Elliptic zastava

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Zastava

- X a smooth complex projective curve. G a simply connected semisimple group. $T \subset B \subset G$ a Cartan torus and Borel subgroup; N_- the opposite unipotent subgroup. $\alpha = \sum_{i \in I} a_i \alpha_i \in \mathbb{X}_*(T)_{\mathrm{pos}}$ a coroot.
- The (open) zastava $\mathring{Z}^\alpha_X$: the moduli space of G-bundles on X with a flag (a B-structure) of degree α and a generically transversal N_--structure. A smooth variety of dimension $2|\alpha| = 2\sum_{i \in I} a_i$.
- The factorization projection $\pi_\alpha \colon \mathring{Z}^\alpha_X \to X^\alpha$ to the colored configuration space on X: remembers where the N_-- and B-structures are not transversal. Has a local nature: $\pi_\alpha^{-1}(D^\alpha)$ is independent of X for any analytic disc $D \subset X$.

Additive case

- $X = \mathbb{P}^1$, and we additionally require that the N_-- and B-structures are transversal at $\infty \in \mathbb{P}^1$. We obtain a smooth affine variety $\mathring{Z}^\alpha_{\mathbb{G}_a} \to \mathbb{A}^\alpha$. It is an algebraic-geometric incarnation of the moduli space of euclidean G_c-monopoles with maximal symmetry breaking at infinity, of topological charge α. So it carries a hyperkähler structure and hence a holomorphic symplectic form.
- From the modular point of view, the classifying stack BG has a 2-shifted symplectic structure, and $BB \to BG$ has a coisotropic structure. $\mathring{Z}_{\mathbb{G}_a}$ is the space of based maps from $(\mathbb{P}^1, \infty)$ to G/B, that is a fiber of $\mathrm{Maps}(\mathbb{P}^1, \infty; BB) \xrightarrow{p} \mathrm{Maps}(\mathbb{P}^1, \infty; BG)$. The latter space has a 1-shifted symplectic structure, and p is coisotropic as well as $\mathrm{pt} \to \mathrm{Maps}(\mathbb{P}^1, \infty; BG)$. Hence the desired Poisson (symplectic) structure on $\mathring{Z}_{\mathbb{G}_a}$ [T.Pantev, T.Spaide].

Explicit formula

- *Factorization property:* the addition of divisors $X^\beta \times X^\gamma \to X^\alpha$ for $\alpha = \beta + \gamma$. A canonical isomorphism

$$\mathring{Z}^\alpha_X \times_{X^\alpha} (X^\beta \times X^\gamma)_{\mathrm{disj}} \cong (\mathring{Z}^\beta \times \mathring{Z}^\gamma)|_{(X^\beta \times X^\gamma)_{\mathrm{disj}}}$$

- For a simple coroot α_i a canonical isomorphism $\mathring{Z}^{\alpha_i}_{\mathbb{G}_a} \cong \mathbb{G}_a \times \mathbb{G}_m$. Hence for arbitrary α away from diagonals in $\mathbb{A}^\alpha$ we have coordinates $(w_{i,r} \in \mathbb{G}_a)_{r=1}^{a_i}$ and $(y_{i,r} \in \mathbb{G}_m)_{r=1}^{a_i}$ on $\mathring{Z}^{\alpha_i}_{\mathbb{G}_a}$ up to simultaneous permutations in $S_\alpha = \prod_{i \in I} S_{a_i}$.
- From now on G is assumed simply laced. Choose an orientation of the Dynkin graph. Coordinate change: $u_{i,r} := y_{i,r} \prod_{i \to j} \prod_{s=1}^{a_j} (w_{j,s} - w_{i,r})^{-1}$. The new coordinates are "Darboux" in the sense that the only nonzero brackets are $\{w_{i,r}, u_{i,r}\} = u_{i,r}$.

Integrable system

- The factorization projection $\mathring{Z}^\alpha_{\mathbb{G}_a} \to \mathbb{A}^\alpha$ is an integrable system. In case $G = \mathrm{SL}(2)$, the degree α is a positive integer d. Then we get the Atiyah-Hitchin system.
- It also coincides with the open Toda system for $\mathrm{GL}(d)$. In particular, $\mathbb{A}^{(d)}$ is the Kostant slice for $\mathfrak{gl}(d)$, and $\mathring{Z}^d_{\mathbb{G}_a}$ is the universal centralizer (pairs: x in the slice, and commuting $g \in \mathrm{GL}(d)$).
- Equivalently, take a surface $S = \mathbb{G}_a \times \mathbb{G}_m \cong \mathring{Z}^1_{\mathbb{G}_a}$. Then $\mathring{Z}^d_{\mathbb{G}_a} \simeq \mathrm{Hilb}^d_{\mathrm{tr}}(S)$: the transversal Hilbert scheme of d points on S. It is an open subscheme of $\mathrm{Hilb}^d(S)$ classifying the subschemes whose projection to $\mathbb{G}_a$ is a closed embedding.
- A symplectic form on S: $\{w, y\} = y$ induces a symplectic form on $\mathrm{Hilb}^d_{\mathrm{tr}}(S)$. It coincides with the above symplectic form on $\mathring{Z}^d_{\mathbb{G}_a}$.

Coulomb branch of a quiver gauge theory

- Recall the oriented Dynkin graph of G. Take the gauge group $\mathbf{G} := \prod_{i\in I} \mathrm{GL}(a_i)$ acting on $\mathbf{N} := \oplus_{i\to j} \mathrm{Hom}(\mathbb{C}^{a_i}, \mathbb{C}^{a_j})$. It gives rise to a certain space of triples $\mathcal{R}_{\mathbf{G},\mathbf{N}}$ over the affine Grassmannian $\mathrm{Gr}_{\mathbf{G}}$, and the Coulomb branch $\mathcal{M}_C(\mathbf{G},\mathbf{N}) := \mathrm{Spec}\, H^{\mathbf{G}[\![t]\!]}(\mathcal{R}_{\mathbf{G},\mathbf{N}})$ (symplectically dual to Nakajima quiver variety $(\mathbf{N}\oplus\mathbf{N}^*)/\!\!/\mathbf{G}$).
- We have $\mathcal{M}_C(\mathbf{G},\mathbf{N}) \simeq \mathring{Z}^{\alpha}_{\mathbb{G}_a}$, and the integrable system $\mathring{Z}^{\alpha}_{\mathbb{G}_a} \to \mathbb{A}^{\alpha}$ corresponds to the embedding $\mathbb{C}[\mathbb{A}^{\alpha}] \cong H^{\mathbf{G}[\![t]\!]}(\mathrm{pt}) \subset H^{\mathbf{G}[\![t]\!]}(\mathcal{R}_{\mathbf{G},\mathbf{N}})$.

Multiplicative case

- $X = \mathbb{P}^1$, and we additionally require that the N_-- and B-structures are transversal at $\infty \in \mathbb{P}^1$ and $0 \in \mathbb{P}^1$. We obtain a smooth affine variety $\mathring{Z}^\alpha_{\mathbb{G}_m} \to \mathbb{G}_m^\alpha$.
- Its symplectic structure can be again defined in modular terms, but it *is not* the restriction of the symplectic structure of $\mathring{Z}^\alpha_{\mathbb{G}_a}$ under the open embedding $\mathring{Z}^\alpha_{\mathbb{G}_m} \subset \mathring{Z}^\alpha_{\mathbb{G}_a}$. For a simple coroot, $\mathring{Z}^{\alpha_i}_{\mathbb{G}_m} \cong \mathbb{G}_m \times \mathbb{G}_m$, and $\{w, y\} = wy$ (G is ADE).
- The (quasi)-Hamiltonian reduction $\mathring{Z}^\alpha_{\mathbb{G}_m} /\!\!/ T$ is an algebraic-geometric incarnation of the moduli space of *periodic* euclidean G_c-monopoles of topological charge α in one of its complex structures. It is the multiplicative analogue of *centered* euclidean monopoles, the Coulomb branch with gauge group $\prod_{i \in I} \mathrm{SL}(a_i)$.

Integrable system and cluster structure

- The factorization projection $\mathring{Z}^{\alpha}_{\mathbb{G}_m} \to \mathbb{G}_m^{\alpha}$ is an integrable system. In case $G = \mathrm{SL}(2)$, degree d, it coincides with the relativistic open Toda system for $\mathrm{GL}(d)$. In particular, $\mathring{Z}^{d}_{\mathbb{G}_m}$ is the universal group-group centralizer. Also, $\mathring{Z}^{d}_{\mathbb{G}_m} \simeq \mathrm{Hilb}^{d}_{\mathrm{tr}}(S')$, where $S' = \mathbb{G}_m \times \mathbb{G}_m$. Finally, $\mathring{Z}^{\alpha}_{\mathbb{G}_m}$ is isomorphic to a K-theoretic Coulomb branch and carries a natural cluster structure.

Elliptic case

- $X = E$ an elliptic curve, $G = \mathrm{SL}(2)$, $S'' = E \times \mathbb{G}_m$ with an invariant symplectic structure. Then $\mathrm{Hilb}^d_{\mathrm{tr}}(S'') \subset T^*E^{(d)}$, an open subvariety of the cotangent bundle.
- Surprise: $\mathring{Z}^d_E$ is an open subvariety of the *tangent* bundle $TE^{(d)}$, *not* isomorphic to $\mathrm{Hilb}^d_{\mathrm{tr}}(S'')$; *does not* carry any symplectic structure.
- Still there is a relation between $\mathring{Z}^d_E$ and the symplectic $\mathrm{Hilb}^d_{\mathrm{tr}}(S'')$. To describe it we need a compactification of $\mathring{Z}^\alpha_E$. Generically transversal N_-- and B-structures on a G-bundle on E define its generic trivialization (away from a colored divisor $D = \pi_\alpha(\phi),\ \phi \in \mathring{Z}^\alpha_E$). Thus we obtain an embedding of $\mathring{Z}^\alpha_E$ into a version of Beilinson-Drinfeld Grassmannian of E (partially symmetrized to live over $E^\alpha = E^{|\alpha|}/S_\alpha$). The desired compactification $\overline{Z}^\alpha_E$ is the closure of $\mathring{Z}^\alpha_E$ in the Beilinson-Drinfeld Grassmannian. In case of $\mathrm{SL}(2)$, degree d, it is a fiberwise compactification of the tangent bundle $TE^{(d)}$.

Compactified zastava

- $\overline{Z}_E^\alpha$ is the moduli space of G-bundles on E equipped with generically transversal *generalized* N_-- and B-structures. We also allow a twist of N_--structure. For $G = \mathrm{SL}(2)$, degree d, we consider the data

  $$\mathcal{L} \subset \mathcal{V} \xrightarrow{\xi} \mathcal{K},$$

  where $\mathcal{V}$ is a rank 2 vector bundle, $\det \mathcal{V} \cong \mathcal{O}_E$;
  $\mathcal{L}$ an invertible subsheaf (not necessarily a line subbundle);
  ξ a morphism to a line bundle $\mathcal{K}$ (not necessarily surjective).
  $\xi|_{\mathcal{L}}$ is not zero, and $\mathrm{length}(\mathcal{K}/\xi(\mathcal{L})) = d$.
  We fix $\mathcal{K}$ and obtain the (twisted) compactified zastava $\overline{Z}_{\mathcal{K}}^d$.
- For general G we consider the similar data for the associated (to all irreducible representations of G) vector bundles and impose Plücker relations. We get $\overline{Z}_{\mathcal{K}}^\alpha$, where $\mathcal{K}$ is a T-bundle.

Mirković approach

- The relatively very ample determinant line bundle on the Beilinson-Drinfeld Grassmannian restricted to $\overline{Z}^{\alpha}_{\mathcal{K}}$ gives a very explicit projective embedding. *Reason:* restriction to the T-fixed points in $\overline{Z}^{\alpha}_{\mathcal{K}}$ gives an isomorphism on sections of the determinant line bundle [X.Zhu]
- The T-fixed points components are $E^{\beta} \times E^{\gamma},\ \beta + \gamma = \alpha$. The contribution of a component is

$$\mathbf{q}_* \left(\mathbf{p}^* \Big(\mathcal{K}^{\beta} \big(\sum_{i \in I} \Delta^{\beta}_{ii} - \sum_{i \to j} \Delta^{\beta}_{ij} \big) \Big) \big(\sum_{i \in I} \Delta^{\beta,\gamma}_{ii} \big) \right),$$

where $E^{\beta} \xleftarrow{\mathbf{p}} E^{\beta} \times E^{\gamma} \xrightarrow{\mathbf{q}} E^{\alpha}$ (addition of colored divisors); $\Delta^{\beta,\gamma}_{ij} \subset E^{\beta} \times E^{\gamma}$ is the incidence divisor; $\Delta^{\beta}_{ii} \subset E^{\beta}$ is the incidence divisor; $\mathcal{K}^{\beta} = \boxtimes_i \mathcal{K}_i^{(b_i)}$ (symmetric powers), and $\mathcal{K}_i$ is the line bundle associated to the character $-\alpha_i^{\vee} \colon T \to \mathbb{C}^{\times}$.

Mirković approach

- Summing up the above vector bundles on E^α over all partitions $\beta+\gamma=\alpha$ we obtain a factorizable vector bundle $\mathbb{V}^\alpha_{\mathcal{K}}$ of rank $2^{|\alpha|}$. When $\alpha=\alpha_i$, we get $\mathbb{V}^{\alpha_i}_{\mathcal{K}}=\mathcal{K}_i\oplus\mathcal{O}_E$, and $\overline{Z}^{\alpha_i}_{\mathcal{K}}=\mathbb{P}\mathbb{V}^{\alpha_i}_{\mathcal{K}}$.
- Away from diagonals in E^α, we get the fiberwise Segre embedding (from factorization):
  a fiber of compactified zastava $\simeq(\mathbb{P}^1)^{|\alpha|}\hookrightarrow$ a fiber of $\mathbb{P}\mathbb{V}^\alpha_{\mathcal{K}}$.
  The whole of $\overline{Z}^\alpha_{\mathcal{K}}$ is the closure in $\mathbb{P}\mathbb{V}^\alpha_{\mathcal{K}}$ of the off-diagonal Segre embedding image.
- $\mathring{Z}^\alpha_{\mathcal{K}}\subset\overline{Z}^\alpha_{\mathcal{K}}$ is the complement to 2 hyperplane sections. One hyperplane $\mathbb{V}^\alpha_{\mathcal{K},\mathrm{low}}\subset\mathbb{V}^\alpha_{\mathcal{K}}$ is the direct sum of all contributions from partitions $\beta+\gamma=\alpha,\ \beta\neq0$. The other hyperplane $\mathbb{V}^{\alpha,\mathrm{up}}_{\mathcal{K}}\subset\mathbb{V}^\alpha_{\mathcal{K}}$ is the direct sum of all contributions from partitions $\beta+\gamma=\alpha,\ \gamma\neq0$.

Coulomb version

- Instead of $\mathbb{V}^\alpha_\mathcal{K}$ consider

$$\mathbb{U}^\alpha_\mathcal{K} = \bigoplus_{\beta+\gamma=\alpha} \mathbf{q}_* \left(\mathbf{p}^* \mathcal{K}^\beta \otimes \mathcal{O}_{E^\beta \times E^\gamma} \big(\sum_{i\to j} \Delta^{\beta,\gamma}_{ij} \big) \right),$$

  dual to $\oplus$ of equivariant elliptic homology of all the positive minuscule parts of $\mathcal{R}_{\mathbf{G},\mathbf{N}}$, space of triples over $\prod_{i\in I} \mathrm{Gr}_{\mathrm{GL}(a_i)}$.
- It is a factorizable vector bundle of rank $2^{|\alpha|}$, and away from diagonals in E^α we get the fiberwise Segre embedding of $(\mathbb{P}^1)^{|\alpha|}$ into a fiber of $\mathbb{P}\mathbb{U}^\alpha_\mathcal{K}$. The closure is the *Coulomb* elliptic zastava ${}^C\overline{Z}^\alpha_\mathcal{K}$. Removing the two hyperplane sections we get the *open* Coulomb zastava ${}^C\mathring{Z}^\alpha_\mathcal{K} \simeq \operatorname{Spec} H^{\mathbf{G}[\![t]\!]}_{\ell\ell}(\mathcal{R}_{\mathbf{G},\mathbf{N}})$.
- In type A_1, ${}^C\mathring{Z}^d_\mathcal{K}$ is isomorphic to the transversal Hilbert scheme of d points in the total space of line bundle $\mathcal{K}$ with zero section removed.
  ${}^C\overline{Z}^d_{\mathcal{O}_E}$ is the fusion of minuscule $\mathbb{P}^1$-orbits in $\mathrm{Gr}_{\mathrm{PGL}(2),E^{(d)}}$.

Hamiltonian reduction

- The total space of any line bundle $\mathcal{K}_i$ without zero section carries a symplectic form invariant with respect to dilations. Away from the diagonals in E^α, ${}^C\overset{\circ}{Z}{}^\alpha_\mathcal{K}$ is étale covered by a product of $\mathcal{K}_i$, and the direct sum of the above forms extends through the diagonals as a symplectic form on ${}^C\overset{\circ}{Z}{}^\alpha_\mathcal{K}$.
- The action of T is hamiltonian, and we perform the hamiltonian reduction. Consider the composition

$$\mathrm{AJ}_Z\colon {}^C\overset{\circ}{Z}{}^\alpha_\mathcal{K} \xrightarrow{\pi_\alpha} E^\alpha \to \prod_{i\in I} \mathrm{Pic}^{a_i} E$$

  of the factorization projection with the Abel-Jacobi morphism. The reduction ${}^C_\mathcal{D}\overset{\circ}{Z}{}^\alpha_\mathcal{K} = {}^C\overset{\circ}{Z}{}^\alpha_\mathcal{K}/\!\!/T := \mathrm{AJ}_Z^{-1}(\mathcal{D})/T$ is conjecturally isomorphic to the moduli space of doubly periodic G_c-monopoles (*monowalls*) of topological charge α. It is the elliptic analogue of *centered* euclidean monopoles, the Coulomb branch with gauge group $\prod_{i\in I}\mathrm{SL}(a_i)$.

Mock Hamiltonian reduction

- Though the elliptic zastava $\overset{\circ}{Z}{}^{\alpha}_{\mathcal{K}}$ is not symplectic, we can mimic the hamiltonian reduction procedure and define the reduced zastava ${}_{\mathcal{D}}\overset{\circ}{Z}{}^{\alpha}_{\mathcal{K}} := \mathrm{AJ}_Z^{-1}(\mathcal{D})/T$. In case T-bundle $\mathcal{K}$ has degree 0 and is *regular*, the reduced zastava is the moduli space of G-bundles of fixed type $\mathrm{Ind}_T^G \mathcal{K}$ with B-structure of fixed type (fixed isomorphism class of the bundle induced from B to the abstract Cartan T).
- Both Bun_G and $\mathrm{Bun}_{\mathrm{T}}$ carry 1-shifted symplectic structures. The Lagrangian structures on $\mathrm{Bun}_B \to \mathrm{Bun}_G \times \mathrm{Bun}_{\mathrm{T}}$ and on the stacky point $[\mathcal{V}] \times [\mathcal{L}] \to \mathrm{Bun}_G \times \mathrm{Bun}_{\mathrm{T}}$ give rise to a symplectic structure on their cartesian product ${}_{\mathcal{D}}\overset{\circ}{Z}{}^{\alpha}_{\mathcal{K}}$:

$$\begin{array}{ccc} {}_{\mathcal{D}}\overset{\circ}{Z}{}^{\alpha}_{\mathcal{K}} & \longrightarrow & \mathrm{Bun}_B \\ \downarrow & & \downarrow \\ [\mathcal{V}] \times [\mathcal{L}] & \longrightarrow & \mathrm{Bun}_G \times \mathrm{Bun}_T \end{array}$$

Happy end

- Miracle: the reduced zastava are isomorphic: ${}_{\mathcal{D}}\mathring{Z}^\alpha_{\mathcal{K}} \simeq {}^C_{\mathcal{D}}\mathring{Z}^\alpha_{\mathcal{K}'}$ for $\mathcal{K}'_i = \mathcal{K}_i \otimes \mathcal{D}_i \otimes \bigotimes_{i\to j} \mathcal{D}_j^{-1}$.
- Theorem: This isomorphism is a symplectomorphism.
- Explicit formula for the Poisson brackets of the natural étale coordinates $(w_{i,r}, y_{i,r})_{i\in I}^{1\le r\le a_i}$ on ${}_{\mathcal{D}}\mathring{Z}^\alpha_{\mathcal{K}}$ (w-s are constrained to have a fixed sum in E, and y-s are homogeneous coordinates, i.e. only their ratios are well defined on the reduced zastava):

$$\left\{\frac{y_{i,r}}{y_{i,r'}}, w_{i,p}\right\} = (\delta_{rp} - \delta_{r'p})\frac{y_{i,r}}{y_{i,r'}}, \quad \left\{\frac{y_{i,r'}}{y_{i,p'}}, \frac{y_{j,r}}{y_{j,p}}\right\} = \frac{y_{i,r'}}{y_{i,p'}} \cdot \frac{y_{j,r}}{y_{j,p}}.$$

$$\big(\zeta(w_{i,r'}-w_{j,r})-\zeta(w_{i,r'}-w_{j,p})-\zeta(w_{i,p'}-w_{j,r})+\zeta(w_{i,p'}-w_{j,p})\big).$$

in case $i \neq j$ are joined by an edge in the Dynkin diagram of G, and zero otherwise (we assume G simply laced). Here $\zeta(w) = \frac{1}{w} + \sum_{\gamma\in\Gamma\setminus\{0\}} \left(\frac{1}{w-\gamma} + \frac{1}{\gamma} + \frac{w}{\gamma^2}\right)$ is the Weierstraß zeta function (the sum is taken over the period lattice of E).