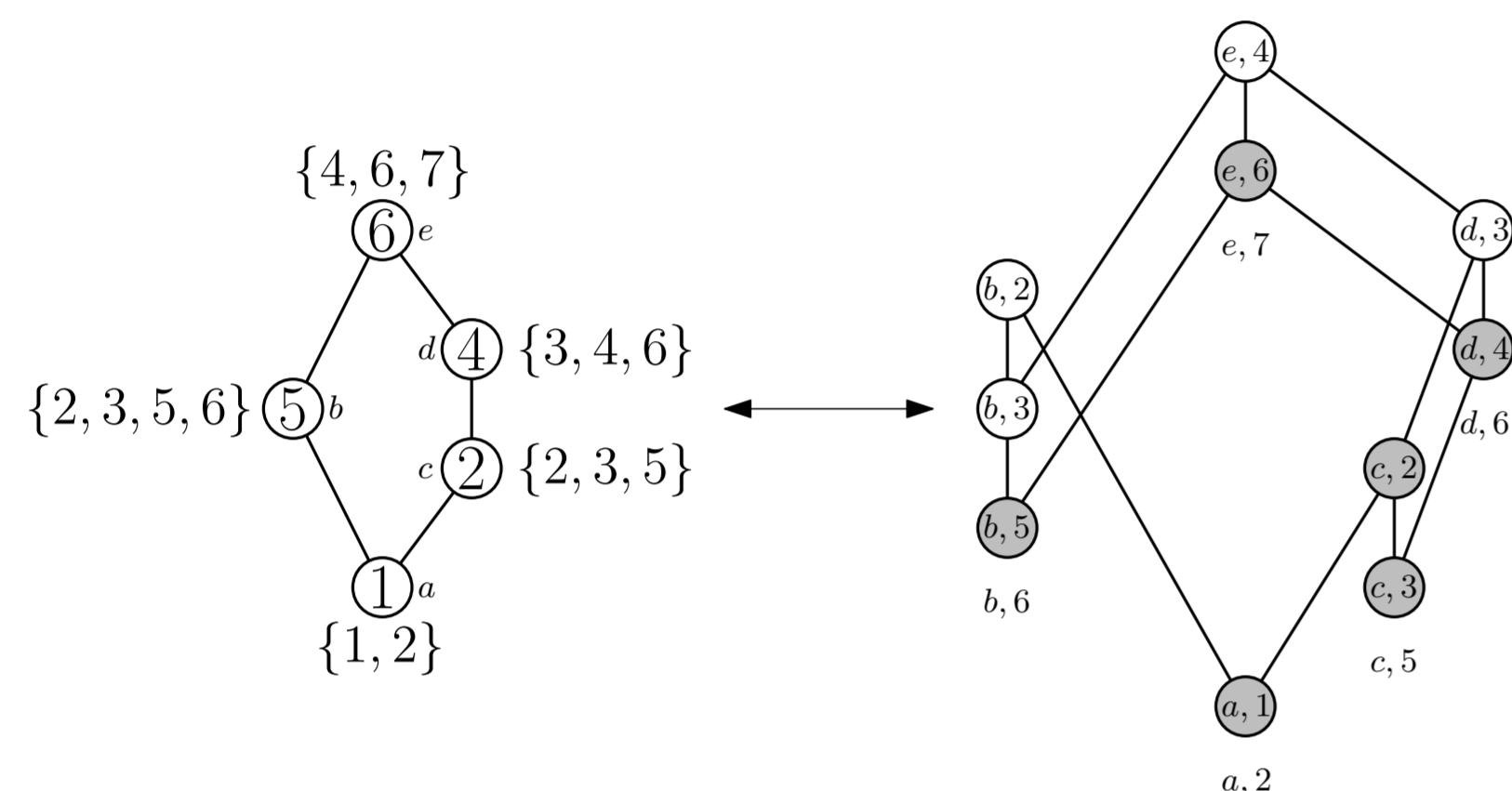# P-strict Promotion and Piecewise-Linear Rowmotion

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Preliminaries

The poset $\Gamma(P, R)$ (right) is defined such that it satisfies the following theorem:


Theorem (Dilks, Striker, Vorland 2017)**.** *Increasing labelings of a poset P with restriction function R under increasing labeling promotion is in equivariant bijection with $J(\Gamma(P, R))$ under toggle-promotion.*

Here, increasing labeling promotion is the composition of *generalized Bender-Knuth involutions* and toggle-promotion is defined by toggling at all elements in $\Gamma(P, R)$ with second component 1, then elements with second component 2, and so on.

Developing the Notation

A multichain of order ideals $\mathcal{O}_\ell \leq \mathcal{O}_{\ell-1} \leq \cdots \leq \mathcal{O}_1$ in $J(\Gamma(P, R))$ corresponds to a $\Gamma(P, R)$-partition, and, since each order ideal corresponds to an increasing labeling, the multichain is also in correspondence with a labeling of $P \times [\ell]$ that strictly increases on copies of P (*layers*) and weakly increases on copies of $[\ell]$ (*fibers*). We call this a **P-strict labeling** of $P \times [\ell]$.
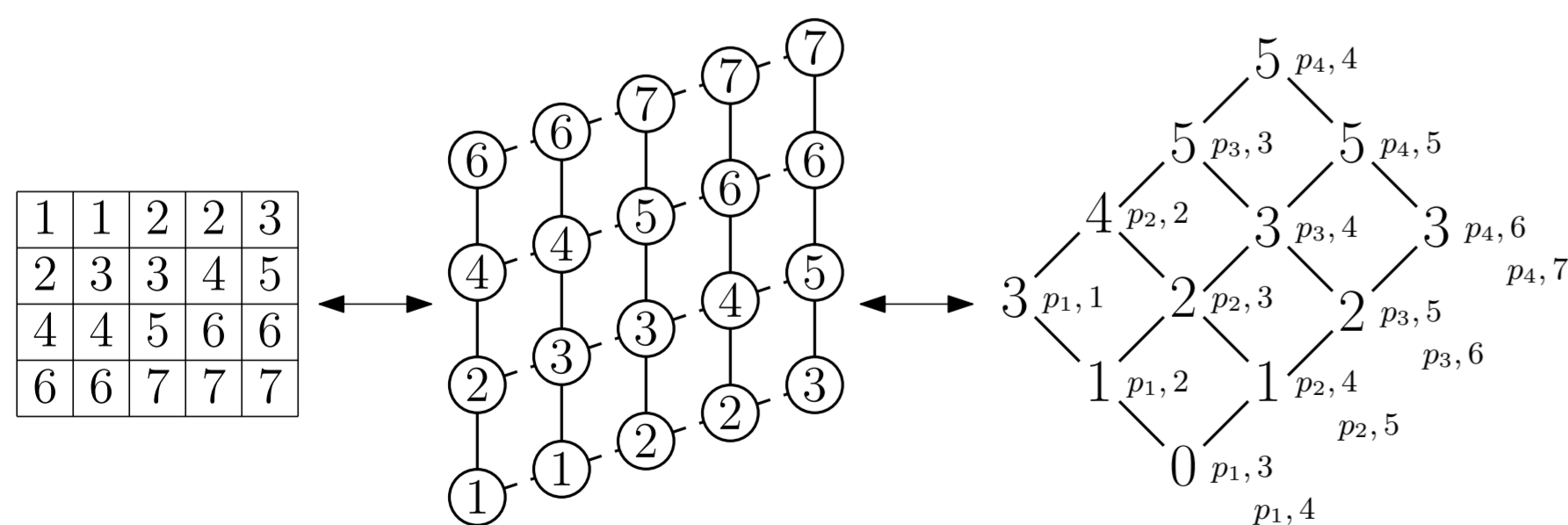
In the case where $P = [n]$ and the restriction function R is induced by a global maximum, this is exactly a semistandard Young tableau of rectangular shape.

We should be able to find something similar for skew semistandard tableaux, but these shapes can change the restriction function, such as by allowing 1 in the rows other than the first.
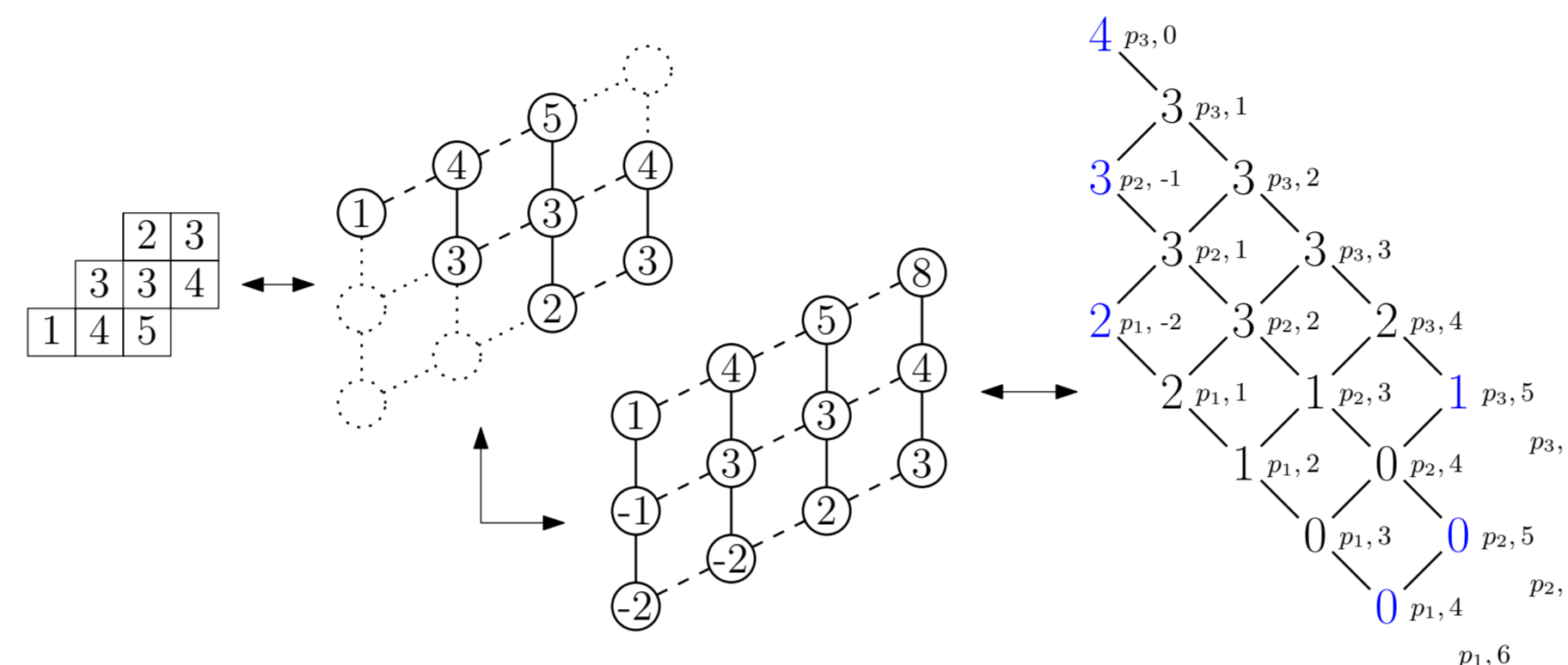
We accommodate this by defining a new restriction function $\hat{R}$ that adds a new maximum and minimum element to $R(p)$ for each $p \in P$, and filling the "missing" boxes with those new values. Now we can treat the general case the same as the rectangular case.

However, because these new values are in every filling of the rectangle, this fixes some elements in the $\Gamma(P, \hat{R})$-partition (shown in blue).

In this perspective, a skew semistandard tableau of shape λ/μ is a $[n]$-strict labeling of a convex subposet of $[n] \times [\ell]$ where $n = |\lambda|$ and $\ell = \lambda_1$. We write a general convex subposet of $P \times [\ell]$ as $P \times [\ell]_u^v$, where $u, v : P \to \{0, 1, \ldots, \ell\}$ denote the number of "missing" elements at the beginning and the end of the fiber $\{(p, i) \in P \times [\ell] \mid 1 \leq i \leq \ell\}$, respectively.

We can now define the objects used in the main theorem:

Definition. Let $\mathcal{L}_{P \times [\ell]}(u, v, R)$ denote the set of all P-strict labelings on $P \times [\ell]_u^v$ with restriction function R.

Definition. Let $\mathcal{A}^{\hat{B}}(\Gamma(P, \hat{R}))$ denote the set of $\Gamma(P, \hat{R})$-partitions σ where $\sigma(p, \min \hat{R}(p)) = \ell - u(p)$ and $\sigma(p, \max R(p)) = v(p)$.

The Main Theorem

Theorem. $\mathcal{L}_{P \times [\ell]}(u, v, R)$ *under* Pro *is in equivariant bijection with $\mathcal{A}^{\hat{B}}(\Gamma(P, \hat{R}))$ under* TogPro.

Here, Pro is the composition of Bender-Knuth involutions and TogPro is defined by performing the piecewise-linear toggle at all elements of $\Gamma(P, \hat{R})$ with second component 1, then 2, and so on.

Note that we drop adjectives modifying *promotion* and *Bender-Knuth involutions*, as our general definitions are consistent with previous applications.

The example below displays the full generality of the theorem.
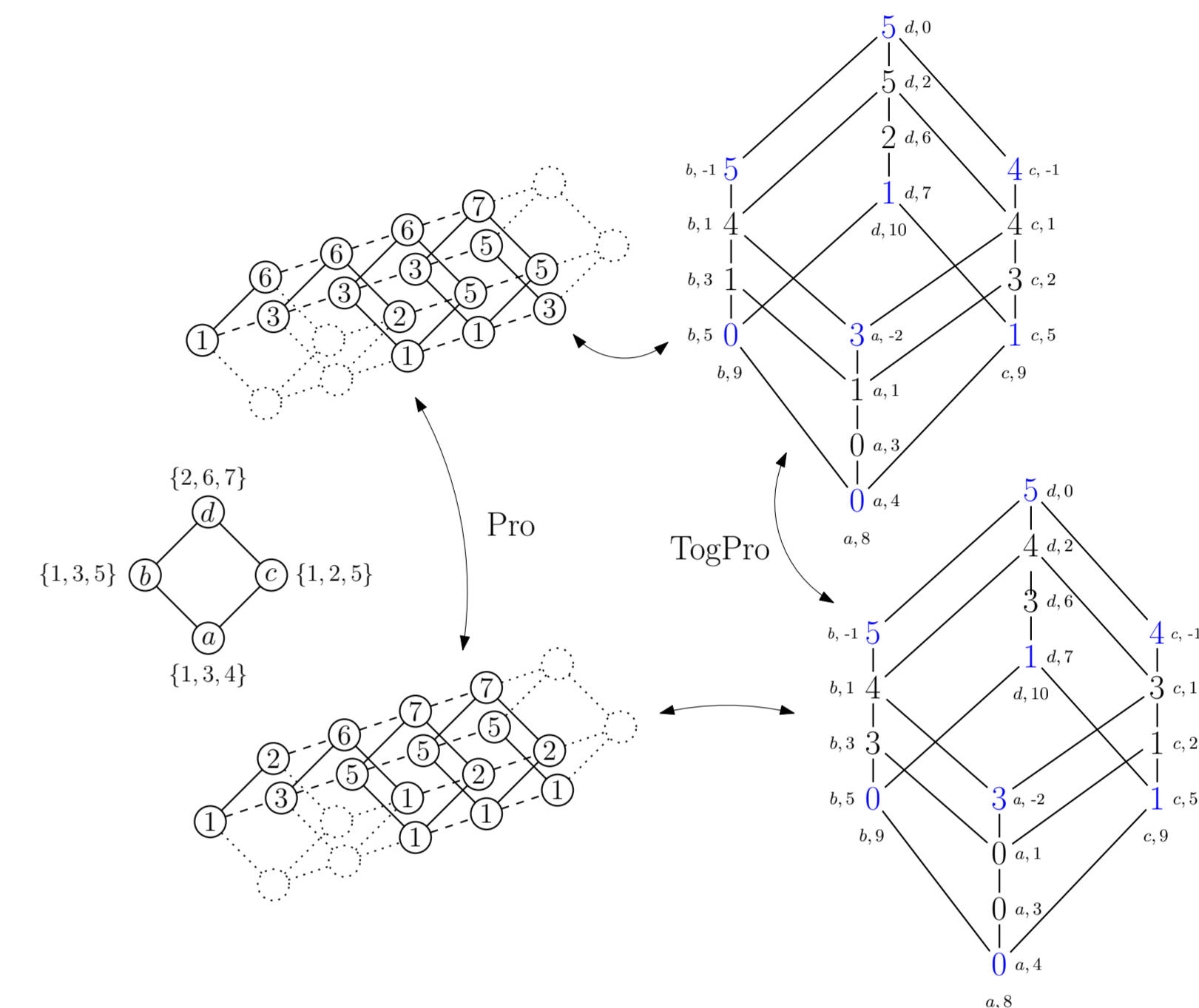
A few remarks:

- If whenever $(p_1, k_1) \lessdot_{\Gamma(P, \hat{R})} (p_2, k_2)$ and $|k_2 - k_1| \neq 1$ either (p_1, k_1) or (p_2, k_2) is fixed in $\mathcal{A}^{\hat{B}}(\Gamma(P, \hat{R}))$, piecewise-linear promotion is conjugate to piecewise-linear rowmotion.

- We can use appropriately defined *jeu-de-taquin* slides to define a promotion action on P-strict labelings equivalent to the composition of Bender-Knuth involutions.

- In the skew semistandard tableaux case (i.e. $P = [n]$), the other side of theorem is equivalent to the composition of elementary transformations over rows of Gelfand-Tsetlin patterns (a result of Kirillov and Berenstein).

And, finally, a specific application involving *flagged tableaux*:
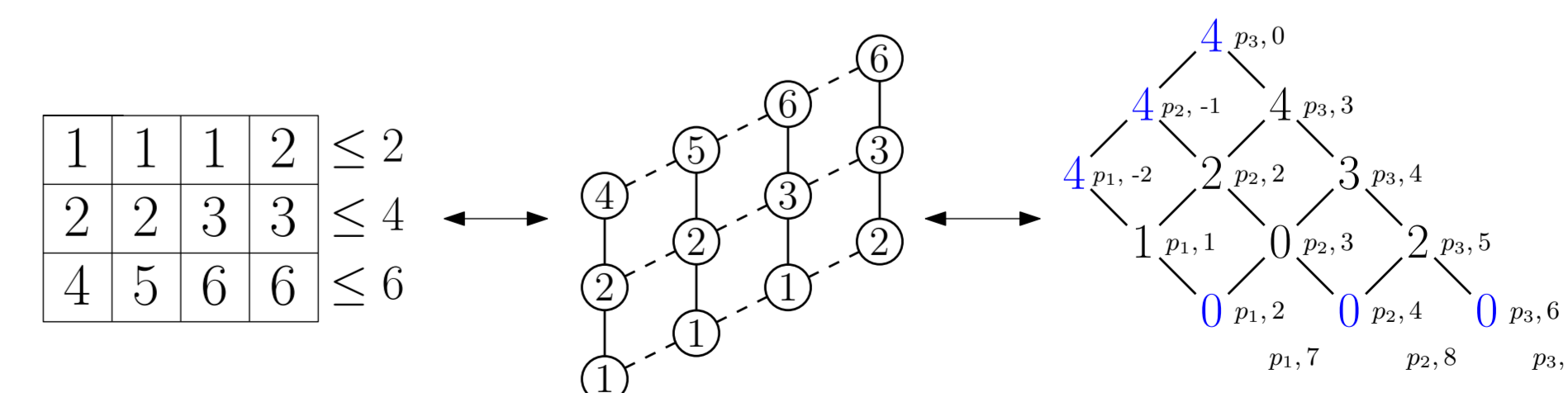
Theorem (Grinberg and Roby)**.** Row *on* $\mathcal{A}^\ell(\Phi^+(A_n))$ *has order dividing* $2(n+1)$.

Corollary. Pro *on* $FT(\ell^n, (2, 4, \ldots, 2n))$ *has order dividing* $2(n+1)$.