Model Theory of Differential Equations, Algebraic Geometry, and their Applications to Modeling

Marisa Eisenberg (University of Michigan), Rahim Moosa (University of Waterloo),
Alexey Ovchinnikov (City University of New York), Gleb Pogudin (École Polytechnique),
Thomas Scanlon (University of California, Berkeley)

05/31/2020–06/05/2020

1 Context and goals of the workshop

Growing complexity of models used in the sciences makes it harder to use conventional numeric and simulation-based techniques (e.g., due to the computational complexity and curse of dimensionality) and get reliable results. One way to tackle this challenge is to empower these techniques by combining them with a *structural analysis* of the models. It is natural to expect that suitable tools for structural analysis come from the areas of Algebra and Logic. For example, *algebraic geometry* and *differential algebra* have been recently successfully applied to problems of parameter identifiability of dynamical models, multistationarity of biochemical reaction networks, and reconstruction of phylogenetic trees. Moreover, these applications gave rise to intriguing questions in differential algebra and algebraic geometry of practical importance and theoretical interest together with valuable intuition behind the approaches to these questions.

On the other hand, in the past decades, algebraic geometry and differential algebra themselves interacted a lot with *model theory*, and this interaction resulted in new impressive directions and results on both sides. In particular, model theory of differential and difference fields has been developed and used to provide insights about the Galois theory of differential and difference equations and deep structural results for differential-algebraic varieties (that is, solution sets of systems of differential equations).

Thus, there are two actively studied connections

$$\begin{array}{lll} \text{differential algebra \& algebraic geometry} \longleftrightarrow \text{modeling} & & \text{(see [2, 5])} \\ \text{differential algebra \& algebraic geometry} \longleftrightarrow \text{model theory} & & \text{(see [6, 12, 3]).} \end{array}$$

Moreover, resent results on multi-experiment parameter identifiability [8, 9] combined all three areas. It turned out that the similarity between definability, a fundamental notion in model theory, and identifiability, a structural property of dynamical models, goes beyond phonetics: in an appropriate context, identifiability is a special case of definability.

The goal of the workshop was to bring together researchers working in between these areas and discuss existing and potential interactions of these areas altogether. In short: what could be a model theory of modeling?

2 Format of the workshop: going online

Since the scope of the workshop was broad and the mathematical background of the participants was ranging from pure logic to applied mathematics, the program included three tutorials:

1. *Model theory, quantifier elimination, and differential algebra* by D. Marker;
2. *Challenges in the study of algebraic models of biochemical reaction networks* by E. Feliu;
3. *Structural parameter identifiability with a view towards model theory* by G. Pogudin.

The official program of the workshop consisted of five three-hour slots (one slot for each day from Monday to Friday). Each slot consisted of either two tutorials or three research talks followed by "coffeebreaks". Each coffeebreak was a discussion dedicated to one of the talks or the tutorials, and it was always moderated by one of the organizers or a senior participant. The coffeebreaks in the first days usually started with the participants introducing themselves. A Slack workspace of the workshop turned out to be a great platform for following up on the talks, discussions, and exchanging the materials.

3 Outcomes of the meeting

Based on the discussions during the talks, coffeebreaks and in the Slack workspace, we would like to outline the following potential topics for three-way interactions between model theory, algebraic geometry/differential algebra, and modeling:

- *O-minimal structures and nonpolynomial functions.* One of the limitations of many popular algebraic methods used in modeling is the fact that they typically manipulate with polynomial functions. On the other hand, numerous nonpolynomial functions (e.g., exponential, logarithmic) appear in practice. For example, in chemical reactor design, one is often interested in considering convex hulls of trajectories of solutions of ODE systems [7]. Such convex hull while being inconvenient from the algebraic standpoint can be naturally viewed from the point of view of the theory of bounded analytic functions, its o-minimality and quantifier elimination [4].
- *Valued fields and initial conditions.* The majority of dynamical models used in sciences consist of equations of some type together with the initial conditions describing the initial state of the system. Many important properties of the model (boundedness, identifiability, controllability) may substantially depend on the values of the initial conditions. A natural way to treat the initial conditions from the algebraic point of view is using valued differential fields. Model theoretic approach has already been proved to be fruitful for these objects [1], so it is natural to conjecture that these tools can now be applied to study dynamical models with initial conditions.
- *Real algebraic geometry, ordered fields, and models over the real numbers.* Many existing algorithms for studying structural properties of dynamical models using techniques from algebraic geometry work over the field of complex numbers. This is not always realistic in the context of applications in the sciences in which most of the unknowns are real (or even nonnegative) functions. A natural approach would be to refine and extend the existing algorithms using tools from real algebraic geometry and model theory of ordered differential fields [1, 11].
- *Symmetries of models and differential Galois theory.* Some structural questions about dynamical models (such as identifiability and order reduction) can be viewed as questions about the symmetries of models. One common framework to study symmetries of differential equations is differential Galois theory, in which model theoretic techniques have been already successfully used [10]. Turning these theory into algorithms can help gain deep insights into the structure of dynamical models.

References

[1] M. Aschenbrenner, L. van den Dries, J. van der Hoeven, *Asymptotic differential algebra and model theory of transseries*, Princeton University Press, 2017.

[2] F. Boulier and F. Lemaire, Differential Algebra and System Modeling in Cellular Biology, *International Conference on Algebraic Biology* (2008), 22-39.

[3] E. Bouscaren (editor), *Model theory and algebraic geometry*, Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 1998.

[4] J. Denef and L. van den Dries, P-adic and real subanalytic sets, *Annals of Mathematics*, **128** (1988), 79–138.

[5] A. Dickenstein, Biochemical reaction networks: an invitation for algebraic geometers, In *Mathematical Congress of the Americas*, Contemp. Math., **656**, 65-83. American Mathematical Society, 2016.

[6] D. Marker, Model theory of differential fields, In D. Marker, M. Messmer, and A. Pillay, *Model Theory of Fields*, Lecture Notes in Logic, Cambridge University Press (2017), 38-113.

[7] D. Ming, D. Glasser, D. Hildebrandt, B. Glasser, and M. Metzger, *Attainable region theory: an introduction to choosing an optimal reactor*, Wiley, 2016.

[8] A. Ovchinnikov, A. Pillay, G. Pogudin, T. Scanlon, Computing all identifiable functions for ODE models, arXiv:2004.07774, 2020.

[9] A. Ovchinnikov, A. Pillay, G. Pogudin, T. Scanlon, Multi-experiment parameter identifiability of ODEs and model theory, arXiv:2011.10868, 2020.

[10] A. Pillay, Differential Galois theory I, *Illinois Journal of Mathematics*, **42:4** (1998), 678–699.

[11] M. Singer, The model theory of ordered differential fields, *The Journal of Symbolic Logic*, **43:1** (1978), 82–91.

[12] T. Scanlon, Model theory and differential algebra, *Differential Algebra and Related Topics*, World Scientific (2002), 125-150.