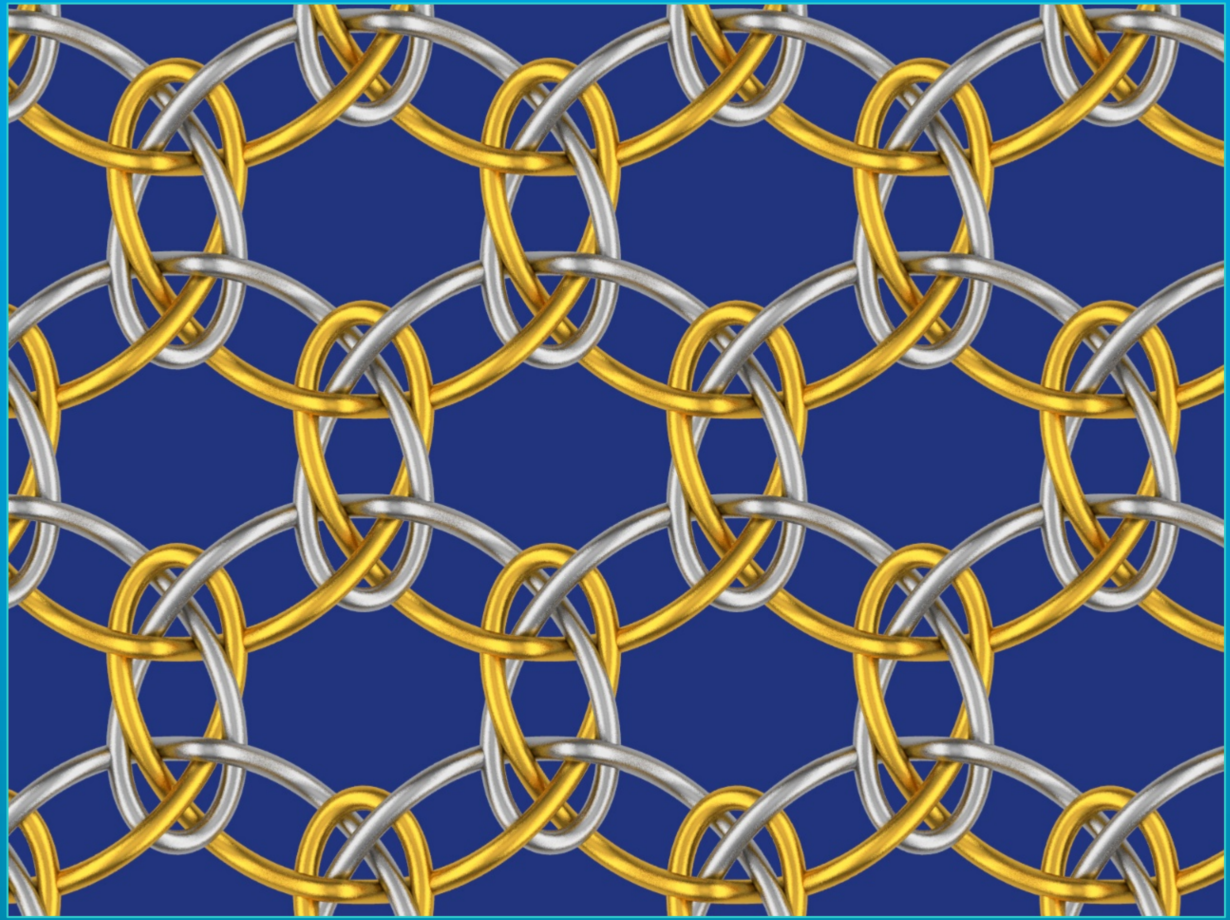
Wallpaper Patterns from Looping Strands: The Layer Groups



Frank Farris,
Santa Clara U

GEAR 2021 at Banff

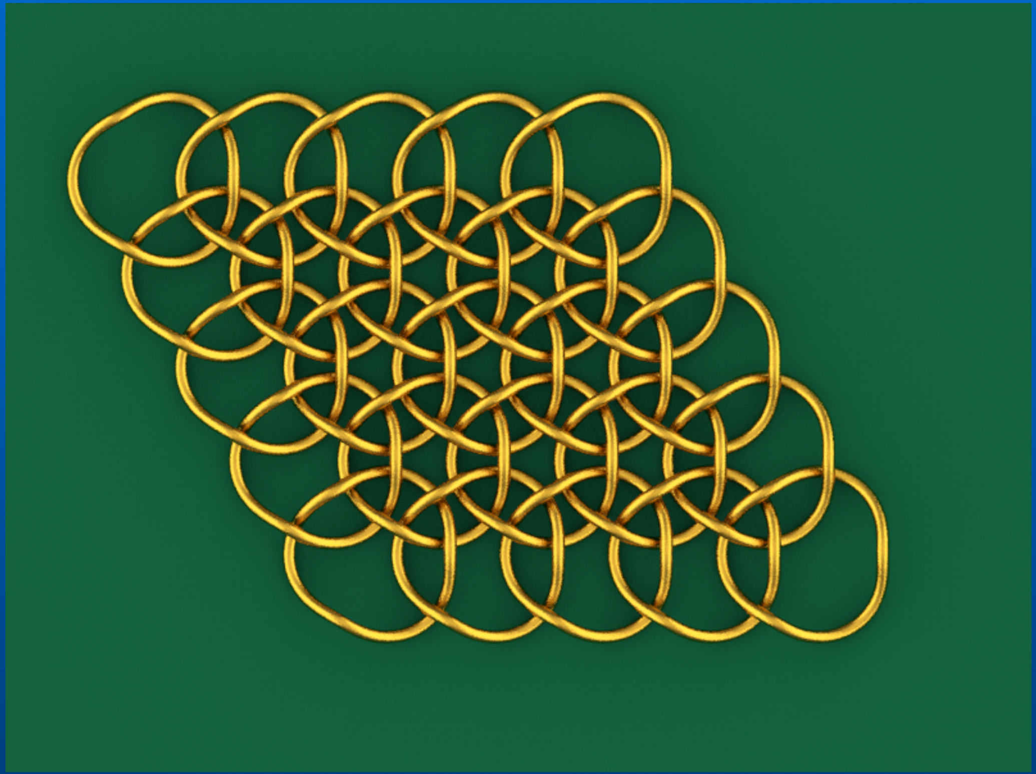
If you do anything with wallpaper patterns,
you can generalize it to 3D.

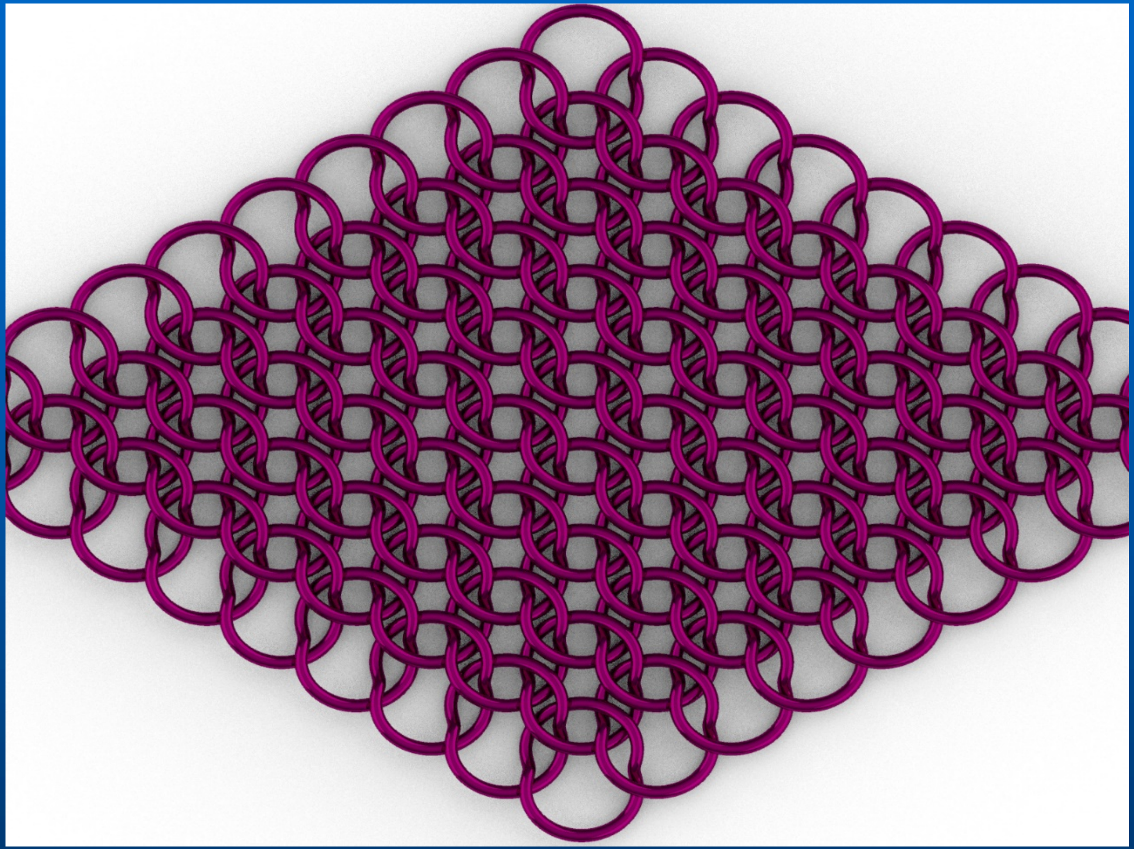
Before we begin

- *We pause to acknowledge that Santa Clara University sits on the land of the Ohlone and the Muwekma Ohlone people, who trace their ancestry through the Missions Dolores, Santa Clara, and San Jose. We remember their connection to this region and give thanks for the opportunity to live, work, learn and pray on their traditional homeland.* (Juristac http://www.protectjuristac.org/)

- The Golden Section of the MAA (Northern CA, NV, and HI) has its section meeting next weekend! Register and attend for free. Meeting held in Zoom and Gather.town

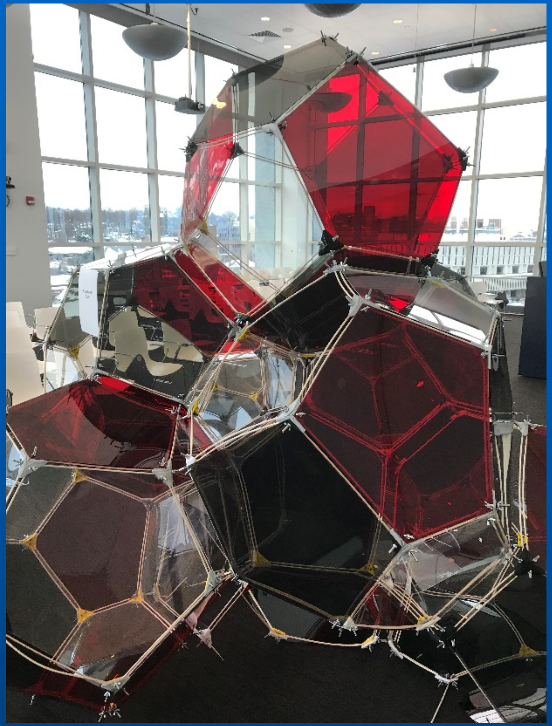
Earlier project: Mathematical chain mail



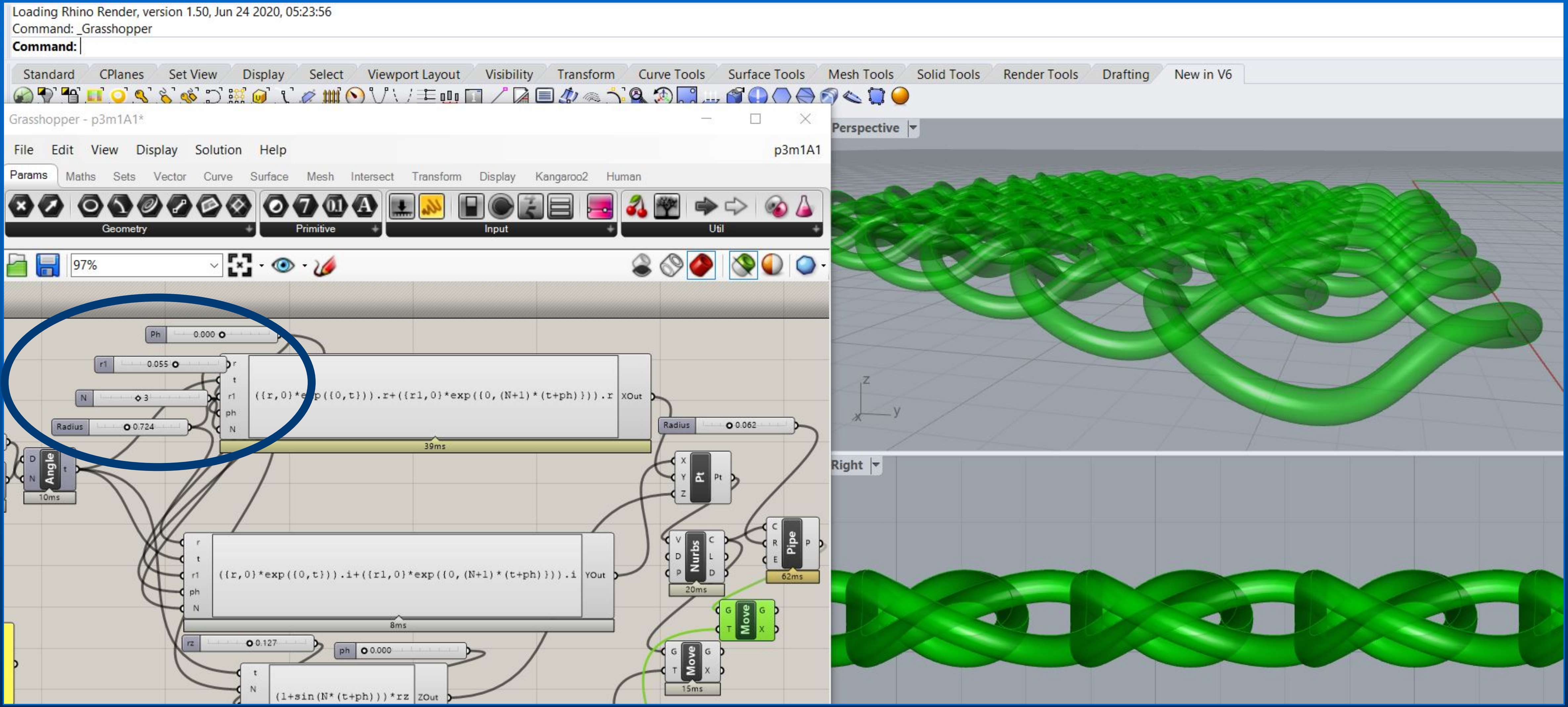


“Wallpaper Patterns from Nonplanar Chain Mail Links” FF, Bridges, 2020.

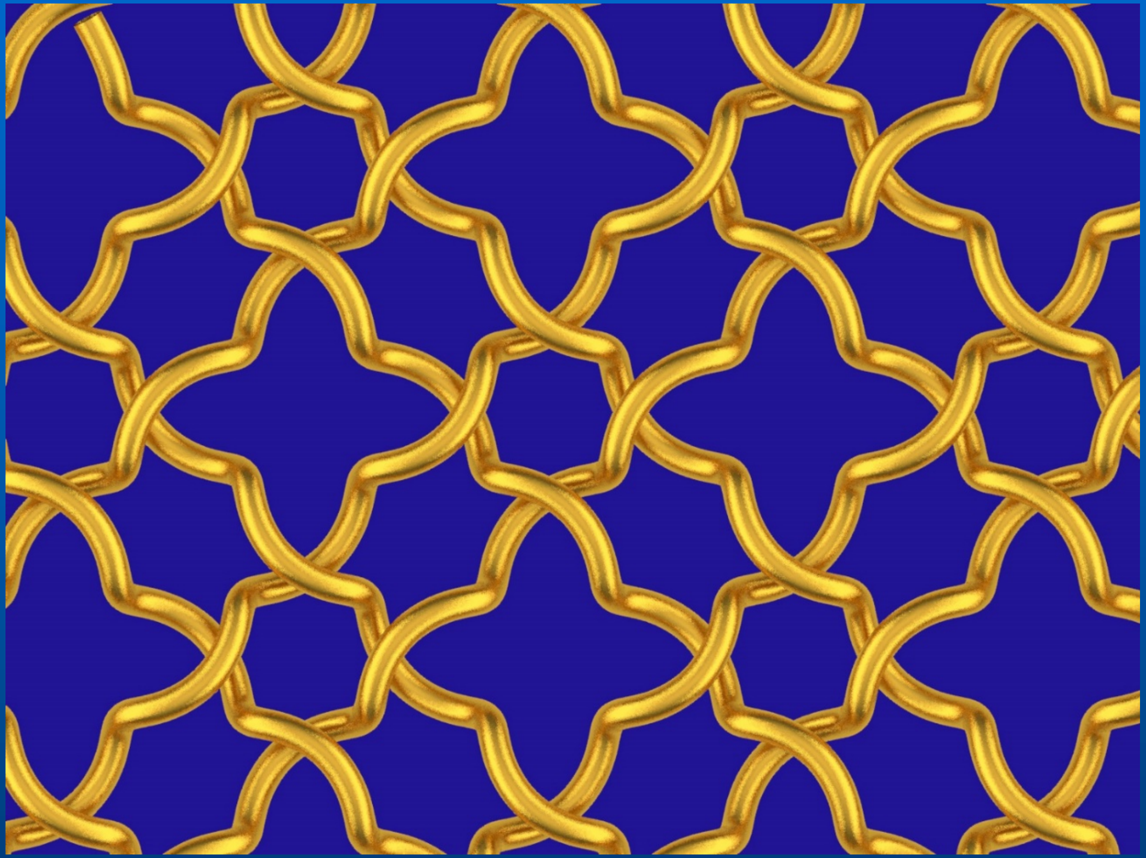
Fall ’19 semester at ICERM!
“Illustrating Mathematics”



Grasshopper as Rhino plug-in

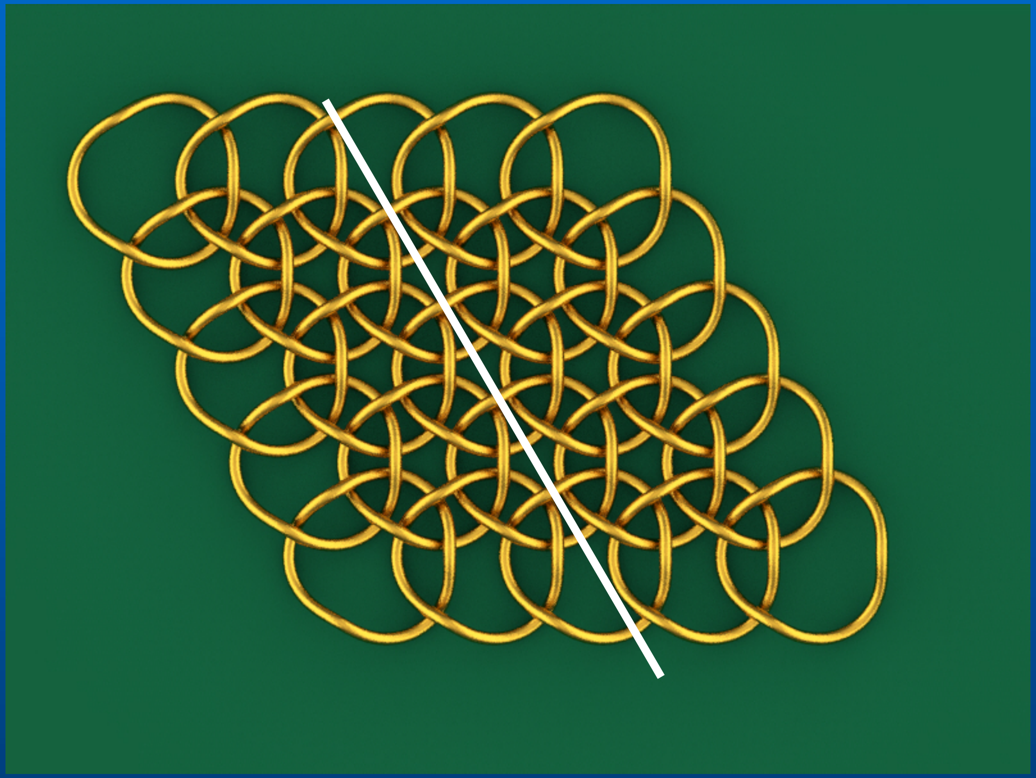


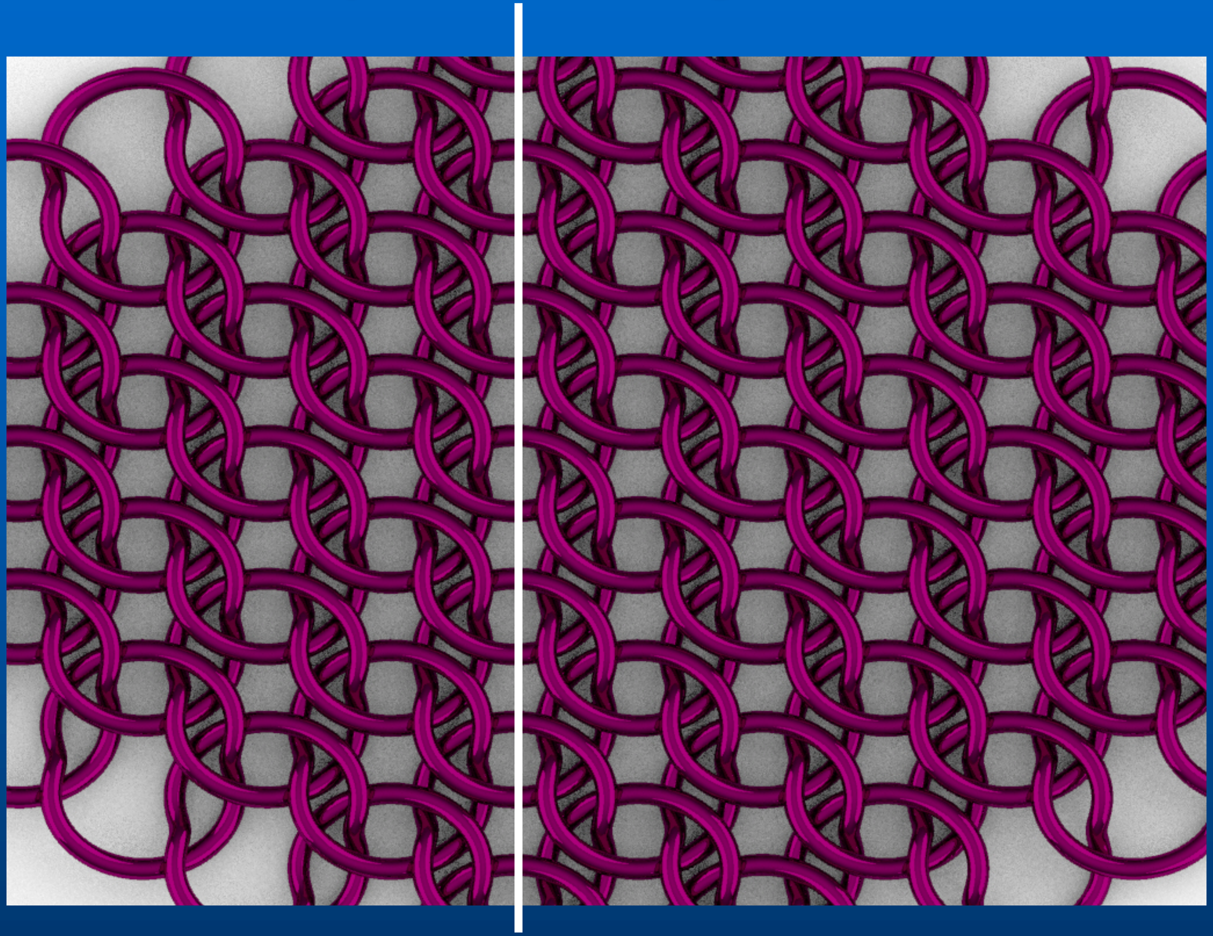
Seen last summer



What is the symmetry of this pattern?

Wallpaper is not exactly the right term





These patterns are invariant under *layer group* actions.

Wallpaper symmetry is for plane patterns

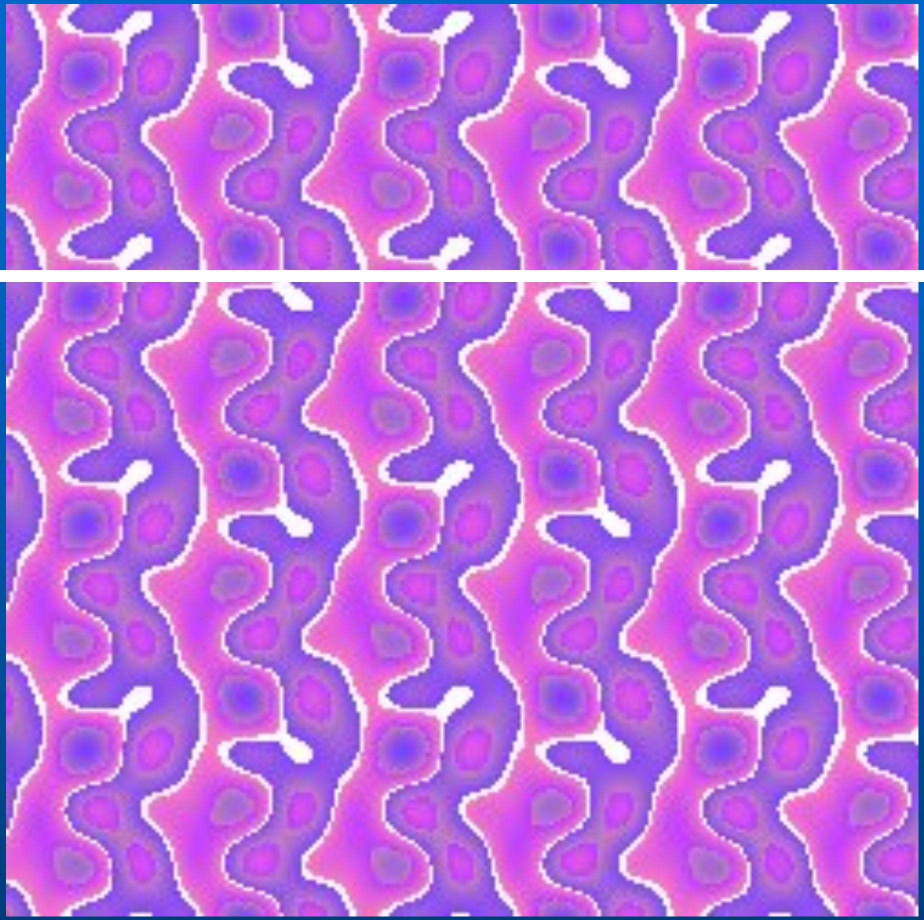
A *symmetry* of a pattern is a transformation that leaves it unchanged.

The symmetries of a pattern form a *group*. (Closed under composition)

There are only 17 (isomorphism classes of) wallpaper groups!

This pattern has symmetry group pg.

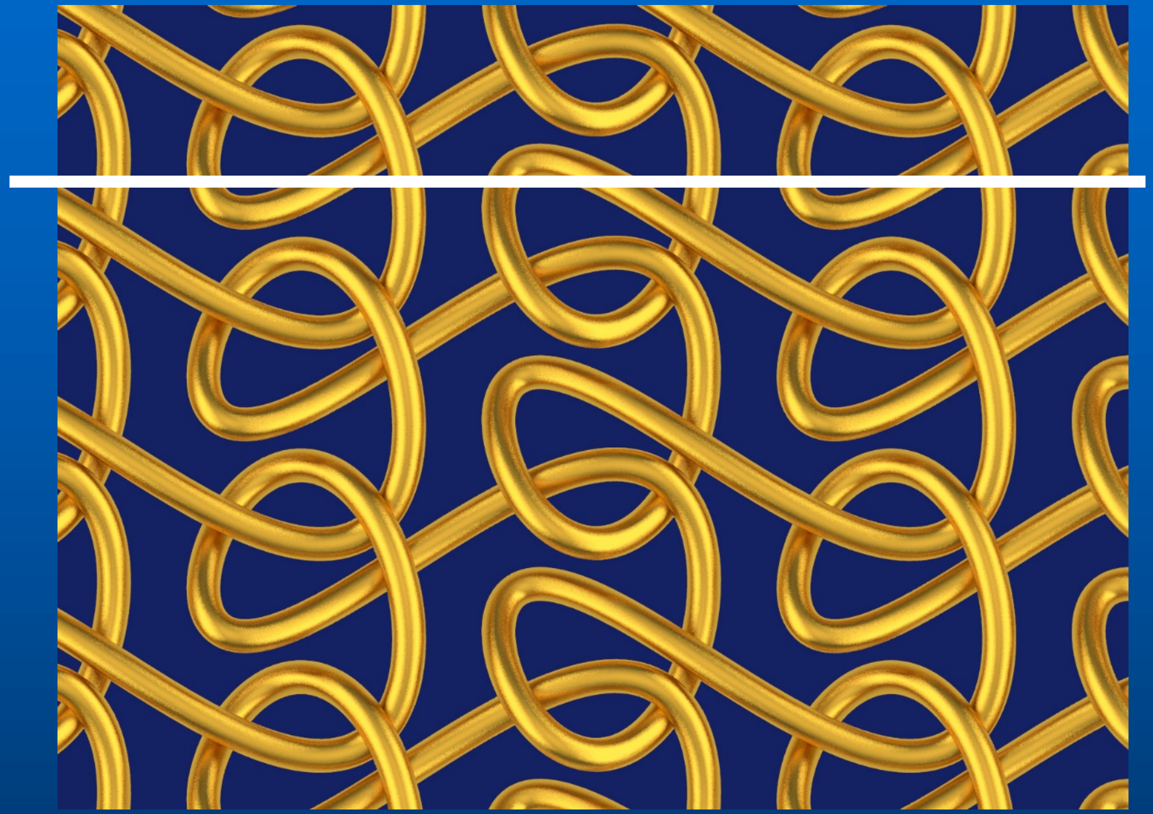
Wallpaper symmetry is for plane patterns

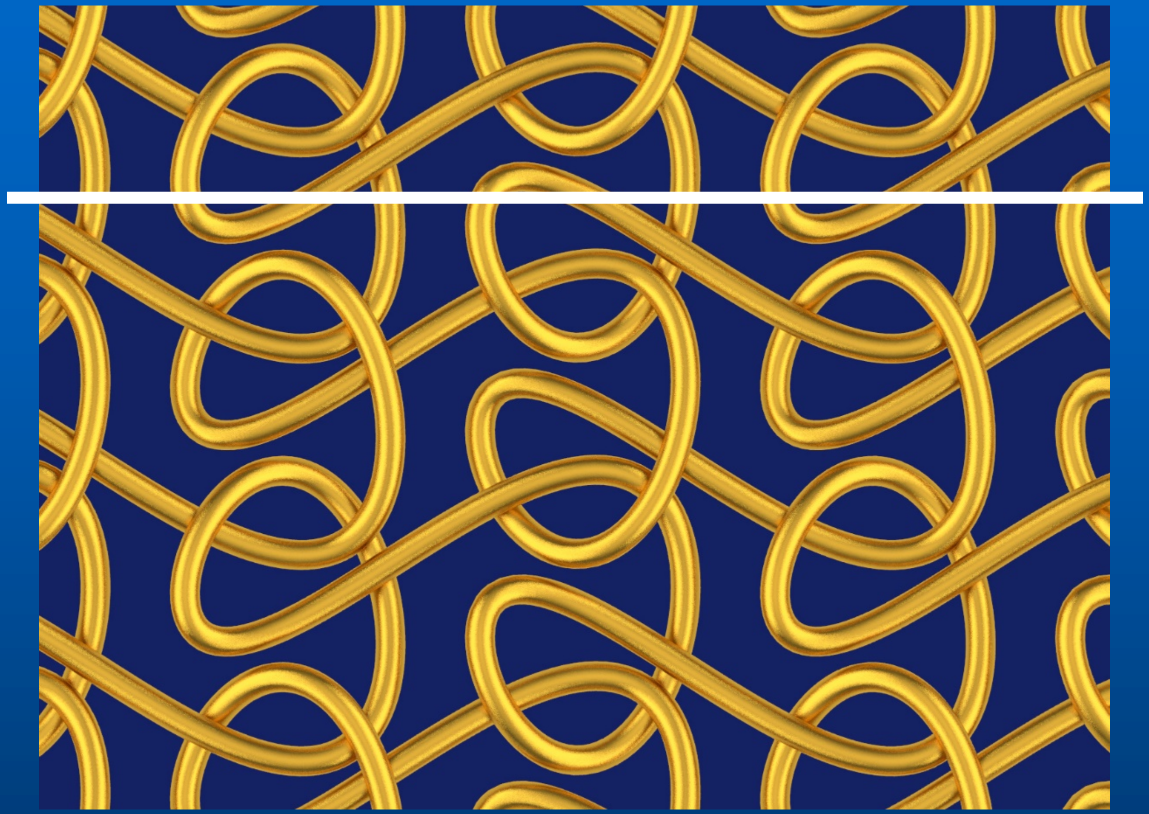


Cited in Wikipedia article "Layer Groups"

A pg pattern from "Vibrating Wallpaper" FF, 1998, CVM.

3D patterns have “layer group” symmetry





Two different layer group extensions of wallpaper group pg

From wallpaper groups to layer groups

- A *wallpaper group* is a gp of Euclidean isometries of the plane whose translations can be generated by 2 linearly independent translations.
(17 isomorphism classes.)
- A *layer group* is a gp of Euclidean isometries of space whose translations can be generated by 2 linearly independent translations
(and hence has an invariant plane). 80 of these!
- The restriction of a layer group to its invariant plane is a wallpaper group.

Wallpaper group notation

- Individual transformations are written in complex notation.
- For instance, the glide reflection in our examples is

$$\gamma_x(z) = \bar{z} + 1/2$$

- Groups are named by the International Crystallographic Union symbols, such as p3m1 (*333 in orbifold notation).

Extending transformations to space

- The trivial extension of a planar transformation is

$$\check{\alpha}(z,w) = (\alpha(z), w)$$

- The only other possible extension is a composition with

$$\sigma_z(z,w) = (z, -w)$$

- Call that the *flip extension.*

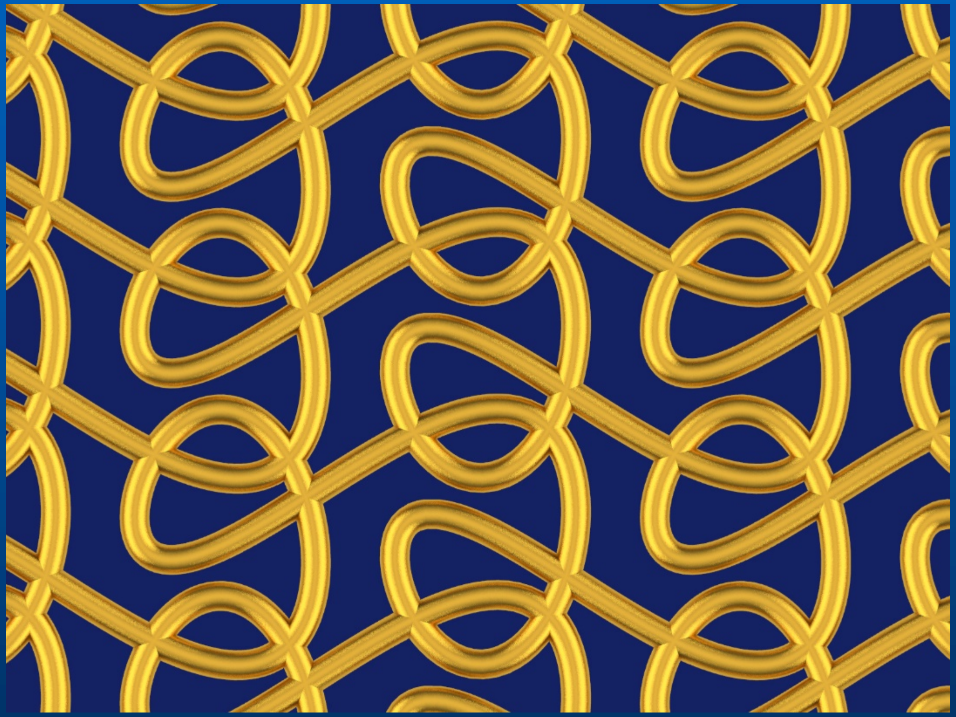
Counting the 80 layer groups

- Each of 17 wallpaper gps has a trivial extension.
- Each of 17 wallpaper gps has a double extension.

The pg pattern flattened.

If round on top, flat on bottom: trivial extension of pg

If round on top and bottom: double extension of pg



That leaves 46 layer groups

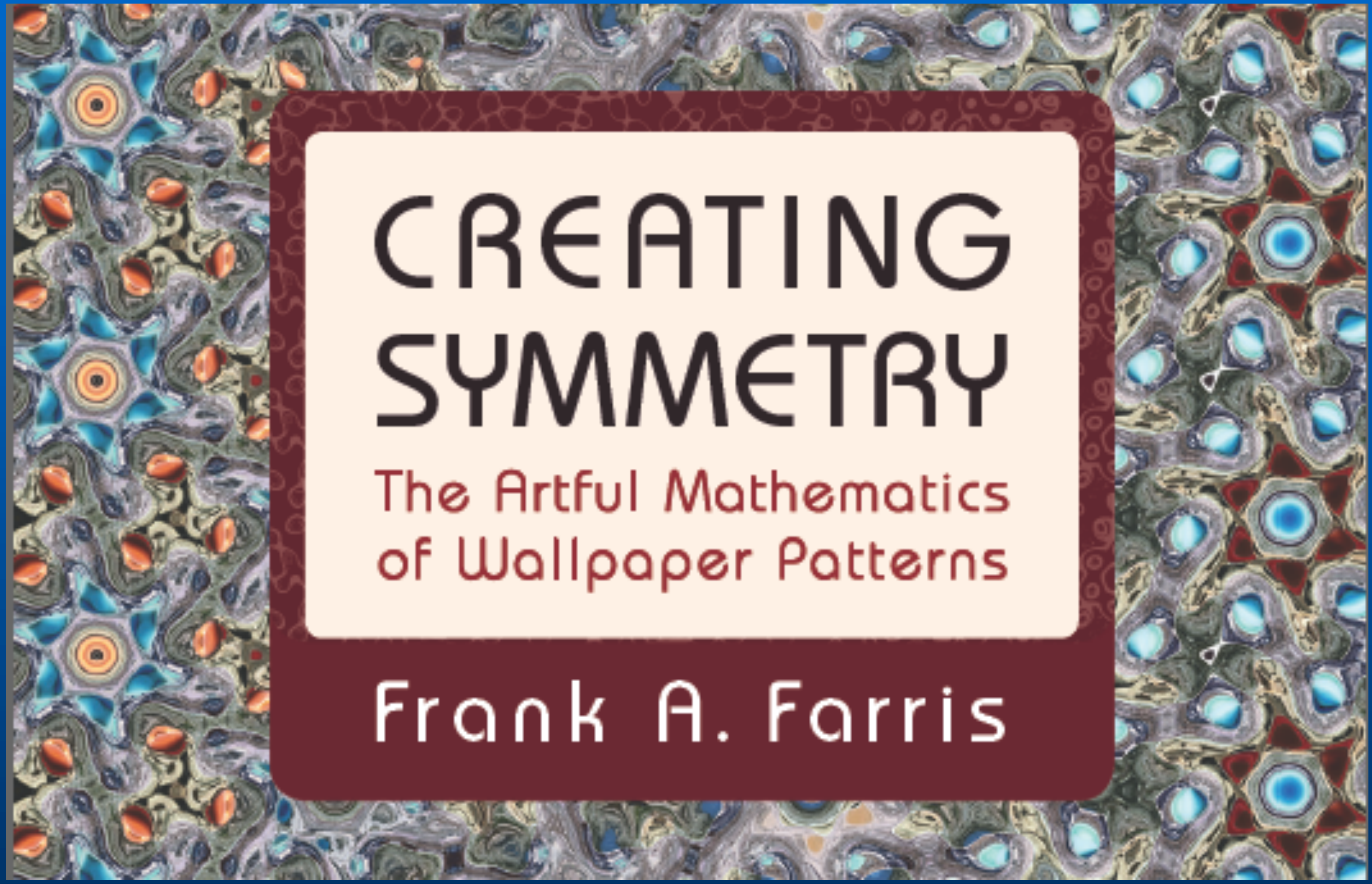
- Theory exactly the same as that of color symmetry
- To extend G, tag some elements for trivial extension, some for flip extension.

Given a homomorphism $\phi : G \to \{0, 1\}$

Extend elements of G by $g \to g \circ \sigma_z^{\phi(g)}$

There are 46 equivalence classes of such homomorphisms

Read about color symmetry here:

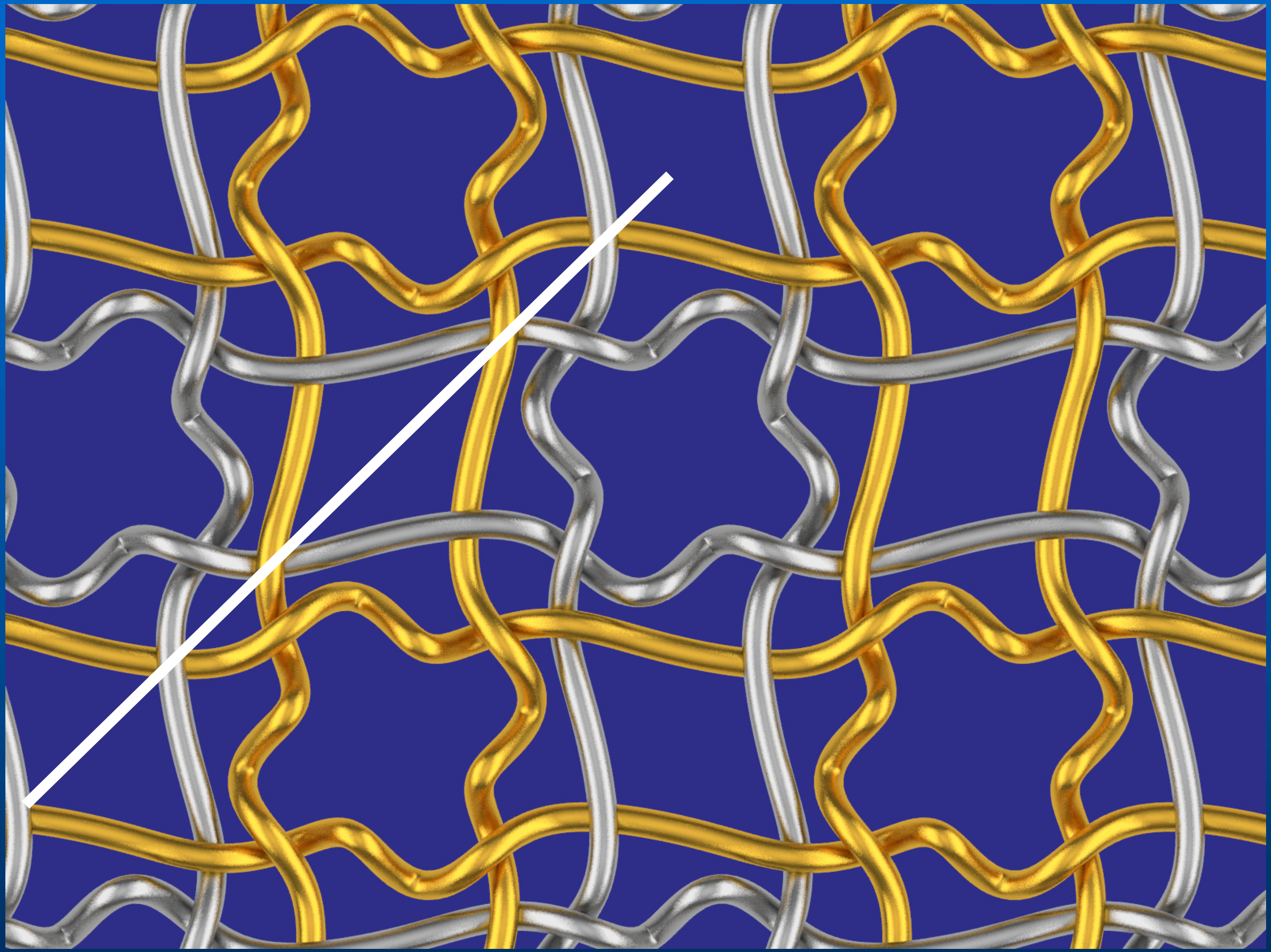


Working through an example

Turn gold strands over to create silver strands

New sym. gp is an extension of p4g

Coxeter notation p4g/p4



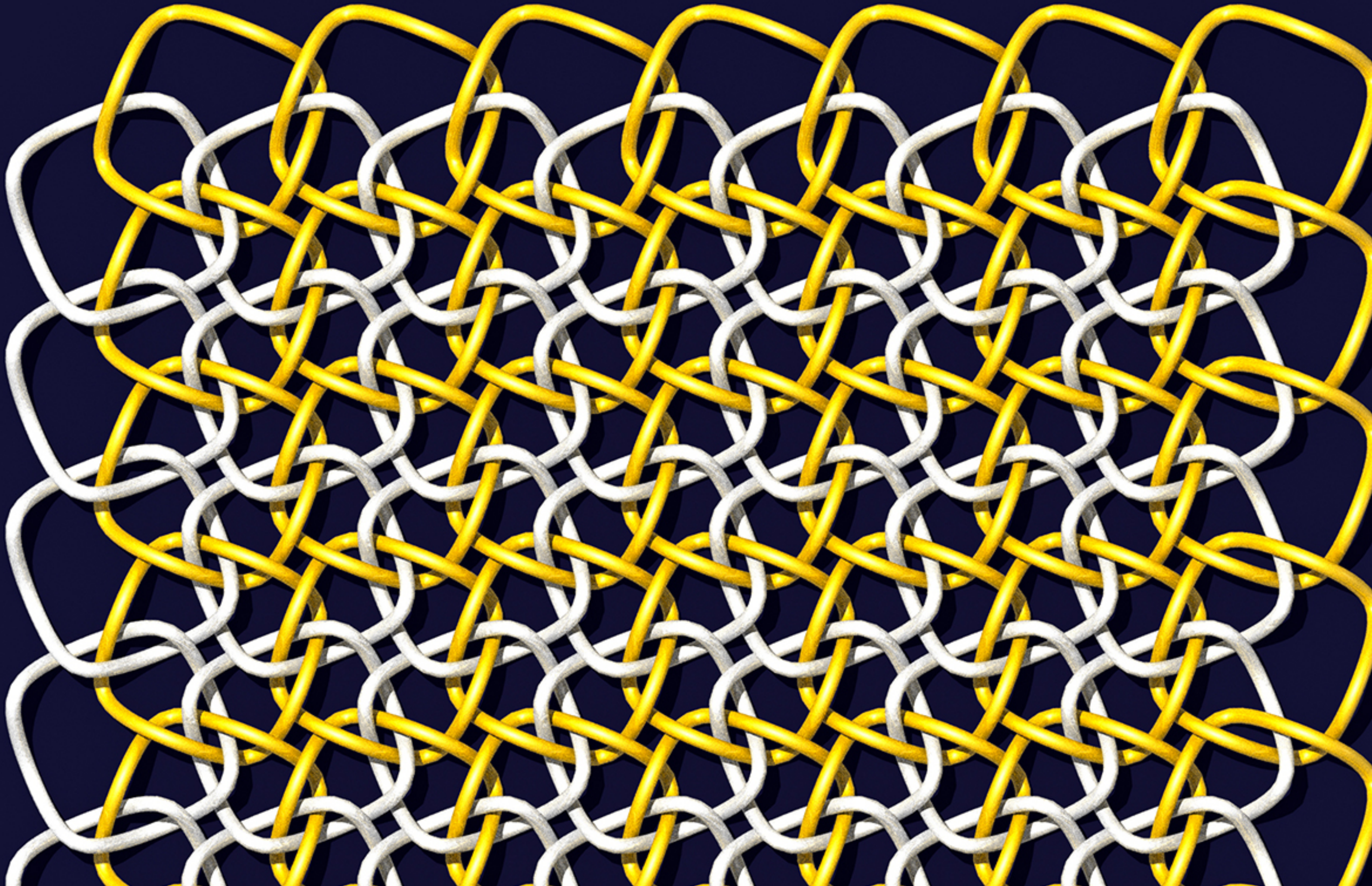
Focus on gold strands

Symmetry group is p4

$p42_12$

in IUC notation

The same symmetry type in chain mail



Tables to translate between terminologies

Supplement to Bridges '21 paper

Post to Wikipedia?

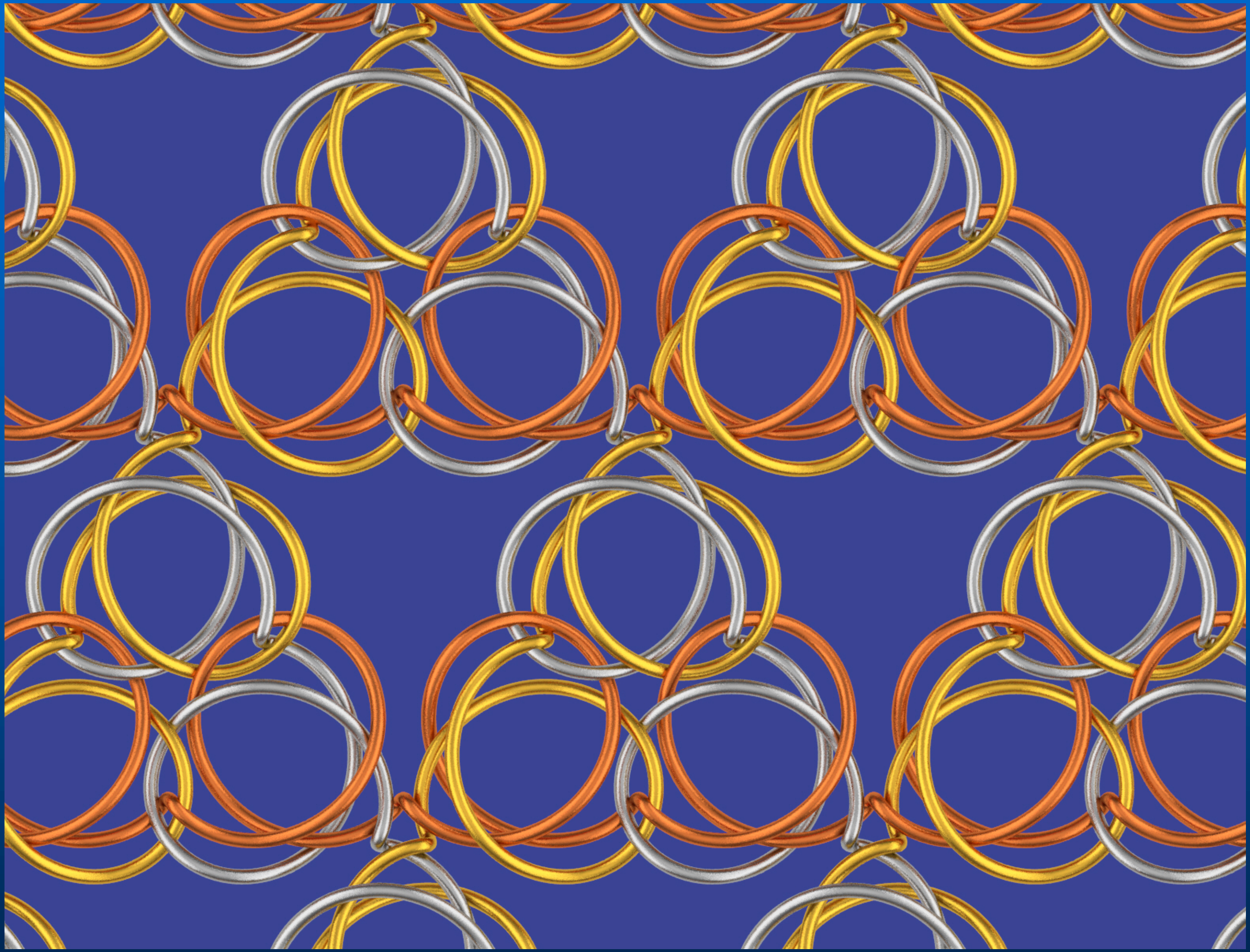
Table 3: *A key to the layer groups, part 3: lattices with higher symmetry*

Lattice	**Group #**	**IUC Symbol**	**Wallpaper group interpretation**
Tetragonal	49	p4	p4
	50	$p\bar{4}$	p4/p2
	51	p4/m	$p4\times\mathbb{Z}_2$
	52	p4/n	p4/p4
	53	p422	p4m/p4
	54	$p42_12$	p4g/p4
	55	p4mm	p4m
	56	p4bm	p4g
	57	$p\bar{4}2m$	p4m/cmm
	58	$p\bar{4}2_1m$	p4g/cmm
	59	$p\bar{4}m2$	p4m/pmm
	60	$p\bar{4}b2$	p4g/pgg

See you at Bridges 2021?

Each strand made with Fourier series that has the desired symmetry

Wiggle parameters until the pattern weaves



“Rod” symmetry p1m1/p111

Out of left field: A 7-color torus