Monopoles: construction, dynamics, transforms

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1st February 2021

Save pure maths at Leicester:
https://www.ipetitions.com/petition/mathematics-is-not-redundant

The world in 1974

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BPS monopoles

$\Phi : \mathbb{R}^3 \to \mathfrak{su}(2)$, $A \in \Omega^1(\mathbb{R}^3) \otimes \mathfrak{su}(2)$,
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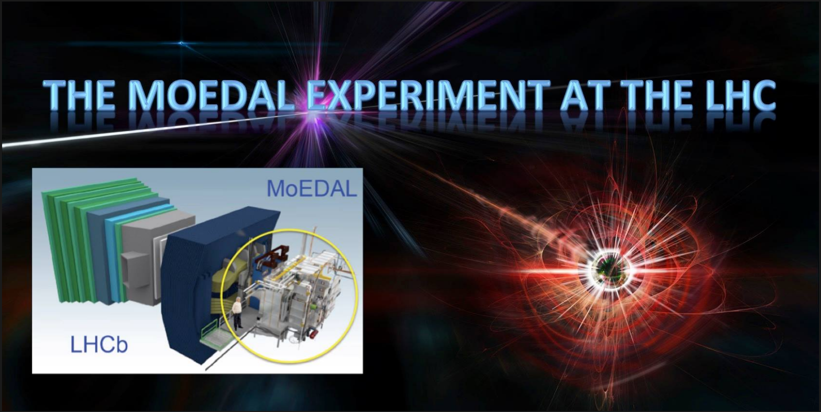
't Hooft, Polyakov 1974: $*F^A \approx \begin{pmatrix} iN & 0 \\ 0 & -iN \end{pmatrix} \frac{dr}{2r^2}$ in gauge where Φ is diagonal $\implies$ magnetic pole of charge $2\pi N$ in U(1) gauge theory.

The search for monopoles continues. . .



Holy grail of particle physics?

The Prasad-Sommerfield solution (1975)

$$\begin{aligned} \Phi &= \left(\coth(2r) - \frac{1}{2r}\right) Q \\ A &= \frac{1}{2}\left(1 - \frac{2r}{\sinh(2r)}\right) Q \mathrm{d} Q \\ Q &= \frac{x_j}{r} \mathrm{i}\sigma_j \end{aligned}$$

Spherically symmetric, $N = 1$.

Moduli spaces

Theorem (Taubes (1980s))

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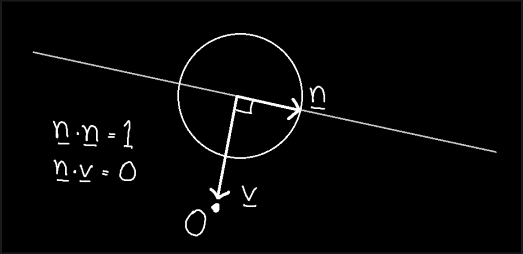
$E = \frac{1}{2}\int_{\mathbb{R}^3} |d^A\Phi|^2 + |F^A|^2$ is the static energy of a (dynamical) Lagrangian field theory.

Theorem (Stuart (1994))

Geodesics on M_N approximate low-energy dynamics of this field theory.

Spectral curves

Minitwistor space = $\{\text{oriented lines in } \mathbb{R}^3\} = TS^2 = T\mathbb{CP}^1$.



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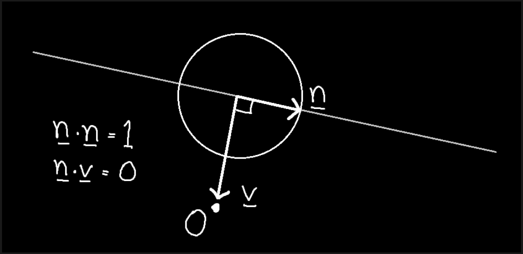

A line L with coordinate $s \in \mathbb{R}$ is called *spectral* if

$$\frac{\partial}{\partial s} \lrcorner \, \mathrm{d}^A v + \mathrm{i}\Phi v = 0$$

has a solution $v : L \to \mathbb{C}^2$ that decays as $s \to \pm\infty$.

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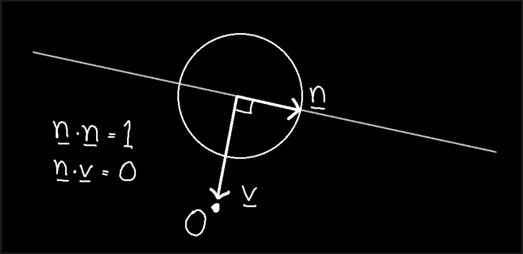

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The *spectral curve* of a monopole is the set of all spectral lines. It is an algebraic variety $S \subset T\mathbb{CP}^1$.

Spectral curves

Theorem (Hitchin 1982)

M_N is in bijection with the set of irreducible curves $S \subset T\mathbb{CP}^1$ of the form

$$\eta^N + \eta^{N-1} a_1(\zeta) + \ldots + a_N(\zeta) = 0$$

for polynomials a_i of degree $2i$, satsifying:

1. *S is invariant under the antipodal map;*
2. *L^2 is trivial and $L^1(N-1)$ is real on S;*
3. *$H^0(S, L^s(N-2)) = 0$ for $0 < s < 2$.*

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NB S has genus $(N-1)^2$.

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Hard to recover monopole from S... but can easily recover ϕ s.t. $\Phi = 1 - \phi + O(e^{-\epsilon r})$ (Hurtubise 1985).

Nahm transform

$T_1, T_2, T_3 : (-1,1) \to \mathfrak{u}(N)$ are called *Nahm data* if:

$$\frac{\mathrm{d}T_i}{\mathrm{d}s} = \frac{1}{2}\varepsilon_{ijk}[T_j, T_k]$$

$$T_i(s) = \frac{R_i^{\pm}}{\pm 1 - s} + O(1) \text{ as } s \to \pm 1.$$

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Nahm data $\to$ monopole: for $\mathbf{x} \in \mathbb{R}^3$ let

$$E_{\mathbf{x}} = \left\{ v : [-1,1] \to \mathbb{C}^N \otimes \mathbb{C}^2 \ : \ \frac{dv}{ds} = (x_j 1_N - \mathrm{i} T_j) \otimes \sigma_j \, v \right\}.$$

Then $E \to \mathbb{R}^3$ is a rank 2 vector bundle. If A is the induced connection and $\Phi : E \to E$ is the orthogonal projection of the operator $v(s) \to \mathrm{i}sv(s)$ then (A, Φ) is a monopole.

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- Nahm data $\to$ monopoles is a bijection (Hitchin 1983)
- Implementing this requires integration
- The spectral curve $S \subset T\mathbb{CP}^1$ can be written in coordinates (ζ, η):

$$\det(T_1 + iT_2 - 2iT_3\zeta + (T_1 - iT_2)\zeta^2 + \eta 1_N) = 0$$

Charge 2 monopoles

Up to translations and rotations, the spectral curve of a 2-monopole is (Hurtubise 1983):

$$\eta^2 + \frac{K^2}{4}\left(\zeta^4 + 2(k^2 - k'^2)\zeta^2 + 1) + 1\right) = 0.$$

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The associated Nahm data are known explicitly: $T_j = \frac{\sigma_j}{2\mathrm{i}} f_j(s)$ (no sum) with

$$f_1(s) = K\frac{\mathrm{dn}(Ks)}{\mathrm{cn}(Ks)}, \quad f_2(s) = Kk'\frac{\mathrm{sn}(Ks)}{\mathrm{cn}(Ks)}, \quad f_3(s) = Kk'\frac{1}{\mathrm{cn}(Ks)}.$$

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What about the associated monopole?

The axially symmetric 2-monopole

When $k = 0$ the monopole has axial symmetry about the x_2-axis. Ward (1981) obtained:

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\begin{aligned}
|\Phi| &= \left| \tanh(2r) - \frac{16r}{16r^2 + \pi^2} \right| \text{ on the } x_2\text{-axis} \\
|\Phi| &= 1 + \frac{2\pi^2 \cos\rho(\sin\rho - \rho\cos\rho)}{\rho(\pi^2\cos^2\rho - 16r^2)} \text{ in the } x_1, x_3\text{-plane}
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where $\rho = \sqrt{\pi^2/4 - 4r^2}$. This yields a formula for $\mathcal{E} = \frac{1}{2}(|d^A\Phi|^2 + |F^A|^2)$ at $\mathbf{x}=0$ using the identity $\mathcal{E} = -\frac{1}{2}\triangle|\Phi|^2$:

$$\mathcal{E}|_{\mathbf{x}=0} = \frac{8}{\pi^4}(\pi^2-8)^2$$

Method: construct an associated holomorphic bundle over $T\mathbb{CP}^1$ using the $\mathcal{A}_k$-ansatz.

Constructing the general 2-monopole ($k \in [0, 1)$)

- The $\mathcal{A}_k$-ansatz (1981–1983): Corrigan, Fairlie, Goddard, Yates, Prasad, Rossi, Brown, O Raifeartaigh, Rouhani, Singh.
- Forgács, Horváth, Palla (1980–1983): Ernst equation and Bäcklund transformations. Later used to make first video of 2-monopole scattering.
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$$\mathcal{E}|_{\mathbf{x}=0} = \frac{32}{k^8 k'^2 K^4}\left[k^2(K^2k'^2 + E^2 - 4EK + 2K^2 + k^2) - 2(E-K)^2\right]^2$$

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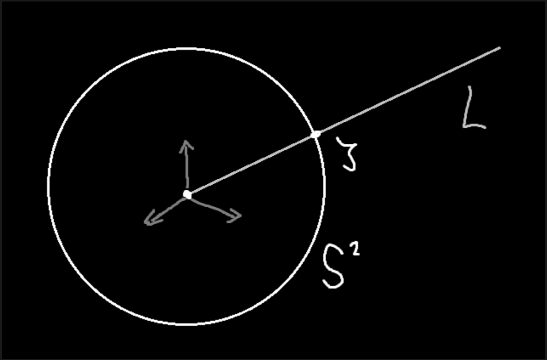
The zeros of Φ are *not* at $(\pm kK/2, 0, 0)$.

Rational maps

Given a monopole, construct $R : \mathbb{CP}^1 \to \mathbb{CP}^1$ as follows:

1. Let $L \subset \mathbb{R}^3$ be the half-line starting at 0 defined by $\zeta \in \mathbb{CP}^1$.
2. Let $v : L \to \mathbb{C}^2$ be a non-zero solution to $\frac{\partial}{\partial r} \lrcorner d^A v - \Phi v = 0$ that decays as $r \to \infty$
3. $v(0) \in \mathbb{C}^2 \setminus \{0\}$ determines a point $R(\zeta) \in \mathbb{CP}^1$.



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Theorem (Jarvis (2000))

The map $(A, \Phi) \mapsto R$ is a bijection from M_N to the space of degree N rational maps $\mathbb{CP}^1 \to \mathbb{CP}^1$, modulo rotations of the target $\mathbb{CP}^1$.

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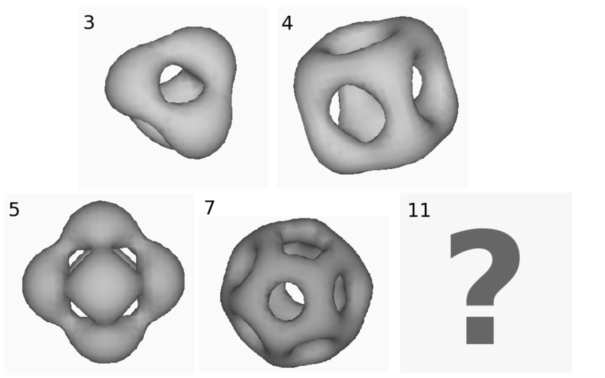
Jarvis' construction allows classification of monopoles invariant under subgroups $\Gamma \subset SO(3)$ (Houghton–Manton–Sutcliffe 1998). Much easier than working with spectral curves (Hitchin–Manton–Murray 1995).

Platonic monopoles

Houghton–Sutcliffe 1996: solve Nahm equation *explicitly*, construct monopole *numerically*



$$\eta^3 - \frac{2i\pi^6}{3^{\frac{9}{2}}\Gamma(\frac{2}{3})^9}\zeta(\zeta^4-1) \qquad \eta^4 + \frac{3\pi^6}{2^8\Gamma(\frac{3}{4})^8}(\zeta^8+14\zeta^4+1)$$

$$\eta^5 - \frac{3\pi^6}{2^6\Gamma(\frac{3}{4})^8}(\zeta^8+14\zeta^4+1)\eta \qquad \eta^7 - \frac{16\pi^{12}}{729\Gamma(\frac{2}{3})^{18}}(\zeta^{11}-11\zeta^6-\zeta)$$

Magnetic bags

Bolognesi conjecture (2006): the “smallest” charge N is approximately spherical, with

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Theorem (Taubes)

Let (A, Φ) be a monopole and $\Omega_\epsilon = \{|\Phi| < \epsilon\} \subset \mathbb{R}^3$. Then

$$\mathrm{diam}(\Omega_\epsilon) > \frac{N}{1-\epsilon}.$$

Here $\mathrm{diam}(\Omega) := \inf\{d \in \mathbb{R} \,:\, \Omega \subset B_{d/2}\}$.
Taubes also constructs monopoles that come close to saturating the bound.

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Other ways to measure the size of a monopole?

Simple gauge groups G

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Spectral curve construction (Hurtubise–Murray 1990): curves in $T\mathbb{CP}^1 \leftrightarrow$ nodes in Dynkin diagram of G. Intersections $\leftrightarrow$ lines in Dynkin diagram.

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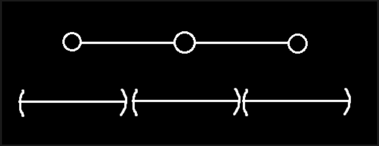
Nahm transform for *classical groups* only (Hurtubise–Murray 1989). For SU(n), get Nahm equations on intervals $\leftrightarrow$ nodes, with gluing at ends $\leftrightarrow$ lines.



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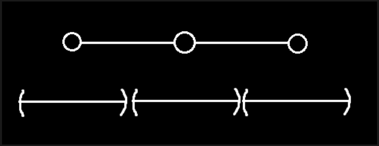

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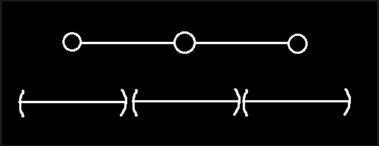

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Nahm transform for non-maximal symmetry breaking: work in progress (Charbonneau–Nagy)
Nahm transform for non-classical groups unknown (but see Shnir–Zhilin 2015).

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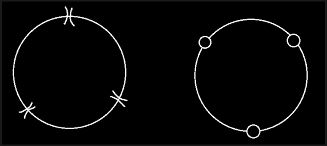

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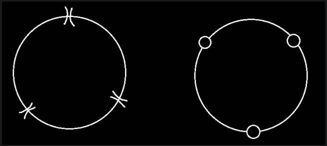


Explicit (1,1)-calorons (Harrington-Shepard 1978; Kraan–van Baal, Lee–Lu 1998)

Classification of charge (N, N) SU(2) calorons with cyclic symmetry (Cork 2018) – involves automorphisms of Dynkin diagram.

Monopoles on $\mathbb{R}^2 \times S^1$ (“monopole chains”)

Nahm transform relates monopoles on $\mathbb{R}^2 \times S^1$ to Hitchin’s equations on a cylinder (Cherkis–Kapustin 2001) and parabolic Higgs bundles (Harland 2020).

$\exists$ N distinct charge N monopoles on $\mathbb{R}^2 \times S^1$ with $\mathbb{Z}_{2N}$ symmetry (Harland 2020).

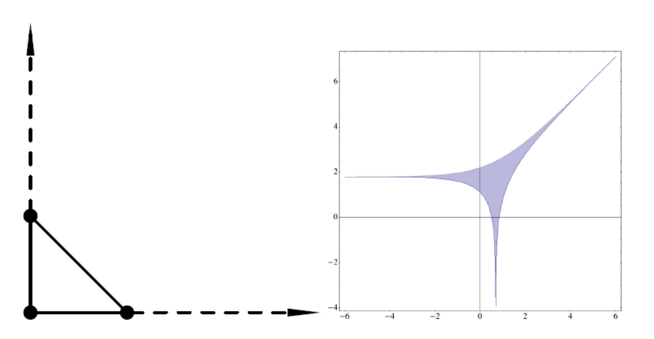
Dynamics: Maldonado–Ward 2013

Monopoles on $\mathbb{R} \times T^2$ "monowalls"

Nahm transform: monowalls $\leftrightarrow$ monowalls (Cherkis–Ward 2012).

Nahm transform part of a SL$(2, \mathbb{Z})$ action on moduli spaces of monowalls.

Perturbative explicit solution involving θ-functions.



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Discrete Nahm data known for 2-monopole, but not platonic monopoles.

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$$|\Phi|^2 = \frac{r^2(1+r^2)^2 - \rho^2(1+r^4) + \frac{1}{4}\rho^4}{\left((1+r^2)^2 - \rho^2\right)^2}$$

$N = 11$ icosahedral, along x_3-axis:

$$|\Phi|^2 = \frac{x_3^2(25x_3^8 + 20x_3^6 - 218x_3^4 + 20x_3^2 + 25)^2}{(75x_3^{10} + 55x_3^8 - 2x_3^6 - 2x_3^4 + 55x_3^2 + 75)^2}.$$

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These approaches also yield spectral curves (Bolognesi–Cockburn–Sutcliffe 2015, Sutcliffe 2020), e.g. for dodecahedral 7-monopole.