The stable cohomology of symplectic groups over the integers

joint with

M. Land (Copenhagen)

T. Nikolaus (Münster)

time !!

#9

based on work with

B. Calmès (Lens)

E. Dotto (Warwick)

Y. Harpaz (Paris XIII)

K. Moi (Oslo)

D. Nardin (Regensburg)

W. Steimle (Augsburg)

= take with a grain of salt!

Definition $\mathcal{O}$ number ring

$O_{g,g}(\mathcal{O})$ is the isometry group of

$$\left(\mathcal{O}^{2g}, \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}\right)$$

$Sp_g(\mathcal{O})$ is the isometry group of

$$\left(\mathcal{O}^{2g}, \begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix}\right)$$

Question:

$$H^*(Sp_g(\mathcal{O})) = ?$$

$$H^*(O_{g,g}(\mathcal{O})) = ?$$

This talk $\mathcal{O} = \mathbb{Z}$!

Quadratic variants:

$$Sp_g^q(\mathbb{Z}) \leq Sp_g(\mathbb{Z})$$

index
$2^{2g-1} + 2^{g-1}$

stabiliser of quadratic refinement

$$q: \mathbb{Z}^{2g} \longrightarrow \mathbb{Z}/2, \quad (a_i, b_i) \longmapsto \sum_{i=1}^{g} a_i b_i$$

$O_{g,g}$ in Nathalie Wahl's talk

$$O_{g,g}^q(\mathbb{Z}) = \mathrm{Isom}\left(\mathbb{Z}^{2g}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

the polarisation of

$$q: \mathbb{Z}^{2g} \longrightarrow \mathbb{Z}, \quad (a_i, b_i) \longmapsto \sum_{i=1}^{g} a_i b_i$$

$$\Gamma_g = \pi_0 \mathrm{Diff}^+(\Sigma_g)$$

surface of genus g

$$\rightsquigarrow \quad 1 \to \mathrm{Tor}_g \to \Gamma_g \twoheadrightarrow Sp_g(\mathbb{Z}) \to 1$$

$$\varphi : \Sigma_g \circlearrowleft \;\longmapsto\; \varphi_* : H_1(\Sigma_g) \circlearrowleft$$

Similarly, for n odd

$$\mathrm{Diff}^+((S^n \times S^n)^{\# g}) \xrightarrow{H_n} Sp_g(\mathbb{Z})$$

has image $Sp_g^q(\mathbb{Z})$ for $n \neq 1, 3, 7$ (Kreck '87)

For n even

$$\mathrm{Diff}^+((S^n \times S^n)^{\# g}) \twoheadrightarrow O_{g,g}^q(\mathbb{Z})$$

$\rightsquigarrow$ Consider all four of

$$O_{g,g}(\mathbb{Z}),\quad O_{g,g}^q(\mathbb{Z}),\quad Sp_g(\mathbb{Z}),\quad Sp_g^q(\mathbb{Z}) \;=:\; G_g .$$

What is known in low degrees?

$$H_1(Sp_g(\mathbb{Z})) = \begin{cases} \mathbb{Z}/12 & g=1 \\ \mathbb{Z}/2 & g=2 \\ 0 & g\geq 3 \end{cases}$$

= abelianisation

$$H_1(Sp_g^q(\mathbb{Z})) = \begin{cases} \mathbb{Z}/4 \oplus \mathbb{Z} & g=1 \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2 & g=2 \\ \mathbb{Z}/4 & g\geq 3 \end{cases}$$

$$H_2(Sp_g(\mathbb{Z})) = \begin{cases} 0 & g=1 \\ \mathbb{Z}\oplus\mathbb{Z}/2 & g=2,3 \\ \mathbb{Z} & g\geq 4 \end{cases}$$

Meyer '72, measures the signature of surfaces bundles over surfaces

$$H_2(Sp_g^q(\mathbb{Z})) = \begin{cases} \mathbb{Z}/2 & g=1 \\ \mathbb{Z}\oplus ? & 2\leq g\leq 6 \\ \mathbb{Z} & g\geq 7 \end{cases}$$

Krannich-Kupers '19

Thm (Chesney, Mirzaii, van der Kallen, ... Friedrich '17)

The tautological map
$$H_*(G_g) \longrightarrow H_*(G_{g+1})$$
is an isomorphism for $* \leq g/2 - 3$

Goal

$$\text{stable part of } H^*(G_g) =: H^*(G).$$

λ_{4i-2} in Orsola Tommasi's talk

Thm (Borel, '74)

$$H^*(Sp^{(q)}(\mathbb{Z}); \mathbb{Q}) = \mathbb{Q}[c_{4i-2} \mid i \geq 1]$$

$$H^*(O^{(q)}(\mathbb{Z}); \mathbb{Q}) = \mathbb{Q}[c_{4i} \mid i \geq 1]$$

$$H^*(GL(\mathbb{Z}); \mathbb{Q}) = \Lambda_{\mathbb{Q}}[p_{4i+1} \mid i \geq 1]$$

Thm (Karoubi, '80)

$$H^*(O(\mathbb{Z}); \mathbb{Z}[1/2]) =$$

$$\mathbb{Z}[1/2, c_{4i}, i \geq 1] \otimes \operatorname{im}\left(H^*(GL(\mathbb{Z})) \to H^*(O(\mathbb{Z}))\right)$$

$$H^*(Sp(\mathbb{Z}); \mathbb{Z}[1/2]) =$$

$$\mathbb{Z}[1/2, c_{4i-2}, i \geq 1] \otimes \operatorname{im}\left(H^*(GL(\mathbb{Z})) \to H^*(Sp(\mathbb{Z}))\right)$$

Theorem (Dwyer-Mitchell '98)

p odd + regular

$$H^*(GL(\mathbb{Z}); \mathbb{F}_p) = \mathbb{F}_p[d_{2(p-1)i} \mid i \geq 1] \otimes \Lambda[p_{4i+1}, e_{2(p-1)i-1}, i \geq 1]$$

(assuming Quillen-Lichtenbaum conjecture now proven by Voevodsky, Rost, ...)

Theorem (maybe H-Land-Nikolaus)

p odd + regular

$$H^*(O^{(q)}(\mathbb{Z}); \mathbb{F}_p) = \mathbb{F}_p[c_{4i}, d_{2(p-1)i} \mid i \geq 1] \otimes \Lambda[e_{2(p-1)i-1} \mid i \geq 1]$$

$$H^*(Sp^{(q)}(\mathbb{Z}); \mathbb{F}_p) = \mathbb{F}_p[c_{4i-2}, d_{2(p-1)i} \mid i \geq 1] \otimes \Lambda[e_{2(p-1)i-1} \mid i \geq 1]$$

Thm (H-Land-Nikolaus)

$$H^*(O^q(\mathbb{Z});\mathbb{F}_2) = \mathbb{F}_2[w_i, v_i, a_{2i+1} \mid i \geq 1]$$

$$H^*(O(\mathbb{Z});\mathbb{F}_2) = \mathbb{F}_2[w_i, v_i, a_{2i+1}, y_1 \mid i \geq 1]$$

$$H^*(Sp(\mathbb{Z});\mathbb{F}_2) = \mathbb{F}_2[c_{2i} \mid i \geq 1] \otimes \Lambda[b_{4i-1} \mid i \geq 1]$$

$$H^*(Sp^q(\mathbb{Z});\mathbb{F}_2) = \mathbb{F}_2[x_2, x_4, z_{2^j+1}, c_{2i} \mid i \geq 1, j \geq 1] \otimes \Lambda[x_1, x_5, b_{4i-1} \mid i \geq 2]$$

Thm (Dwyer-Mitchell '98)

$$H^*(GL(\mathbb{Z});\mathbb{F}_2) = \mathbb{F}_2[w_i \mid i \geq 1] \otimes \Lambda[a_{2i+1} \mid i \geq 1]$$

more integral statements are possible,
e.g.

1) $\mathcal{O}_{g,g}(\mathbb{Z}) \to GL_{2g}(\mathbb{Z}) \to GL_{2g}(\mathbb{F}_3)$

$\mathcal{O}_{g,g}(\mathbb{Z}) \to \mathcal{O}^{top}_{g,g}(\mathbb{R}) \overset{\sim}{\leftarrow} \mathcal{O}(g) \times \mathcal{O}(g) \xrightarrow{pr_1} \mathcal{O}(g)$

give a 2-local isomorphism

$$\underbrace{H^*(BO \times BGL(\mathbb{F}_3))}_{\text{completely known}} \to H^*(BO(\mathbb{Z}))$$

2)

$$
\begin{aligned}
H_1(S_p(\mathbb{Z})) &= 0 \\
H_2(S_p(\mathbb{Z})) &= \mathbb{Z} \\
H_3(S_p(\mathbb{Z})) &= \mathbb{Z}/24 \\
H_4(S_p(\mathbb{Z})) &= \mathbb{Z} \\
H_5(S_p(\mathbb{Z})) &= \mathbb{Z}/24 \\
H_6(S_p(\mathbb{Z})) &= \mathbb{Z}^2 \\
&\vdots
\end{aligned}
$$

Method of proof

Just like

$$H^*(GL(\mathbb{Z})) = H^*(K(\mathbb{Z})_0)$$

with $K(R) = (\mathrm{Proj}(R), \oplus)^{grp}$

we have $\lambda \in \{\pm s, \pm q\}$

$$H^*(G^\lambda(\mathbb{Z})) = H^*(GW^\lambda(\mathbb{Z})_0)$$

with $GW^\lambda(R) = (\mathrm{Unimod}^\lambda(R), \oplus)^{grp}$

$$\left(GW^{-s}(\mathbb{Z}) = KSp(\mathbb{Z}) \right)$$

Tony Feng's talk

Thm (#9, '20) $\mathcal{O}$ number ring

i) There is a fibre sequence

$$K(\mathcal{O},\varepsilon)_{hC_2} \xrightarrow{hyp} GW^{\varepsilon s}(\mathcal{O}) \xrightarrow{bord} L^{\varepsilon s}(\mathcal{O})$$

that splits after inverting 2

Ranicki's L-spaces

ii) $$GW^{\varepsilon s}(\mathcal{O}) \xrightarrow{fgt} K(\mathcal{O},\varepsilon)^{hC_2}$$

is a 2-local equivalence

iii) $$\pi_* GW^{\varepsilon q}(\mathcal{O}) \xrightarrow{pol} \pi_* GW^{\varepsilon s}(\mathcal{O})$$

is an isomorphism past degree 3

nb. e.g. fibre sequence

$$GW^q(\mathbb{Z})_0 \longrightarrow GW^s(\mathbb{Z})_0 \longrightarrow B\mathbb{Z}/2$$

iv) has kernel and cokernel 8-torsion.
(for any ring)

Thm (Voevodsky, Rost, Bökstedt, Rognes, Weibel, ... 00's)

p a regular prime.

$$K(\mathbb{Z})_{(p)} \simeq \mathrm{fib}\left(BO_{(p)} \xrightarrow{c\cdot(\psi^{\ell}-\mathrm{id})} \tau_{\geq 4} BU_{(p)}\right)$$

ℓ top. generator of $\mathbb{Z}_p^\times$, p odd
$\ell = 3$, $p = 2$.

Thm (Ranicki-Sullivan '80s)

- $L^s(\mathbb{Z})[1/2] \simeq BO[1/2]$.
- $L^{-s}(\mathbb{Z}) \simeq \Omega^2 L^s(\mathbb{Z})$