Brill-Noether for moduli spaces of sheaves on surfaces

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Set up: X smooth proj surface

H ample on X.

$\mathcal{F}$ coherent, pure dim sheaf

$$P_{\mathcal{F}}(m) = \chi(\mathcal{F}(m)) = a_d \frac{m^d}{d!} + \text{l.o.t.}$$

↑ Hilbert poly.

Reduced Hilbert poly $\quad p_{\mathcal{F}} = \dfrac{P_{\mathcal{F}}}{a_d}$

If $\mathcal{F}$ is torsion free, $\mu = \frac{c_1(\mathcal{F})}{r}$

H-slope $\mu_H = \frac{c_1(\mathcal{F}) \cdot H}{\mathrm{rk}(\mathcal{F})}$

the discriminant

$$\Delta = \frac{1}{2}\left(\frac{c_1}{r}\right)^2 - \frac{ch_2}{r}$$

It will be convenient to use

(r, μ, Δ) instead of

(r, ch_1, ch_2).

Def: $\mathcal{F}$ is <u>H-Gieseker</u> <u>(semi)stable</u> if for all proper $0 \neq \mathcal{E} \subsetneq \mathcal{F}$

$$p_{\mathcal{E}}(m) \underset{(-)}{<} p_{\mathcal{F}}(m) \text{ for } m \gg 0.$$

Semistable sheaves have Jordan-Hölder filtrations. $\mathcal{F}$ and $\mathcal{E}$ are <u>S-equivalent</u> if $\mathcal{F}$ and $\mathcal{E}$ have same associated graded.

Theorem: (Gieseker-Maruyama)
There exists projective moduli spaces $M_{X,H}(v)$ parameterizing S-equiv. classes of H-Gieseker ss sheaves on X with Chern character v.

Question: · What is the cohomology of a general sheaf $\mathcal{F} \in M_{X,H}(v)$?

· Describe the loci of sheaves with 'unexpected' cohomology.

Remarks: · If we fix r, μ and let $\Delta \gg 0$, then by Theorem of O'Grady $M_{X,H}(v)$ is irreducible.

If Δ is sufficiently large, then

$H^i(\mathcal{F}) = 0 \quad i = 0, 2$

$h^1(\mathcal{F}) = -\chi(\mathcal{F})$.

· If μ is sufficiently ample, $\mathcal{F} \in M_{X,H}(v)$ is general,

$H^i(\mathcal{F}) = 0 \quad i = 1, 2, \quad h^0(\mathcal{F}) = \chi(\mathcal{F})$.

Theorem (Göttsche–Hirschowitz)

$M_{\mathbb{P}^2, \mathcal{O}(1)}(r, \mu, \Delta)$ with $r \geq 2$.

μ under the μ: $\cup$

Then general $\mathcal{F}$ has at most one nonzero cohomology group

If $\chi(\mathcal{F}) < 0$, then $H^0(\mathcal{F}) = \underset{\text{"}0\text{"}}{H^2(\mathcal{F})}$

$h^1(\mathcal{F}) = -\chi(\mathcal{F})$.

If $\chi(\mathcal{F}) \geq 0$ and $\mu \geq 0$,

$h^0(\mathcal{F}) = \chi(\mathcal{F}) \quad h^i(\mathcal{F}) = 0 \quad i = 1, 2.$

If $\chi(\mathcal{F}) \geq 0$ and $\mu < 0$

$h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0 \quad h^2(\mathcal{F}) = \chi(\mathcal{F})$.

Example: $\chi(I_p(-3)) = 0$

$p \in \mathbb{P}^2$ is a point.

But $H^1(I_p(-3)) \simeq H^2(I_p(-3)) \simeq \mathbb{C}$.

Weak Brill-Noether holds if there exists $\mathcal{F} \in M_{X,H}(v)$ such that $\mathcal{F}$ has at most one nonzero cohomology group.

If $M_{X,H}(v)$ is irreducible, then this is equivalent to saying that the general sheaf has at most one nonzero cohomology group.

$\mathbb{P}^2$ is the dream situation. WBN holds for all moduli spaces of bundles.

Already when there are curves with negative self-intersection, the picture is not as simple.

Example: Let $\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$

Let E be the section with $E^2 = -e$.

$$\chi(\mathcal{O}_{\mathbb{F}_e}(E)) = 2-e$$

In particular, if $e \geq 2$, $\chi \leq 0$, but $\mathcal{O}(E)$ clearly has a section.

More generally, it can happen that

$$\chi(\mathcal{F}(E)) \leq 0 \quad , \text{ but } \quad \chi(\mathcal{F}) > 0$$

so the Euler char. alone cannot predict the cohomology.

Theorem (C–Huizenga)

$M_{\mathbb{F}_e, H}(r, \mu, \Delta) \ni \mathcal{F}$ general sheaf.

(1) $F \cdot \mu \geq -1 \implies h^2(\mathcal{F}) = 0$

(2) $F \cdot \mu \leq -1 \implies h^0(\mathcal{F}) = 0$

(In particular, if $\mu \cdot F = -1$, $h^1(\mathcal{F}) = -\chi(\mathcal{F})$ is the only possible nonzero coh.)

By Serre duality, assume $F \cdot \mu > -1$.

(3) If $E \cdot \mu \geq -1$, then $\mathcal{F}$ has at most one nonzero cohomology group.

(4) If $E \cdot \mu < -1$, then

$$H^0(\mathcal{F}(-E)) \simeq H^0(\mathcal{F}).$$

and the computation inductively reduces to (1) or (3).

K3 surfaces:

New feature: $\mathcal{O}_X$ has both h^0 and h^2.

Set up: X K3 surface

$\mathrm{Pic}(X) \cong \mathbb{Z}H$, $H^2 = 2n$.

Examples:

① $0 \to \mathcal{O} \to \mathcal{O}(H)^{n+2} \to \mathcal{F} \to 0$
on X $(H^2 = 2n)$

$\mathcal{F}$ is a spherical, stable bundle (i.e. it is the unique point in the moduli space)

$H^0(\mathcal{F}) \cong \mathbb{C}^{(n+2)^2-1}$ and $H^1(\mathcal{F}) \cong \mathbb{C}$.

② Let X be a degree 2 K3.

Let $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$ ....

be the Fibonacci numbers.

$$0 \to \mathcal{O}^{f_{2k-2}} \longrightarrow \mathcal{O}(H)^{f_{2k}} \longrightarrow \mathcal{F} \to 0$$

E is a stable spherical bundle.
It is the unique point in the moduli space.

$$H^0(\mathcal{F}) = \mathbb{C}^{3f_{2k}-f_{2k-2}} \qquad H^1(\mathcal{F}) \simeq \mathbb{C}^{f_{2k-2}}$$

③ X a K3 of degree $2n$.

Mukai vector $v = (r,\ rk+1,\ nk^2r + 2nk - i)$

$M_{X,H}(v) \ni \mathcal{F}$ general

$$h^1(\mathcal{F}) = \max(0,\ r - 2nk - ki)$$

$$h^0(\mathcal{F}) = (nk^2+1)r + 2nk - i + h^1(\mathcal{F})$$

Example with $i=0$

$$0 \to \mathcal{O}(kH)^r \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_H(D) \to 0$$

Here $\mathcal{O}_H(D)$ is a line bundle on a curve in $|H|$.

$$\chi(\mathcal{O}_H(D)) = 2nk - r$$

If $r > 2nk$, then $H^1(\mathcal{O}_H(D)) \neq 0$

We see that $H^0(\mathcal{F}) \simeq \mathbb{C}^{(2+k^2n)r}$

$$H^1(\mathcal{F}) = \mathbb{C}^{2nk-r}.$$

<u>Set up</u>: X K3 $\rho(X)=1$

$\mathrm{Pic}(X) = \mathbb{Z}H$, $H^2 = 2n$.

Mukai vector

$$v(E) = (r, c_1, a) = ch(E)\sqrt{td(X)}$$

$$\chi(E) = a + r$$

Mukai pairing

$$\langle v(E), v(F)\rangle = c_1(E)c_1(F) - r_F a_E - r_E a_F = -\chi(E,F)$$

Basic facts about moduli spaces

$v = m v_0$, v_0 primitive Mukai vector

<u>Theorem</u> (Mukai, Yoshioka, ...)

$M_{X,H}(v)$ is nonempty $\iff v_0^2 \geq -2$.

① If $m=1$ or $v_0^2 > 0$, $\dim M_{X,H}(v) = v^2 + 2$

② If $v_0^2 = -2$, then $M_{X,H}(v)$ is a single point.

If $v_0^2 = 0$, then $\dim M_{X,H}(v) = 2m$

③ Normal, irreducible projective variety with $\mathbb{Q}$-factorial sing.

④ There is a classification of moduli spaces where the general sheaf is not locally free.

⑤ There is a classification of moduli spaces where the general sheaf is not slope stable.

Strategy for studying $H^*(\mathcal{F})$

$$\Delta \subset X \times X$$

$$\Phi^{I_\Delta}_{X \to X} : D^b(X) \to D^b(X)$$

Let $E := \Phi^{I_\Delta}_{X \to X}(\mathcal{F})^\vee$

Assume E is a coherent sheaf. Then

(1) $H^1(X, \mathcal{F}) = H^2(X, \mathcal{F}) = 0$

(2) E is locally free $\iff$ $\mathcal{F}$ is globally generated.

$$0 \to \mathcal{H}^0(\Phi^{I_\Delta}_{X \to X}(\mathcal{F})) \to H^0(\mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$$
$$\to \mathcal{H}^1(\Phi^{I_\Delta}_{X \to X}(\mathcal{F})) \to H^1(\mathcal{F}) \otimes \mathcal{O}_X \to 0$$

$$\hookrightarrow \mathcal{H}^2(\Phi^{I_0}_{X\to X}(\mathcal{F})) \to H^2(\mathcal{F})\otimes \mathcal{O}_X \to 0$$

$$\mathcal{H}^i(\Phi^{I_0}_{X\to X}(\mathcal{F})) = \mathcal{E}xt^i(E, \mathcal{O}_X)$$

We study this using Bridgeland stability.

$\beta, \omega \in NS(X)_{\mathbb{R}}$, ω ample. $E \in D^b(X)$

$$Z_{\beta,\omega}(E) = \langle e^{\beta+i\omega}, E\rangle$$

$$\mathcal{A}_{\beta,\omega} = \left\{ E \in D^b(X) \;\middle|\; \begin{array}{l} \mathcal{H}^p(E) = 0 \quad p \neq -1, 0 \\ \mathcal{H}^{-1}(E) \in \mathcal{F}_{\beta,\omega} \\ \mathcal{H}^0(E) \in \mathcal{T}_{\beta,\omega} \end{array} \right\}$$

(1) $\mathcal{F}_{\beta,\omega}$ torsion free sheaves F st $\forall F' \subseteq F$

$\quad \operatorname{Im} Z_{\beta,\omega}(F') \leq 0$

(2) $\mathcal{T}_{\beta,\omega}$ sheaves $T \twoheadrightarrow Q$

$\quad \operatorname{Im} Z_{\beta,\omega}(Q) > 0.$

Since $\operatorname{Pic}(X) \cong \mathbb{Z}H$, we get upper half plane of stability conditions.

Wall and Chamber decomposition.

Given $\exists$ there is a special chamber C adjacent to the wall defined by $I_x^\vee[1]$

Prop. (Minamide-Yanagida-Yoshioka)

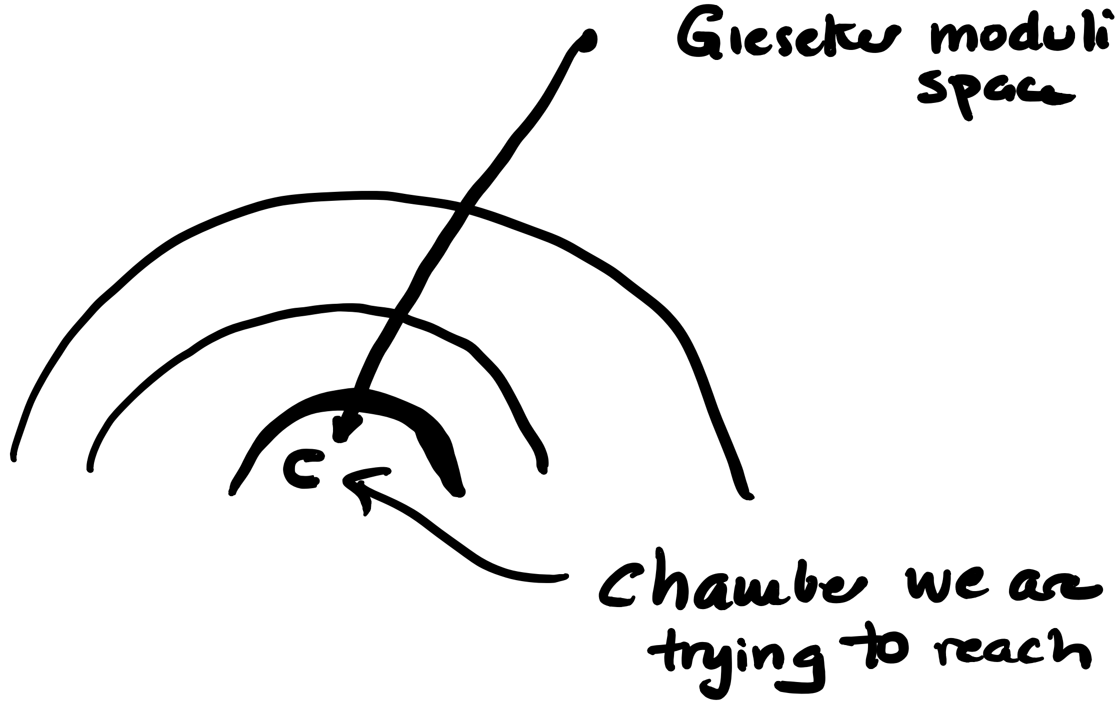
$\sigma \in C$

$$M_\sigma(r, dH, a) \simeq M_{X,H}(a, dH, r)$$

$$E \longmapsto \Phi^{I_D}_{X \to X}(E)^\vee$$

Question: Does $M_\sigma(r, dH, a)$ contain any stable sheaves?

Cartoon.



Problem: We may cross a totally semistable wall and destabilize all sheaves.

Bayer-Macrì have classified the totally semistable walls.

The relevant walls are defined by spherical objects and can be enumerated.

Theorem: (C-Nuer-Yoshioka)

X K3 surface. $\mathrm{Pic}(X) = \mathbb{Z}H$ $H^2 = 2n$

Let $\mathcal{F} \in M_{X,H}(r, dH, a)$ general sheaf.

(1) If $n \geq r$, then $\mathcal{F}$ has at most one nonzero cohomology group.

(2) If $n \geq 2$, $\mu \geq r+1$, then $\mathcal{F}$ has at most one nonzero cohomology group.

(3) If $\chi(\mathcal{F}) \leq r$, then $\mathcal{F}$ has at most one nonzero cohomology group.

(4) For each rank $r \geq 2$, there are only finitely many moduli spaces $M_{X_n,H}(r, dH, a)$ where the general sheaf has more than one nonzero cohomology group.

Cor. There exists an Ulrich bundle of rank r on (X, mH) $\iff$ $2 \mid rm$.

The general sheaf in

$$M_{X,H}\left(r, \frac{3rm}{2}H, r(2m^2n-1)\right)$$

is Ulrich.

More importantly, one can compute the cohomology of the general sheaf

The biggest strictly semistable wall gives a resolution of the general sheaf

$n=1$	$(2,3,5)$	$h^1=1$
$n=1$	$(3,4,5)$	$h^1=1$
$n=2$	$(3,4,11)$	$h^1=1$
$n=1$	$(4,5,6)$	$h^1=2$

$n=1$	$(4,6,9)$	$h^1=2$
$n=3$	$(4,5,19)$	$h^1=1$
$n=1$	$(5,3,2)$	$h^1=1$
$n=1$	$(5,6,7)$	$h^1=3$
$n=1$	$(5,7,10)$	$h^1=3$
$n=1$	$(5,8,13)$	$h^1=3$
$n=2$	$(5,6,14)$	$h^1=1$
$n=1$	$(5,11,24)$	$h^1=1$
$n=4$	$(5,6,29)$	$h^1=1$
$n=1$	$(5,12,29)$	$h^1=1$

Example: $v=(9,5,14)$ $\quad n=5$

$v_1=(2,1,3)$

$$0 \to \mathcal{O} \to E^5_{2,1,3} \to \mathcal{F} \to 0$$

$$H^1(\mathcal{F}) = \mathbb{C}.$$