Representation problem by Quadratic Forms

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- Introduction and History

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- Binary Quadratic Forms

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- Triangular Numbers

Definition

A Quadratic form in n variables is a homogenous polynomial of degree 2 in $x_1, x_2, \cdots, x_n$.

- An integral binary quadratic form is given by

$$ax^2 + bxy + cy^2$$

  for integers a, b an c.
- The form is called primitive if a, b and c are relatively prime.
- The discriminant of the above form is defined to be $b^2 - 4ac$.
- Two binary quadratic forms $Q_1(x, y)$ and $Q_2(x, y)$ are said to be equivalent if

$$Q_1(x, y) = Q_2(px + qy, rx + sy)$$

  for some p, q, r, s such that $ps - qr = 1$.

- For a negative discriminant, a primitive form $Q(x, y) = ax^2 + bxy + cy^2$ is said to be a reduced form if

$$|b| \leq a \leq c, \text{ and } b \geq 0 \text{ if either } |b| = a \text{ or } a = c.$$

- For $D \equiv 3 \pmod 4$, we define the principal form to be

$$Q = x^2 + xy + \frac{(1 + D)}{4} y^2.$$

Class number

Definition

For a negative discriminant D, the class number $h(D)$ is defined to be the number of equivalence classes of binary quadratic forms of discriminant D.

- The class number $h(D)$ is equal to the number of reduced forms of discriminant D.
- The class number $h(D)$ is finite. For example, $h(-23) = 3$.

Definition

The n^{th} triangular number T_n is defined as $T_n = \frac{n(n+1)}{2}$. The first few triangular numbers are $0, 1, 3, 6, \cdots$.

- Suppose

$$n = \frac{x_1(x_1+1)}{2} + \frac{x_2(x_2+1)}{2} + \cdots + \frac{x_k(x_k+1)}{2}.$$

Then,

$$8n + k = (2x_1+1)^2 + (2x_2+1)^2 + \cdots + (2x_k+1)^2.$$

Theorem (Fermat)

An odd prime can be p can be expressed as:

$$p = x^2 + y^2$$

for integers x and y iff $p \equiv 1 \pmod 4$.

Theorem (Jacobi)

The number of representations $r_4(n)$ *of an integer n as a sum of 4 squares is given by*

$$r_4(n) = 8\sigma_1(n) - 32\sigma_1(n/4),$$

where $\sigma_1(n)$ *is the sum of divisors of n.*

Theta function of a quadratic form

- Let $a(n, Q)$ be the number of representations of n by the quadratic form Q.

Definition

For $q = e^{2\pi iz}$, the theta function associated to Q is defined to be

$$\theta_Q(z) = \sum_{m,n\in\mathbb{Z}} q^{Q(m,n)} = \sum_{n=0}^{\infty} a(n, Q)q^n.$$

Modular forms

Definition

A function f on the upper half plane $\mathbb{H}$ is said to be a modular form of weight k, level N, and character χ if
(I) f is holomorphic on $\mathbb{H}$.
(II) $f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and for any $z \in \mathbb{H}$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}.$$

(III) f is holomorphic at each cusp of $\Gamma_0(N)$.

- $M_k(\Gamma_0(N), \chi)$ denotes the space of weight k modular forms on $\Gamma_0(N)$ with character χ.
- A modular form $f \in M_k(\Gamma_0(N), \chi)$ is said to be a *cusp form* if f vanishes at every cusp of $\Gamma_0(N)$.
- The space of cusp forms in $M_k(\Gamma_0(N), \chi)$ is denoted by $S_k(\Gamma_0(N), \chi)$.

- The space $M_k(\Gamma_0(N),\chi)$ can be written as a direct sum. Thus,

$$M_k(\Gamma_0(N),\chi) = E_k(\Gamma_0(N),\chi) \oplus S_k(\Gamma_0(N),\chi),$$

where $E_k(\Gamma_0(N),\chi)$ is called the Eisenstein space of level N and character χ.
- Let $f \in M_k(\Gamma_0(N),\chi)$. Then, f can be written as a sum

$$f = e + s,$$

where, $e \in E_k(\Gamma_0(N),\chi)$ and $s \in S_k(\Gamma_0(N),\chi)$.

Eta products

Definition

The *Dedekind eta function* is defined as

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i z}.$$

- An *Eta product* is given by

$$f(z) = \prod_{m} \eta(mz)^{a_m}, m \in \mathbb{Z}^+, a_m \in \mathbb{Z}.$$

- The least common multiple N of all the m's is called the *level* of f. We may also write,

$$f(z) = \prod_{m|N} \eta(mz)^{a_m}.$$

A criterion of Gordon, Hughes and Newman

Theorem

Let $f(z) = \prod_{m|N} \eta(mz)^{a_m}$ *be an eta product with* $k = \frac{1}{2}\sum_{m|N} a_m \in \mathbb{Z}$, *with the additional properties that*

$$\sum_{m|N} m a_m \equiv 0 \pmod{24}$$

and

$$\sum_{m|N} \frac{N}{m} a_m \equiv 0 \pmod{24},$$

Then

$$f(z) \in M_k(\Gamma_0(N), \chi),$$

where, the character χ *is defined by* $\chi(d) = \left(\frac{(-1)^k \prod_{m|N} m^{a_m}}{d}\right)$.

Binary quadratic forms

Theorem (Fred van der Blij, 1952)

Let $a(n,Q)$ be the number of representations of n by a quadratic form Q. Then

$$a(n,Q_1) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{23}\right) + \frac{4}{3}t(n)$$

and

$$a(n,Q_2) = a(n,Q_3) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{23}\right) - \frac{2}{3}t(n),$$

where $\left(\frac{\cdot}{23}\right)$ is the Legendre symbol mod 23, the quadratic forms Q_i's of discriminant -23 are

$$Q_1 = x^2 + xy + 6y^2,$$
$$Q_2 = 2x^2 + xy + 3y^2,$$
$$Q_3 = 2x^2 + xy - 3y^2,$$

and

$$\sum_{n=1}^{\infty} t(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)(1-q^{23n}).$$

Class number 3

There are 16 values of $D > 0$ such that $h(-D) = 3$.

D	Q_1	Q_2	Q_3	d_1	d_2
23	$x^2 + xy + 6y^2$	$2x^2 + xy + 3y^2$	$2x^2 - xy + 3y^2$	2	1
31	$x^2 + xy + 8y^2$	$2x^2 + xy + 4y^2$	$2x^2 - xy + 4y^2$	2	1
59	$x^2 + xy + 15y^2$	$3x^2 + xy + 5y^2$	$3x^2 - xy + 5y^2$	2	1
83	$x^2 + xy + 21y^2$	$3x^2 + xy + 7y^2$	$3x^2 - xy + 7y^2$	2	1
107	$x^2 + xy + 27y^2$	$3x^2 + xy + 9y^2$	$3x^2 - xy + 9y^2$	2	1
139	$x^2 + xy + 35y^2$	$5x^2 + xy + 7y^2$	$5x^2 - xy + 7y^2$	2	1
211	$x^2 + xy + 53y^2$	$5x^2 + 3xy + 11y^2$	$5x^2 - 3xy + 11y^2$	2	1
283	$x^2 + xy + 71y^2$	$7x^2 + 5xy + 11y^2$	$7x^2 - 5xy + 11y^2$	4	3
307	$x^2 + xy + 77y^2$	$7x^2 + xy + 11y^2$	$7x^2 - xy + 11y^2$	2	1
331	$x^2 + xy + 83y^2$	$5x^2 + 3xy + 17y^2$	$5x^2 - 3xy + 17y^2$	4	3
379	$x^2 + xy + 95y^2$	$5x^2 + xy + 19y^2$	$5x^2 - xy + 19y^2$	2	1
499	$x^2 + xy + 125y^2$	$5x^2 + xy + 25y^2$	$5x^2 - xy + 25y^2$	2	1
547	$x^2 + xy + 137y^2$	$11x^2 + 5xy + 13y^2$	$11x^2 - 5xy + 13y^2$	2	1
643	$x^2 + xy + 161y^2$	$7x^2 + xy + 23y^2$	$7x^2 - xy + 23y^2$	4	3
883	$x^2 + xy + 221y^2$	$13x^2 + xy + 17y^2$	$13x^2 - xy + 17y^2$	2	1
907	$x^2 + xy + 227y^2$	$13x^2 + 9xy + 19y^2$	$13x^2 - 9xy + 19y^2$	2	1

Table: Values of D with $h(-D) = 3$ and reduced forms Q_1, Q_2, Q_3. Here, $d_1 = \dim(M_1(\Gamma_0(D), \chi))$ and $d_2 = \dim(S_1(\Gamma_0(D), \chi))$.

Theorem (T.)

Let $D > 0$ *be such that* $h(-D) = 3$*. Let* Q_1, Q_2, Q_3 *be the reduced binary quadratic forms, where* Q_1 *represents the principle form. Let* $a(n, Q)$ *be the number of representations of* n *by the quadratic form* Q*. For* $q = e^{2\pi i z}$ *with* $z \in \mathbb{H}$*, let*

$$F_D(z) = \frac{1}{2}\left(\sum_{a,b\in\mathbb{Z}} q^{Q_1(a,b)} - \sum_{a,b\in\mathbb{Z}} q^{Q_2(a,b)}\right) = \sum_{n=0}^{\infty} t(n)q^n.$$

Then F_D *is a cusp form of weight 1, level* D*, character* $\chi = \left(\frac{-D}{\cdot}\right)$ *and* $t(1) = 1$*. Moreover,*

$$a(n, Q_1) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{D}\right) + \frac{4}{3}t(n)$$

and

$$a(n, Q_2) = a(n, Q_3) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{D}\right) - \frac{2}{3}t(n).$$

- Using a result of Eholzer and Skoruppa, we have

$$F_D(z) = q\prod_{n=1}^{\infty}(1-q^n)^{c(n)}, \quad q = e^{2\pi iz},$$

  for some integers $c(n)$ and for sufficiently small q.
- For $D = 23$, F. van der Blij proved that

$$F_{23}(z) = q\prod_{n=1}^{\infty}(1-q^n)(1-q^{23n}).$$

- Note that for $F_{23}(z)$, the coefficients $c(n)$'s are bounded.

Theorem (T.)

Let $D \neq 23$ be such that $h(-D) = 3$. Let Q_1, Q_2, Q_3 be the three reduced forms of discriminant $-D$, where Q_1 is the principal form. Let

$$F_D(z) = \frac{1}{2}\left(\sum_{a,b\in\mathbb{Z}} q^{Q_1(a,b)} - \sum_{a,b\in\mathbb{Z}} q^{Q_2(a,b)}\right).$$

Then the integers $c(n)$ in the expansion

$$F_D(z) = q\prod_{n=1}^{\infty}(1-q^n)^{c(n)}$$

are unbounded.

Lemma

Let $f(q) = \sum_{n=1}^{\infty} a_f(n) q^n$ *be a holomorphic function on the upper half plane and* $a_f(1) = 1$*. Then the complex numbers* $c(n)$ *in the expansion*

$$f(q) = \prod_{n=1}^{\infty} (1 - q^n)^{c(n)}$$

are unbounded if f *has a zero on the upper half plane.*

Lemma

Let $f(q) = \sum_{n=1}^{\infty} a_f(n)q^n$ *be a holomorphic function on the upper half plane and* $a_f(1) = 1$. *Then the complex numbers* $c(n)$ *in the expansion*

$$f(q) = \prod_{n=1}^{\infty} (1 - q^n)^{c(n)}$$

are unbounded if f *has a zero on the upper half plane.*

- This lemma is inspired by an exercise in Section 2.1.3 of Serre's monograph "Lectures on $N_X(p)$".

Triangular numbers

The Psi function

- The *Psi function* is given by

$$\Psi(q) = \sum_{n=0}^{\infty} q^{T_n} = 1 + q + q^3 + q^6 + \cdots .$$

- It is connected to $\delta_k(n)$ by

$$\Psi^k(q) = \sum_{n=0}^{\infty} \delta_k(n) q^n.$$

- It has the product expression

$$\Psi(q) = q^{\frac{-1}{8}} \frac{\eta^2(q^2)}{\eta(q)},$$

where

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

Known results

- In a 1995 article, Ono, Robins, and Wahl describe formulas for $\delta_k(n)$ for $k = 2, 3, 4, 6, 8, 10, 12, 24$.
- For $k = 4$, they note that

$$q\Psi^4(q^2) = \sum_{n=0}^{\infty} \delta_4(n) q^{2n+1} \in M_2(\Gamma_0(4))$$

and observe that it is the Eisenstein series given by

$$\sum_{n=0}^{\infty} \sigma_1(2n+1) q^{2n+1}.$$

- In 2008, Atanasov et al. obtained formulas for $\delta_{4k}(n)$ in terms of a divisor function and the coefficients of a cusp form.

Definition

The *generalised divisor function* is defined as

$$\sigma_{t,\chi,\psi}(n) = \sum_{d|n} \chi(d)\psi(n/d)d^t,$$

where χ and ψ are two Dirichlet characters and t is a non-negative integer.

- When χ_1 and χ_2 are both *trivial*, i.e., $\chi_1(n) = \chi_2(n) = 1$ for all integers n, this reduces to $\sigma_t(n) = \sum_{d|n} d^t$.
- The two generalised divisor functions we use are $\sigma_{t,\chi_1,\chi_0}(n)$ and $\sigma_{t,\chi_{-4},\chi_1}(n)$, where
  χ_0 - principal Dirichlet character mod 4.
  χ_1 - principal Dirichlet character mod 1.
  χ_{-4} - Dirichlet character mod 4 taking the values $\chi_{-4}(1) = 1$ and $\chi_{-4}(3) = -1$.

$\delta_{4k}(n)$

- We give a new proof of Atanasov et al.'s result, by employing a method due to Zafer Selcuk Aygin.

Theorem

For any positive $k > 1$, we have

$$\delta_{4k}(n) = \frac{1}{d_k}\sigma_{2k-1,\chi_1,\chi_0}(2n+k) + c(2n+k),$$

where

$$\sum_{n=1}^{\infty} c(n)q^n \in \mathrm{S}_{2k}(\Gamma_0(4))$$

is a cusp form and

$$d_k = -\frac{(-16)^k\left(4^k-1\right)B_{2k}}{8k}$$

in which B_{2k} is the $2k^{th}$ Bernoulli number.

Theorem (T.)

For any positive k, we have

$$\delta_{4k+2}(n) = \frac{-(2k+1)}{2^{4k} B_{2k+1,\chi_{-4}}} \sigma_{2k,\chi_{-4},\chi_1}(4n+2k+1) + t(4n+2k+1),$$

where

$$\sum_{n=1}^{\infty} t(n) q^n \in \mathrm{S}_{2k+1}(\Gamma_0(8), \chi_{-4})$$

is a cusp form and $B_{n,\chi_{-4}}$ is the n^{th} generalised Bernoulli number associated to χ_{-4}.

Theorem (T.)

The collection

$$\{C(2k, v, 4, z) \quad ; \quad 1 \leq v \leq k-2\}$$

forms a basis of $S_{2k}(\Gamma_0(4))$, *where*

$$C(2k, v, 4, z) = \left(\frac{\eta^{10}(2z)}{\eta^4(z)\eta^4(4z)}\right)^{2k} \left(\frac{\eta^8(z)\eta^{16}(4z)}{\eta^{24}(2z)}\right)^{v} \left(\frac{\eta^{16}(z)\eta^8(4z)}{\eta^{24}(2z)}\right).$$

Cusp part for $\delta_{4k+2}(n)$

Theorem (T.)

The collection

$$\{C(2k+1, v, 8, z) \quad ; \quad 1 \leq v \leq 2k-2\}$$

forms a basis of $S_{2k+1}(\Gamma_0(8), \chi_{-4})$, *where*

$$C(2k+1, v, 8, z) = \left(\frac{\eta^4(z)}{\eta^2(2z)}\right)^{2k+1} \left(\frac{\eta^2(2z)\eta^4(8z)}{\eta^4(z)\eta^2(4z)}\right)^v \left(\frac{\eta^{10}(2z)\eta^6(4z)}{\eta^{12}(z)\eta^4(8z)}\right).$$

New explicit formulas for $\delta_k(n)$

Theorem (T.)

We have

$$\delta_{14}(n) = -\frac{1}{124928}\left(\sigma_{6,\chi_{-4},\chi_1}(4n+7) - c(4n+7)\right),$$
$$\delta_{16}(n) = \frac{1}{17408}\left(\sigma_{7,\chi_1,\chi_0}(2n+4) - d(2n+4)\right),$$
$$\delta_{18}(n) = \frac{1}{45383680}\left(\sigma_{8,\chi_{-4},\chi_1}(4n+9) - e(4n+9)\right),$$

where

$$\sum_{n=1}^{\infty} c(n)q^n = 728\left(\eta^4(z)\eta^2(2z)\eta^8(8z) + 4\frac{\eta^4(2z)\eta^{12}(8z)}{\eta^2(4z)}\right),$$
$$\sum_{n=1}^{\infty} d(n)q^n = 128\eta^8(2z)\eta^8(4z),$$
$$\sum_{n=1}^{\infty} e(n)q^n = \frac{\eta^{20}(z)\eta^4(4z)}{\eta^6(2z)} + 20\frac{\eta^{16}(z)\eta^2(4z)\eta^4(8z)}{\eta^4(2z)} + 144\frac{\eta^{12}(z)\eta^8(8z)}{\eta^2(2z)}$$
$$+448\frac{\eta^8(z)\eta^{12}(8z)}{\eta^2(4z)} + 391168\frac{\eta^4(z)\eta^2(2z)\eta^{16}(8z)}{\eta^4(4z)} + 1562624\frac{\eta^4(2z)\eta^{12}(20z)}{\eta^6(4z)}.$$

Let $r_N(n)$ be the number of representations of n as a sum of N squares and let

$$L_N(s) = \sum_{n=1}^{\infty} \frac{r_N(n)}{n^s} \quad \text{and} \quad R_N(s) = \sum_{n=1}^{\infty} \frac{r_N^2(n)}{n^s}$$

Theorem

The Dirichlet series associated to $r_N(n)$ for $N = 2, 4, 6, 8$ are

$$\begin{aligned} L_2(s) &= 4\zeta(s)L(s, \chi_{-4}), \\ L_4(s) &= 8(1 - 4^{1-s})\zeta(s)\zeta(s-1), \\ L_6(s) &= 16\zeta(s-2)L(s, \chi_{-4}) - 4\zeta(s)L(s-2, \chi_{-4}), \\ L_8(s) &= 16(1 - 2^{1-s} + 4^{2-s})\zeta(s)\zeta(s-3). \end{aligned}$$

Theorem

The Dirichlet series associated to $r_N^2(n)$ for $N = 2, 4, 6, 8$ are

$$
\begin{aligned}
R_2(s) &= \frac{(4\zeta(s)L(s,\chi_{-4}))^2}{(1+2^{-s})\zeta(2s)},\\
R_4(s) &= 64\,\frac{(8.2^{3-3s} - 10.2^{2-2s} + 2^{1-s} + 1)\zeta(s-2)\zeta^2(s-1)\zeta(s)}{(1+2^{1-s})\zeta(2s-2)},\\
R_6(s) &= 16\,\frac{(17-32.2^{-s})\zeta(s-4)L^2(s-2,\chi_{-4})\zeta(s)}{(1-16.2^{-2s})\zeta(2s-4)}\\
&\quad - \frac{128L(s-4,\chi_{-4})\zeta^2(s-2)L(s,\chi_{-4})}{(1+4.2^{-s})\zeta(2s-4)},\\
R_8(s) &= 256\,\frac{\left(32\cdot 2^{6-2s} - 3\cdot 2^{3-s} + 1\right)\zeta(s-6)\zeta^2(s-3)\zeta(s)}{\left(1+2^{3-s}\right)\zeta(2s-6)}.
\end{aligned}
$$

Wagon's conjecture for sums of squares

- Wagon's conjecture states that

$$\sum_{n \leq x} r_N^2(n) \sim W_N x^{N-1}, \text{ as } x \to \infty, N \geq 3,$$

where

$$W_N = \frac{\pi^N}{\left(1 - 2^{-N}\right)(N-1)\Gamma^2\left(\frac{N}{2}\right)} \frac{\zeta(N-1)}{\zeta(N)}.$$

$\zeta(.)$ - Riemann zeta function, $\Gamma(.)$ - gamma function.

First moment of $\delta_N(n)$

Theorem (T.)

For an even value of $N > 2$ and any $\epsilon > 0$, we have

$$\sum_{n \leq x} \delta_N(n) = \frac{\pi^{N/2}}{2^{N/2}\Gamma(N/2+1)} x^{N/2} + O\left(x^{(N-1)/2+\epsilon}\right).$$

Second moment of $\delta_N(n)$

Theorem (T.)

For an even value of $N > 2$ and any $\epsilon > 0$, we have

$$\sum_{n \leq x} \delta_N^2(n) = Y_N x^{N-1} + O\left(x^{N-1/2+\epsilon}\right),$$

where

$$Y_N = \frac{\pi^N}{2^N (N-1)\Gamma^2\left(\frac{N}{2}\right)} \frac{L(N-1, \chi_0)}{L(N, \chi_0)}.$$

Thank You