Billiards and the arithmetic of non-arithmetic groups

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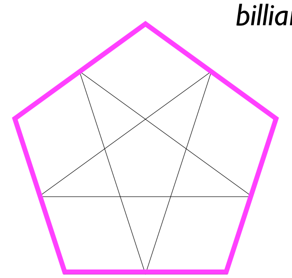
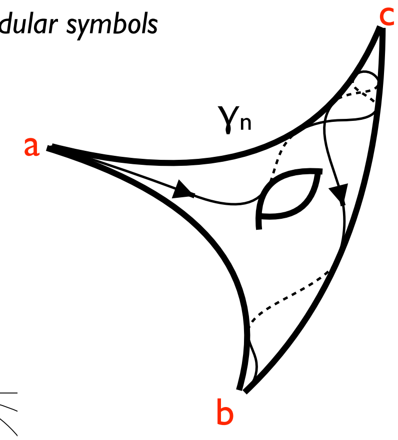
Weil, Manin, Birch, Leutbecher, Veech, Masur, Forni, Möller, Leininger, Hubert, Lanneau, Davis, Lelievre, ….

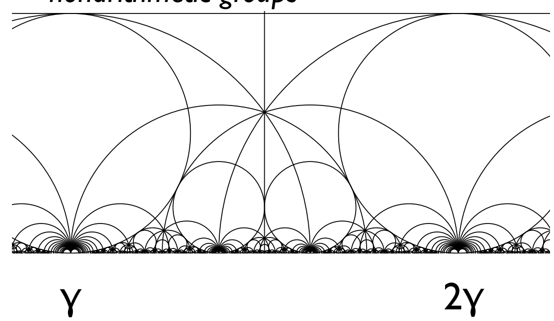
billiards



modular symbols



nonarithmetic groups



heights

$$H_A(x) = \inf_C \prod_{v|\infty} |C|_v$$

I. Triangle groups

Curves on a Hilbert modular surface

cf. M, Möller-Viehweg

K = real quadratic field

$$X_K = (\mathbb{H} \times \mathbb{H}) / \operatorname{SL}(\mathcal{O} \oplus \mathcal{O}^\vee)$$

$V = \mathbb{H}/\Gamma \looparrowright X_K$ geodesic curve

Theorem Q

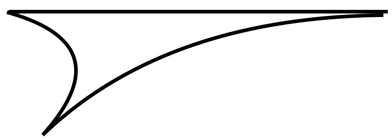
Either V is a Shimura curve, or the cusps of V coincide with $\mathbb{P}^1(K)$ and satisfy quadratic height bounds.

proof by descent

Triangle groups

$\Delta(p,q,\infty) \subset SL_2(\mathbb{R})$ *lattice*

π/p



cusp

$\mathbb{H}/\Delta$

π/q

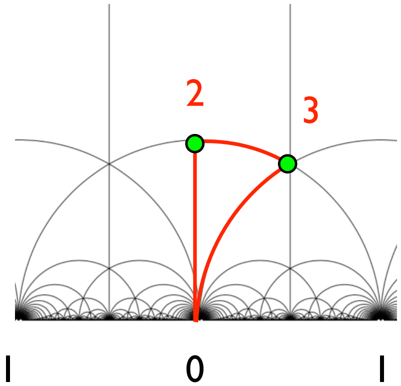
invariant trace field

$$K_{pq} = \mathbb{Q}(\,\mathrm{Tr}(g^2) : g \in \Delta(p,q,\infty))$$

$$= \mathbb{Q}(\cos(2\pi/p),\ \cos(2\pi/q),\ \cos(\pi/p)\cos(\pi/q))$$

$\Delta(p,q,\infty)$ is arithmetic $\Leftrightarrow$ $K_{pq} = \mathbb{Q}$

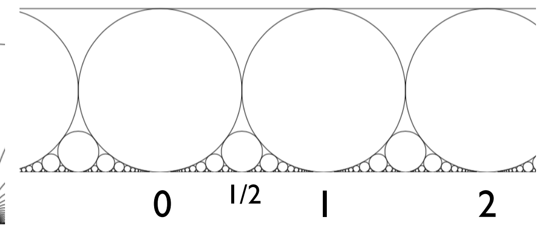
Arithmetic case



$z \to z+1$

$z \to -1/z$

$$\Delta(2,3,\infty) = SL_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$



matrix entries = $\mathbb{Z}$

columns (a,b), gcd=1

cusp = $\mathbb{Q} \cup \{\infty\}$

Non-arithmetic case $\Delta(p,q,\infty)$

is more mysterious!

matrix entries = ?

columns (a,b) ?

cusps = ? $\cup \{\infty\}$

Cor of Thm Q

The cross-ratios of the cusps of $\Delta(p,q,\infty)$ coincide with $P^1(K_{pq}) - \{0,1,\infty\}$, whenever $\deg(K_{pq}/\mathbb{Q}) = 2$.

Proof: Every Δ comes from a geodesic curve V in a Hilbert modular variety X_K.

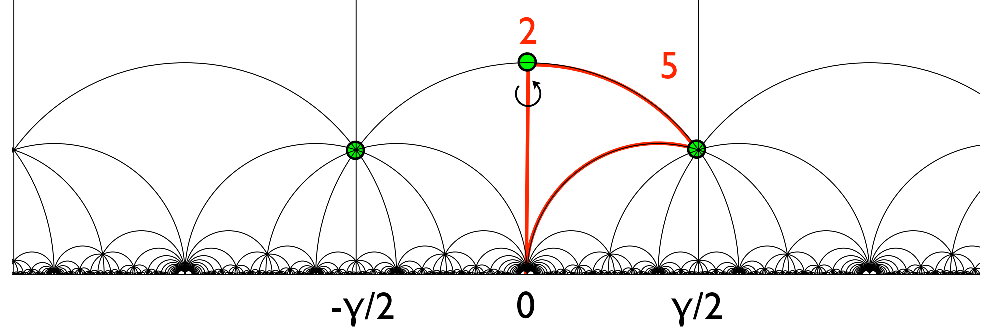
The golden Hecke group

$\gamma = (1+\sqrt{5})/2$

$z \to z+\gamma$

$$\Gamma = \Delta(2,5,\infty) = \left\langle \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

$z \to -1/z$

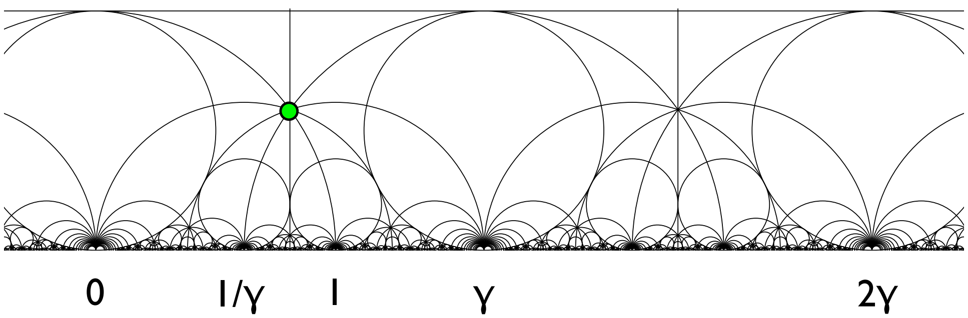


Cor

The cusps of Γ coincide with $K = \mathbb{Q}(\sqrt{5}) \cup \{\infty\}$.

Leutbecher, 1970s

5 packing



Golden Fractions

Cor

Every x in K= $\mathbb{Q}(\sqrt{5})$ can be written uniquely as a `golden fraction' x = a/c, up to sign.

a,c in $\mathcal{O} = \mathbb{Z}[\gamma] \subset K$ relatively prime

(a,c) column of a matrix in Γ

Quadratic height bounds: $h(a)+h(c) = O(1+h(x)^2)$.

h(n) = log n

Golden Continued Fractions

Cor

Every x in $\mathbb{Q}(\sqrt{5})$ can be expressed as a *finite* golden continued fraction:

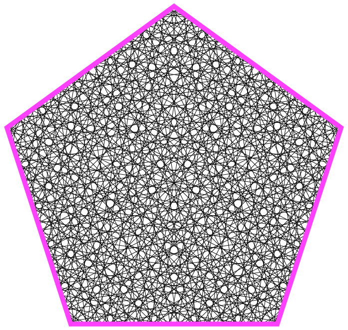
$$x = [a_1, a_2, a_3, \ldots, a_N] = a_1\gamma + \cfrac{1}{a_2\gamma + \cfrac{1}{a_3\gamma + \cdots + \cfrac{1}{a_N\gamma}}},$$

with a_i in $\mathbb{Z}$.

Quadratic height bounds: $N, \max a_i = O(1+h(x))$.

II. Billiards

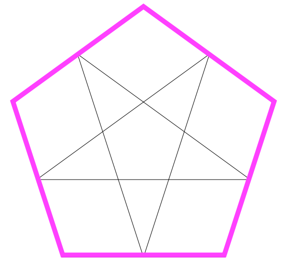
Billiards in a regular pentagon



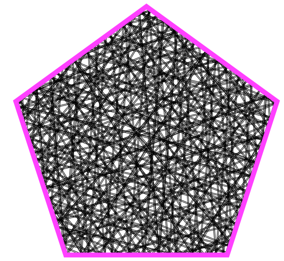
A dense set of slopes are periodic.

How do the periodic trajectories behave?

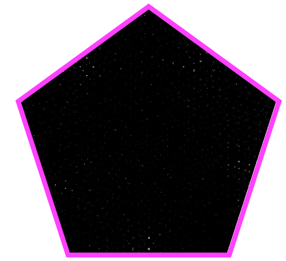
Slopes and lengths



s
L(s) = 5



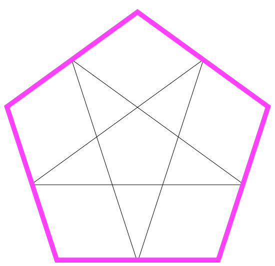
4s
L(s) = 469



20s
L(s) = 2338

6765s
$L(6765s) = 1.734 \times 10^{25}$

Slopes, lengths and heights



s

Cor

The periodic slopes coincide with $\mathbb{Q}(\sqrt{5})s$*, and* $\log L(xs) = O(h(x)^2)$*.*

Example

$$L(10^n s) = O(10^{Cn^2})$$

exponent 2 is sharp

III. Teichmüller curves

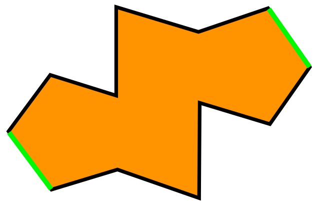
How to describe X in $\mathcal{M}_g$?

g=1: $X = \mathbb{C}/\Lambda$

g>1: X = ? *Uniformization Theorem*

Every X in $\mathcal{M}_g$ can be built from a polygon in $\mathbb{C}$

X = P / gluing by translations



P

How to describe X in $\mathcal{M}_g$?

g=1: $X = \mathbb{C}/\Lambda$

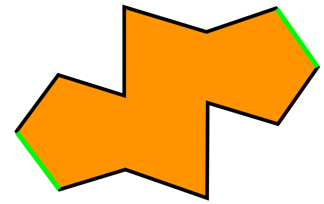
g>1: X = ? *Uniformization Theorem*

Every (X,ω) in $\Omega\mathcal{M}_g$ can be built from a polygon in $\mathbb{C}$

$\downarrow$

$\mathcal{M}_g$

$(X,\omega) = (P,dz)$ / gluing by translations

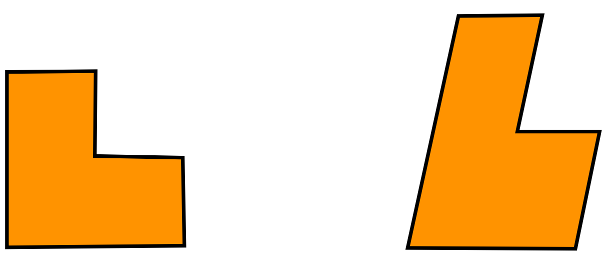


P

Action of g in $SL_2(\mathbb{R})$ on $\Omega\mathcal{M}_g$

$(X,\omega) = (P,dz)$ / gluing

$g \cdot (X,\omega) = (g(P),dz)$ / gluing



Teichmüller curves

$SL(X,\omega)$ = stabilizer of (X,ω) in $SL_2(\mathbb{R})$

$SL(X,\omega)$ lattice $\Rightarrow$ $SL_2(\mathbb{R})$ orbit of (X,ω) generates

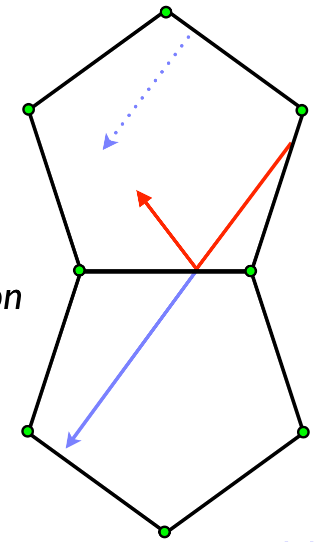
an isometrically immersed *Teichmüller curve:*

$$f : V = \mathbb{H} / SL(X,\omega) \rightarrow \mathcal{M}_g$$

Billiards and Riemann surfaces

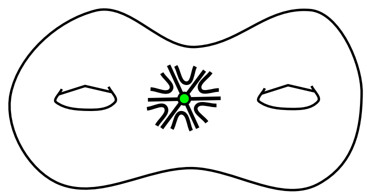
P

octagon



(X,ω) = (P,dz) / gluing

X has genus 2
ω has just one zero!



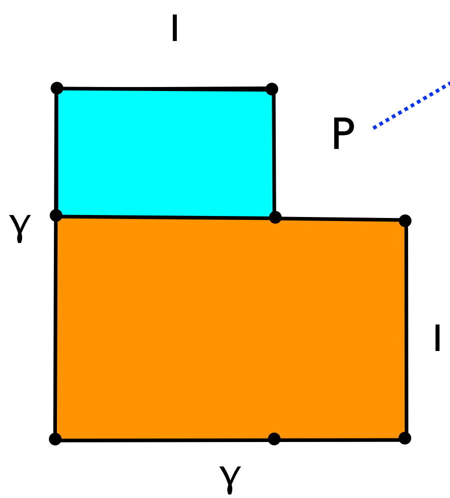
billiards ⟹ geodesics on (X,|ω|)

Theorem
(Veech, Masur):

If SL(X,ω) is a lattice, then billiards in P has optimal dynamics.

(Every trajectory is periodic or uniformly distributed.)

The golden table

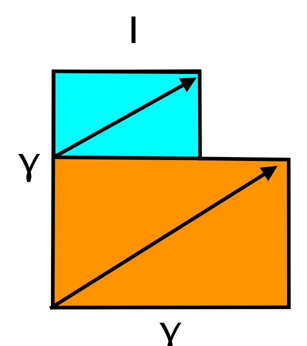


SL(X,ω) = (P,dz)/~

Theorem
SL(X,ω) = Δ(2,5,∞).

Cor
Optimal dyamics.

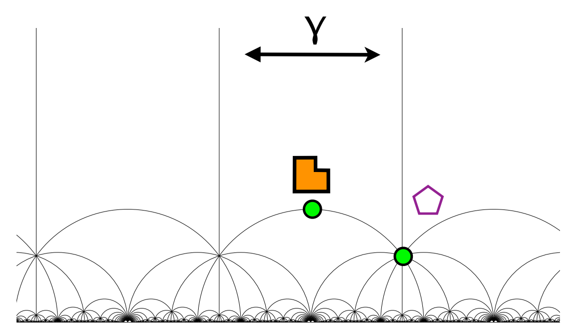
proof:



SL(X,ω) contains:

$$\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

QED



Hilbert modular surfaces

Theorem

For the golden table, we have

$$V = \mathbb{H} / SL(X,\omega) \rightarrow X_K \rightarrow \mathcal{M}_g .$$

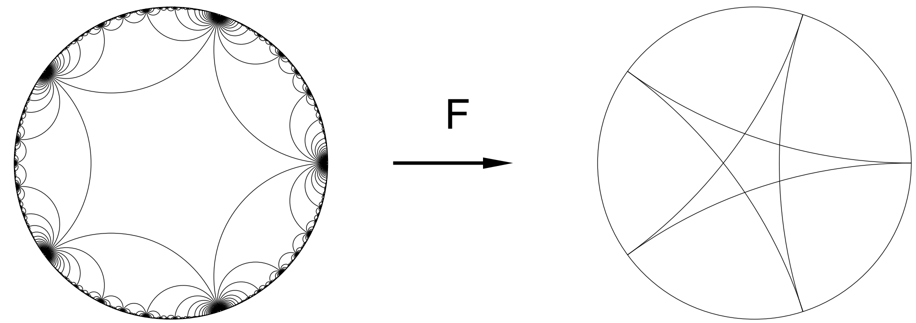
$K = \mathbb{Q}(\sqrt{5}) \qquad g = 2$

Jacobians have real multiplication

Cor: Theorem Q applies.

Holomorphic pentagon-to-star map

$\Gamma = \Delta(2,5,\infty)$ $\qquad$ Γ'



$V \longrightarrow X_K$ covered by $\mathbb{H} \longrightarrow \mathbb{H}\mathrm{x}\mathbb{H}$

via $x \longrightarrow (x,F(x))$.

Pentagon revisited

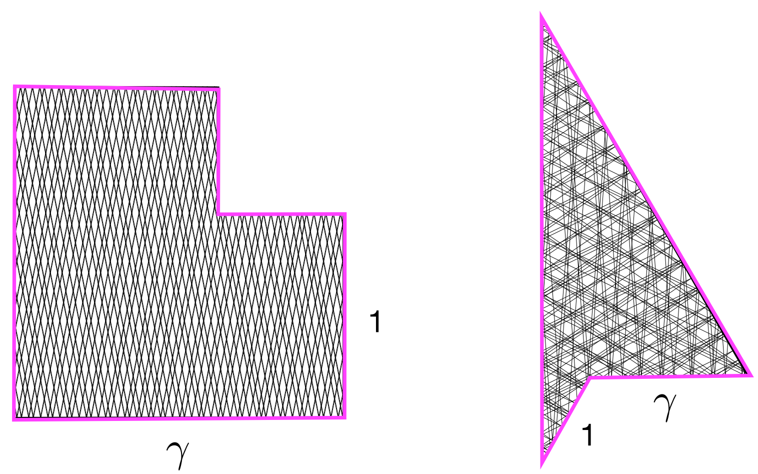
Theorem

Since $SL(X,\omega) = \Delta(2,5,\infty)$:

golden fractions a/c describe unfolded vectors (a,c) of periodic billiards paths.

Cor: *Results on billiards also follow from Theorem Q.*

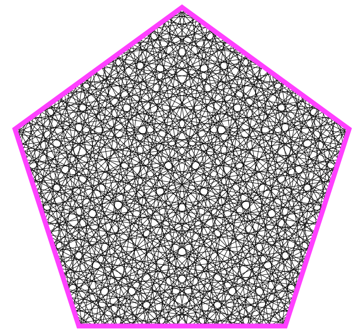
Similarly for all families of optimal billiards



…since these are quadratic: *Eskin - Filip - Wright*

IV. Modular symbols

Billiards in a regular pentagon

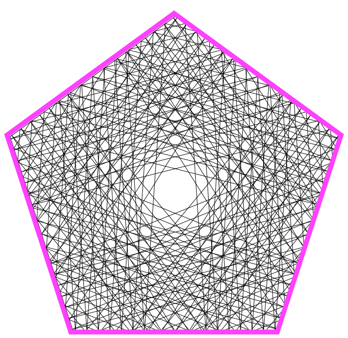


Every trajectory is
periodic or uniformly distributed.

(optimal dynamics)

How are the periodic trajectories distributed?

Billiards in a regular pentagon



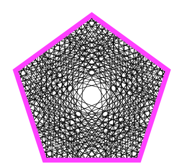
Every trajectory is
periodic or uniformly distributed.

How are the periodic trajectories distributed?

Davis-Lelievre: *Not always uniformly!*

Cantor set of measures?

Limit Measures



Theorem

For each periodic slope s, the limit measures M_s form a countable set, homeomorphic to $\omega^\omega + 1$.

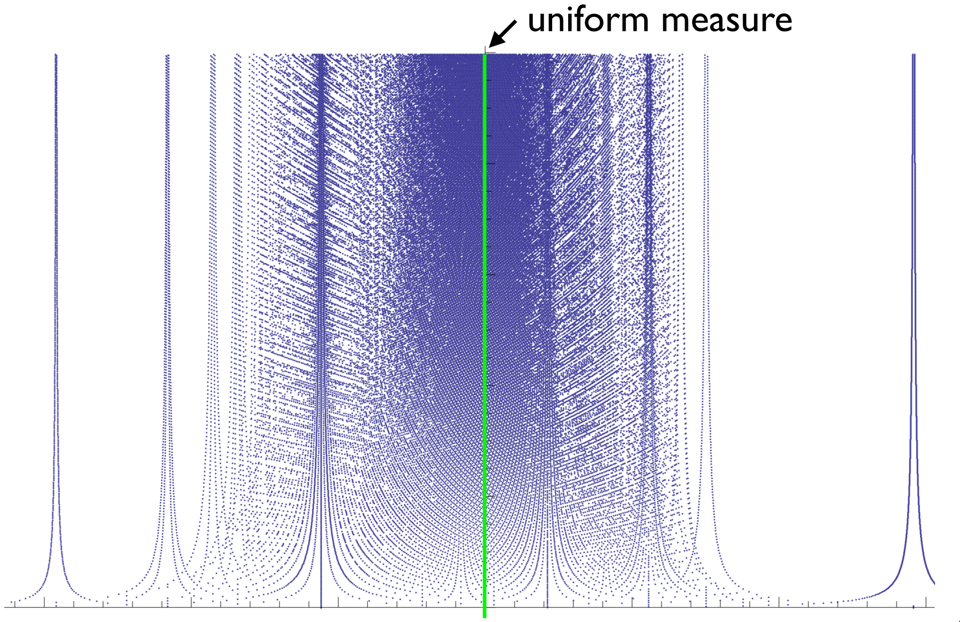
describes scarring

& closure of ergodic measures

Complement

We have uniform distribution iff the lengths of the golden continued fractions of the slopes tend to infinity.

Limit Measures M_0 in golden L



Where does ω^ω come from?

Space of Modular symbols $\mathbb{S}(V)$

$V = \mathbb{H}/\Gamma$ hyperbolic surface

modular symbol of degree d: formal product

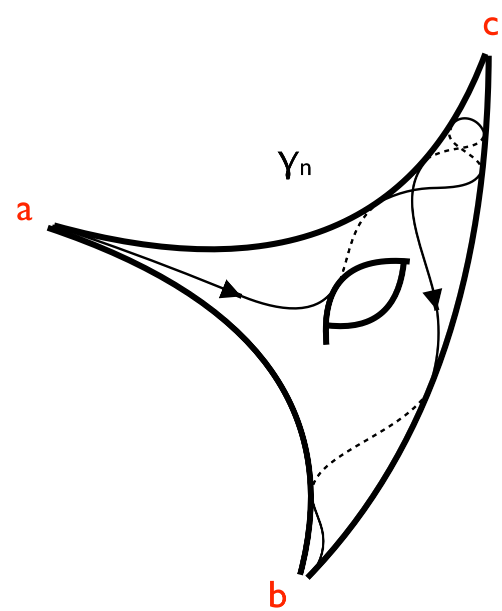
$\sigma = \gamma_1 * \gamma_2 * \ldots * \gamma_d$

$a_0, a_1, \ldots, a_d$ = cusps of V

γ_i geodesic from a_{i-1} to a_i

$a_0 \xrightarrow{\gamma_1} a_1 \xrightarrow{\gamma_2} a_2 \cdots\cdots \xrightarrow{\gamma_i} a_d$

Modular symbols: topology



$a \xrightarrow{\gamma_n} b$

$\gamma_n \longrightarrow \delta_1 * \delta_2$

$a \xrightarrow{\delta_1} c \xrightarrow{\delta_2} b$

symbols of degree one are dense

Algebraically:

$\mathbb{S}(V)$ =

morphisms in a graded category whose objects are the cusps of V

Topologically:

$\mathbb{S}(V) \simeq \omega^\omega$

$$\overline{\mathbb{S}^e(V)} = \bigcup_{d \geq e} \mathbb{S}^d(V)$$

Aside: Classical Modular symbols

$\mathbb{Q} \cup \infty$ = cusps of $\Gamma(N)$ in $SL_2(\mathbb{Z})$

$X(N)$ = completion of $\mathbb{H} / \Gamma(N)$

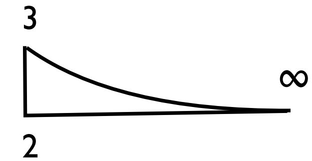
$\{p,q\} : \mathbb{Q}\times\mathbb{Q} \longrightarrow \Omega(X)^* \simeq H_1(X(N), \mathbb{R})$ *abelian*

Theorem (Manin-Drinfeld)

The difference of any 2 cusps of X(N) is torsion in Jac(X(N)).

Modular symbols for $V = \mathbb{H}/SL_2(\mathbb{Z})$

$\infty \xrightarrow{\gamma}$ *p/q in [0,1]*



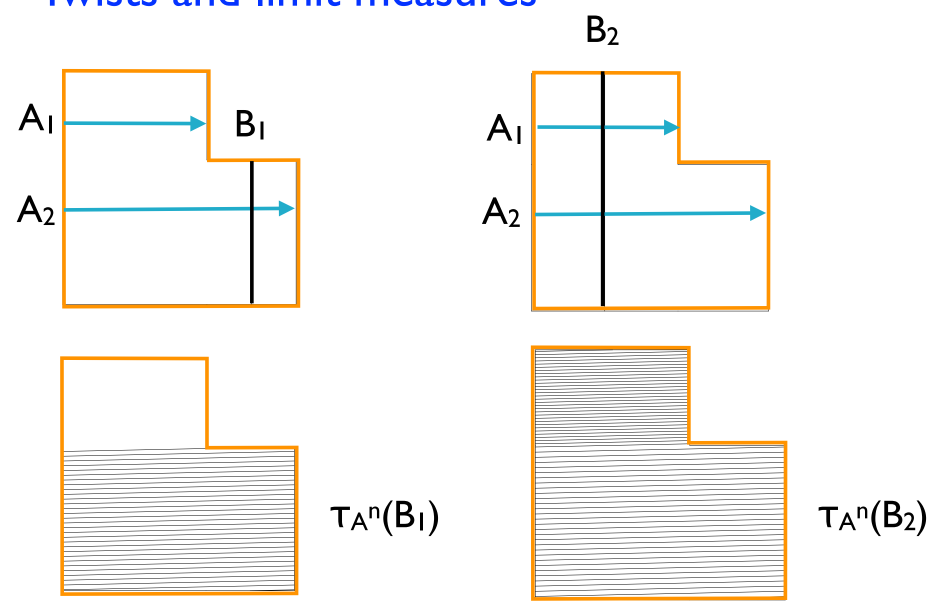
$\mathbb{S}^1(V) = \{ [a_1, \ldots, a_n] \} = \{ 1/a_1 + 1/a_2 + \ldots + 1/a_n \}$

$\mathbb{S}(V) = \{ [a_1, \ldots, a_n] : \text{some } a_i = \infty \}$

$[a_1, \ldots, a_n] * [b_1, \ldots, b_m] = [a_1, \ldots, a_n, \infty, b_1, \ldots, b_m]$

$\{ [a_1, \ldots, a_n] : n \leq N \}$ is *compact*

Twists and limit measures



Measures predicted by i(A,B)

Modular symbols to measures

We have a *continuous functor* $I : \mathbb{S}(V) \longrightarrow \mathbb{L}(V)$

category of matrices up to scale

given by $I(\gamma) = [\mathrm{mod}(A_i)\ i(A_i, B_j)]$.

Decouples as $\gamma \longrightarrow \infty$. $\sim [h(A_i)\ c(B_j)]$

Independent of γ!

$\Rightarrow$ closure of image is ω^ω union a finite set

$\Rightarrow$ limit measures form a copy of $\omega^\omega + 1$.

hidden multiplicative structure QED Theorem II

What about matrix entries in $\Delta(2,5,\infty)$?

M = all nonzero matrix entries

$\delta M = \{m'/m : m \text{ is in } M\}$

$R = -\gamma^{-2} \cdot \delta M$.

Theorem

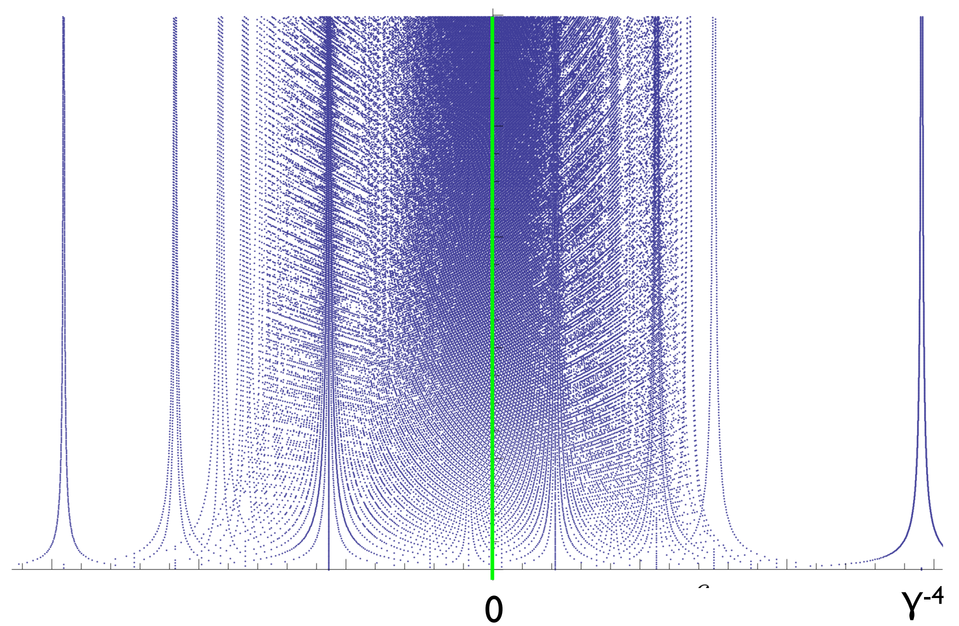
The closure of R is a countable semigroup in [-1,1], homeomorphic to $\omega^\omega + 1$.

(Whereas $\delta\ \mathbb{Z}[\gamma]$ is dense in $\mathbb{R}$.)

cf. Hilbert theorem 90.

IV. Non-arithmetic groups

Image of M under $(m'/m, H(m))$



Compare to ω^ω in

Pisot numbers,
Weyl spectrum,
3D hyperbolic volumes, …

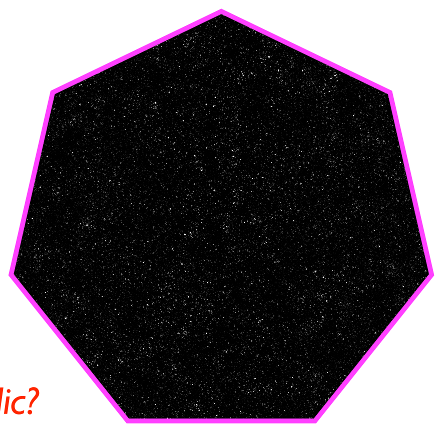
V. The heptagon

Open problems

Open problem

Regular 7-gon

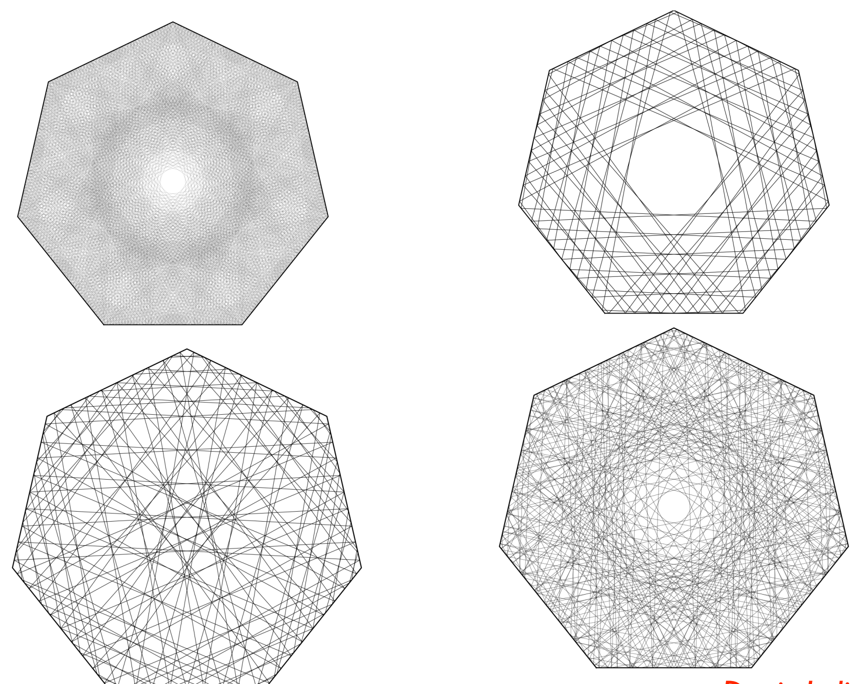
K = $\mathbb{Q}(\cos(2\pi/7))$
(cubic)



(i) Which slopes are periodic?

Shown: L(s)=7, L(2 s) = 2190.

(ii) How long do we have to wait to test periodicity?!



Davis-Lelievre

Bold Conjecture

K = $\mathbb{Q}$(cos(2π/7))

Every x in K is the fixed point of a parabolic or hyperbolic element g in Δ(2,7,∞).

Due independently to Hanson-Merberg-Towse-Yudovina, and Boulanger; further investigations by K. Winsor.

References

Preprints, 2019/2020

Teichmüller dynamics and unique ergodicity via currents and Hodge theory

Modular symbols for Teichmüller curves

Billiards, heights, and the arithmetic of non-arithmetic groups

math.harvard.edu/~ctm/papers

Short Appendix

Heights and Hilbert modular surfaces

Proof of Theorem Q

Curves on a Hilbert modular surface

cf. M, Möller-Viehweg

K = real quadratic field

$$X_K = (\mathbb{H} \times \mathbb{H}) / \operatorname{SL}(\mathcal{O} \oplus \mathcal{O}^\vee)$$

$$V = \mathbb{H}/\Gamma \looparrowright X_K$$ geodesic curve

Theorem

Either V is a Shimura curve, or the cusps of V coincide with $\mathbb{P}^1(K)$ and satisfy quadratic height bounds.

Heights on $\mathbb{P}^n(K)$

$$H(x) = H(x_0 : x_1 : \cdots : x_n) = \prod_v \max_i |x_i|_v.$$

comparable to

$$\widetilde{H}(x) = \inf_a \prod_{v|\infty} \max_i |a_i|_v, \qquad [a_0 : \cdots : a_n] = [x].$$

(a_i are integers)

only requires knowledge of integers and infinite places

Real multiplication

End(A) = ring of endomorphisms of A as a complex Lie group

K totally real field of degree g = dim(A).

A has *real multiplication* by K if we are given a map

$$K \to \mathrm{End}(A) \otimes \mathbb{Q}$$

such that T_k is self-adjoint for all k in K.

Hodge norm at a place v

Diagonalize K on Ω(A) and H_1(A)

$T_k \omega_v = \rho_v(k) \omega_v$ orthonormal eigenforms

$$H_1(A, \mathbb{R}) = \oplus_v S_v \xrightarrow{\pi_v} S_v$$

$$\|C\|_v = \|\pi_v(C)\|_A = \left| \int_C \omega_v \right|$$

$$|C|_v = \|C\|_v^{1/g}$$

\`Hodge valuation'

The projective line $\mathbb{P}^1_A(K)$

$$K \subset \mathrm{End}(A) \otimes \mathbb{Q}$$

$$H_1(A, \mathbb{Q}) \cong K^2$$

$\mathbb{P}^1_A(K)$ = space of K-lines in $H_1(A, \mathbb{Q}) \cong \mathbb{Q}^{2g}$

Height $H_A(x)$ on $\mathbb{P}^1_A(K)$

$$H_A(x) = \inf_C \prod_{v|\infty} |C|_v$$

$x \in \mathbb{P}^1_A(K)$

$C \in H_1(A, \mathbb{Z})$

$[x] = [C]$ (same K line)

Why a height?

$$H_A(x) = \inf_C \prod_{v|\infty} |C|_v$$

$$\widetilde{H}(x) = \inf_a \prod_{v|\infty} \max_i |a_i|_v$$

Theorem. Given a linear isomorphism

$\iota : \mathbb{P}^1_A(K) \to \mathbb{P}^1(K)$

we have $H(\iota(x)) \asymp H_A(x)$.

Proof of Theorem Q

To show a/b in K is a cusp:

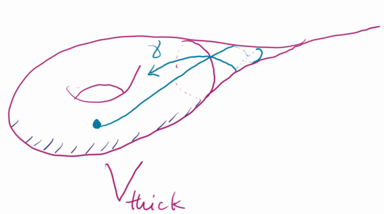
$H_T(a/b) \sim$ (t term) x (F(t) term)

$\lesssim \exp(-s) \exp(|F'| s)$

When t lies over V_{thick} :

$H_T(a/b) \geq 1$

$|F'(t)| < \delta < 1$



So γ spends only a finite amount of time over V_{thick}

$\Rightarrow$

a/b is a cusp

QED Theorem Q