetz SL_2" representation spanned by the cohomology classes.

<u>in our case...</u>

$$\begin{array}{ccc} & h^{N-1,N-1} & \\ h^{N-1,N-2} & & h^{N-2,N-1} \\ \vdots & \vdots & \vdots \\ h^{1,0} & h^{0,0} & h^{0,1} \end{array} = \nu(N-1)$$

irrep of SL_2 of dimension $N-1$

$$\Longrightarrow \rho_\pi = \begin{pmatrix} \boxed{\nu(N-1) \otimes \chi_{N-1}} & \\ & \boxed{\nu(1) \otimes \chi_1} \end{pmatrix}$$

$\Longrightarrow \rho_\pi$ factors through $GL_{N-1}(\mathbb{C}) \times GL_1(\mathbb{C})$

SKETCH OF PROOF

→ cohomology classes $\longleftrightarrow$ fixed vectors in automorphic reps.

→ low degree of cohomology $\Rightarrow$ big SL_2 piece in the Galois rep at infinity.

→ big SL_2 piece in the Galois rep at infinity $\Rightarrow$ the whole representation comes from $U(N-1) \times U(1)$.

→ make this precise using the stable trace formula.

→ $$h^{2,0} \ll \underbrace{Nm(p^n)^{N-1}}_{\text{fundamental lemma}} \cdot \underbrace{Nm(p^n)}_{\text{classes on } U(1)} \asymp Nm(p^n)^N$$

Some Lingering Questions

→ Can one extend this to other groups? Other degrees? Lower bounds?

→ Can one give a geometric proof (e.g. counting cycles?)

Many thanks to Ipsita Datta for the slides inspiration!