Let's birational: Lifting periodicity and homomesy to higher realms

Tom Roby (UConn)

Online Workshop on Dynamical Algebraic Combinatorics (20w5164)
Banff International Research Station

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This talk is being recorded!

Slides for this talk are available online at

http://www.birs.ca/workshops/2020/20w5164/files/
or Google "Tom Roby".

Abstract

Abstract: Maps and actions on sets of combinatorial objects often have interesting extensions to the piecewise-linear realm of order and chain polytopes. These can be further lifted to the birational realm via detropicalization/geometrization, and even to a setting with noncommuting variables. Surprisingly often, properties shown at the "combinatorial shadow" level, such as homomesy and low-order periodicity, lift all the way up to these higher realms.

Acknowledgments

This talks discusses the work of several authors, including joint work, Darij Grinberg, Mike Joseph, Gregg Musiker, and Jim Propp.

I'm grateful to Mike Joseph and Darij Grinberg for sharing source code for slides from their earlier talks, which I shamelessly cannibalized.

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Please feel free to put questions and comments in the chat, and the moderator will convey them with appropriate timing and finesse. Or someone else may answer them!

Outline

In this talk we have two types of actions, which we lift in parallel, four realms for each:

1. Combinatorial Rowmotion on antichains, $\rho_{\mathcal{A}}$;
2. Piecewise-linear rowmotion on chain polytopes, $\rho_{\mathcal{C}}$;
3. Birational Antichain Rowmotion (BAR-motion) on $\mathbb{K}$-labelings of P, BAR;
4. Noncommutative Antichain Rowmotion (NAR-motion) on $\mathbb{S}$-labelings of P, NAR;
5. Combinatorial Rowmotion on order filters, $\rho_{\mathcal{J}}$;
6. Piecewise-linear rowmotion on order polytopes, $\rho_{\mathcal{O}}$;
7. Birational Order Rowmotion (BOR-motion) on $\mathbb{K}$-labelings of P, BOR;
8. Noncommutative Order Rowmotion (NOR-motion) on $\mathbb{S}$-labelings of P, NOR;

THEMES in DAC:

1. Periodicity/order and orbit structure;
2. Homomesy: statistics with the same average over every orbit;
3. Equivariant bijections: often give nice proofs;
4. Lifting to higher realms enriches the subject and fosters connections.

Antichain Rowmotion on Posets

Rowmotion: an invertible operation on antichains

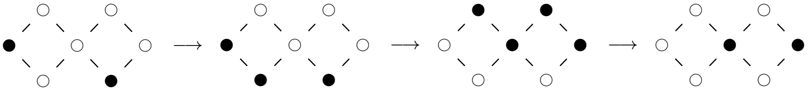
Let $\mathcal{A}(P)$ be the set of antichains of a finite poset P.

Given $A \in \mathcal{A}(P)$, let $\rho_{\mathcal{A}}(A)$ be the set of minimal elements of the complement of the *downward-saturation* of A (the smallest order ideal containing A).

$\rho_{\mathcal{A}}$ is invertible since it is a composition of three invertible operations:

$$\text{antichains} \longleftrightarrow \text{order ideals} \longleftrightarrow \text{order filters} \longleftrightarrow \text{antichains}$$

$\rho_{\mathcal{A}}$:



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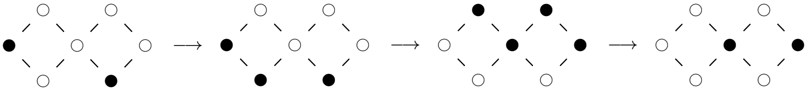


This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it **rowmotion**.

Panyushev's conjecture (AST's theorem)

Let Δ be a (reduced irreducible) root system in $\mathbf{R}^n$. (Pictures soon!)

Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.

Theorem (Armstrong–Stump–Thomas [AST11], Conj. [Pan09])

Let $\mathcal{O}$ be an arbitrary $\rho_{\mathcal{A}}$-orbit. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{n}{2}.$$

In our language: the cardinality statistic is homomesic with respect to the action of rowmotion on antichains in root posets.

Here are the classes of posets included in Panyushev's conjecture.

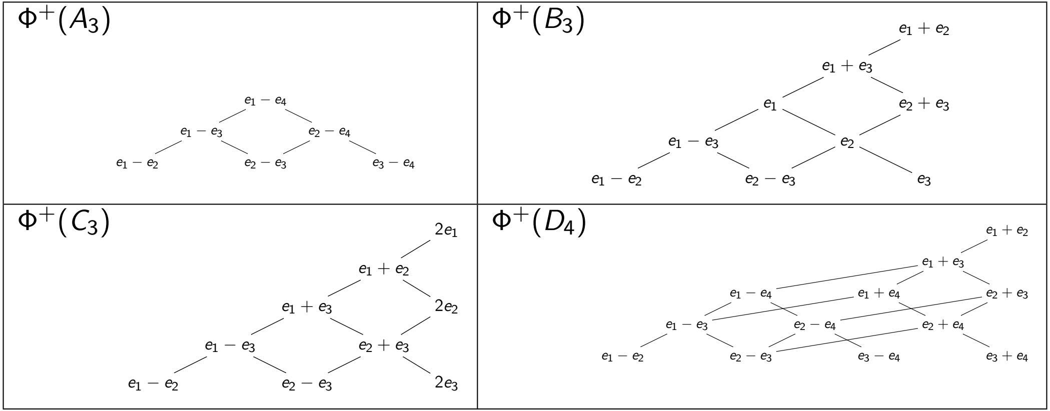
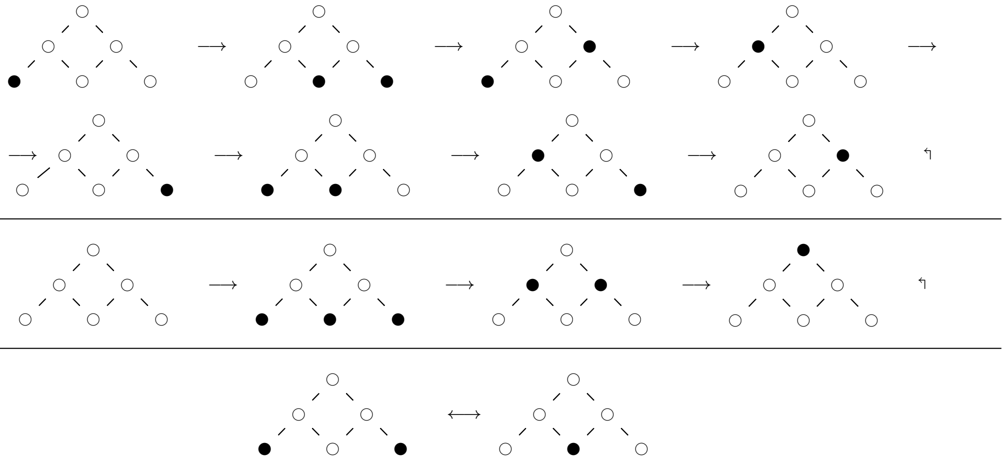


Figure: The positive root posets A_3, B_3, C_3, and D_4.

(Graphic courtesy of Striker-Williams.)

Example of antichain rowmotion on A_3 root poset

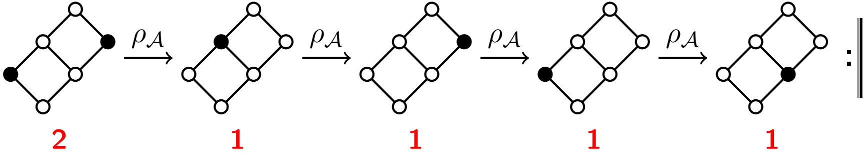
For the type A_3 root poset, there are 3 $\rho_{\mathcal{A}}$-orbits, of sizes 8, 4, 2:



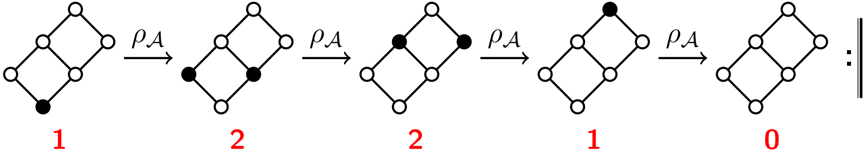
Checking the average cardinality for each orbit we find that

$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{0+3+2+1}{4} = \frac{2+1}{2} = \frac{3}{2}.$$

Orbits of rowmotion on antichains of $[2] \times [3]$



Average cardinality: 6/5



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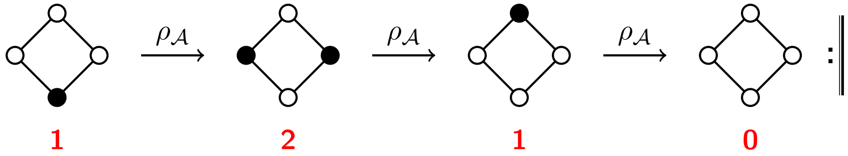

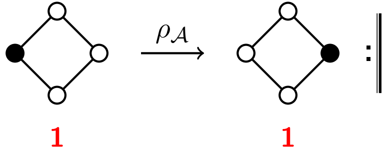

Average cardinality: 1

For antichain rowmotion on this poset, periodicity has been known for a long time:

Theorem (Brouwer–Schrijver 1974)

On $[a] \times [b]$, rowmotion is periodic with period $a + b$.

Theorem (Fon-Der-Flaass 1993)

On $[a] \times [b]$, every rowmotion orbit has length $(a + b)/d$, some d dividing both a and b.

Antichain rowmotion on $[a] \times [b]$: cardinality is homomesic

For rectangular posets $[a] \times [b]$ (the type A *minuscule* poset, where $[k] = \{1, 2, \ldots, k\}$), the cardinality homomesy is easier to show than for root posets.

Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary $\rho_{\mathcal{A}}$-orbit in $\mathcal{A}([a] \times [b])$. Then $\dfrac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \dfrac{ab}{a+b}$.

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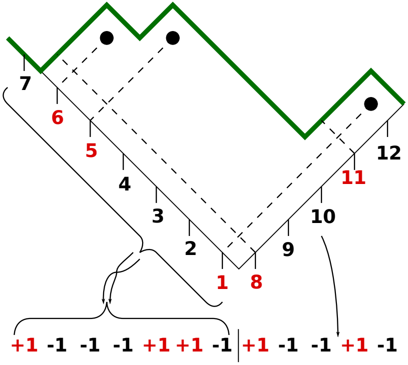

(Graphic courtesy of Ben Young.)

The simplest proof uses an non-obvious equivariant bijection (the "Stanley–Thomas" word [Sta09, §2]) between antichains in $[a] \times [b]$ and binary strings, which carries the $\rho_{\mathcal{A}}$ map to cyclic rotation of bitstrings.

The figure shows the Stanley–Thomas word for a 3-element antichain in $\mathcal{A}([7] \times [5])$. Red $\leftrightarrow +1$, while Black $\leftrightarrow -1$.

Antichain rowmotion on $[a] \times [b]$: cardinality is homomesic

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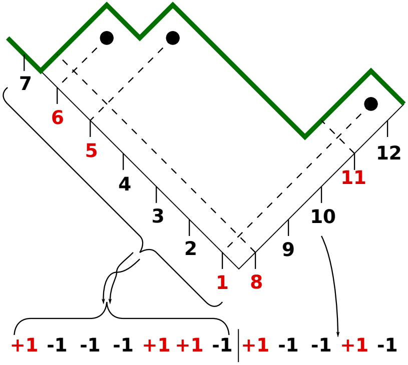

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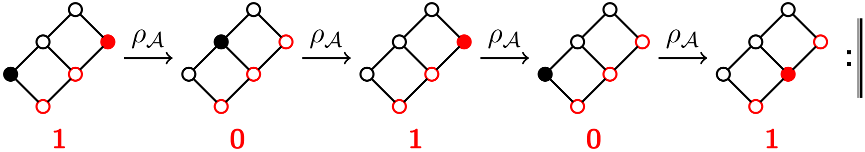
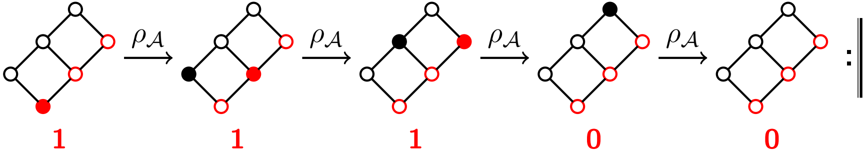
This bijection also allowed Propp–R. to derive refined homomesy results for fibers and antipodal points in $[a] \times [b]$.

Orbits of rowmotion on antichains of $[2] \times [3]$: Refined homomesies

Look at the cardinalities across a **positive fiber** such as the one highlighted in red.



Average: 3/5



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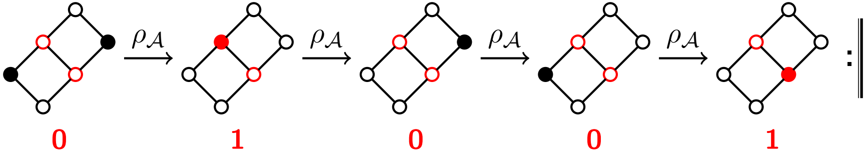
Orbits of rowmotion on antichains of $[2] \times [3]$: Refined homomesies

How about across a **negative fiber** such as the one highlighted in red.



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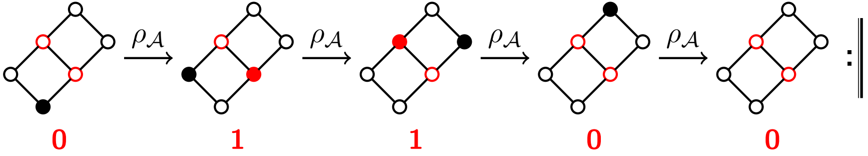


Average: 2/5

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i,j) \in [a] \times [b]$, and A an antichain in $[a] \times [b]$, let $\mathbb{1}_{i,j}(A)$ be 1 or 0 according to whether or not A contains (i,j).

Also, let $f_i(A) = \sum_{j \in [b]} \mathbb{1}_{i,j}(A) \in \{0,1\}$ (the cardinality of the intersection of A with the fiber $\{(i,1),(i,2),\ldots,(i,b)\}$ in $[a] \times [b]$), so that $\#A = \sum_i f_i(A)$.

Likewise let $g_j(A) = \sum_{i \in [a]} \mathbb{1}_{i,j}(A)$, so that $\#A = \sum_j g_j(A)$.

Theorem ([PrRo15])

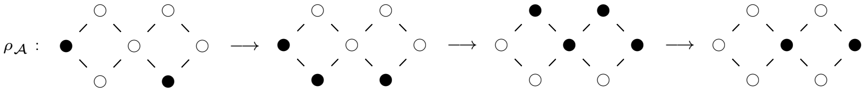
For all i,j,

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \qquad \textit{and} \qquad \frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$

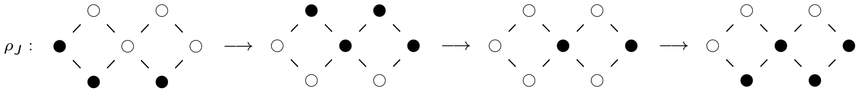
The indicator functions f_i and g_j are homomesic under $\rho_{\mathcal{A}}$, even though the indicator functions $\mathbb{1}_{i,j}$ aren't.

Rowmotion on order ideals and order filters

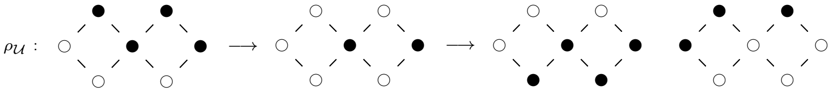
We've already seen examples of Rowmotion on antichains $\rho_{\mathcal{A}}$:



We can also define it as an operator ρ_J on $J(P)$, the set of order ideals (down-sets) of a poset P, by shifting the waltz beat by 1:



Or as an operator on the *order filters (up-sets)* $\mathcal{U}(P)$, of P:



Rowmotion via Toggling (Rowmotion in Slow motion)

Toggling order filters

Cameron and Fond-Der-Flaass showed how to write rowmotion on *order filters* (equivalently *order ideals*) as a product of simple involutions called *toggles*.

Definition (Cameron and Fon-Der-Flaass 1995)

Let $\mathcal{U}(P)$ be the set of order filters of a finite poset P.
Let $e \in P$. Then the **toggle** corresponding to e is the map $T_e : \mathcal{U}(P) \to \mathcal{U}(P)$ defined by

$$T_e(U) = \begin{cases} U \cup \{e\} & \text{if } e \notin U \text{ and } U \cup \{e\} \in \mathcal{U}(P), \\ U \setminus \{e\} & \text{if } e \in U \text{ and } U \setminus \{e\} \in \mathcal{U}(P), \\ U & \text{otherwise.} \end{cases}$$

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom along a linear extension of P gives rowmotion on order filters of P.

Toggling Order filters and order rowmotion

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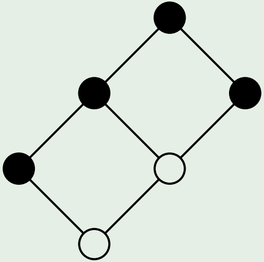
Example



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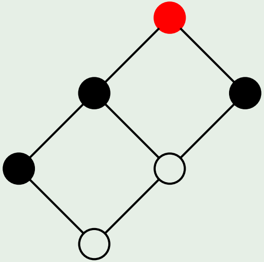

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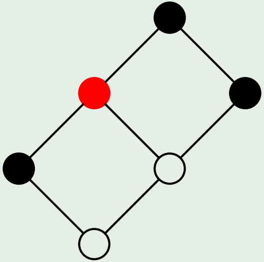

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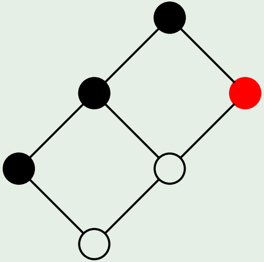

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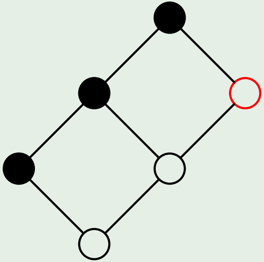

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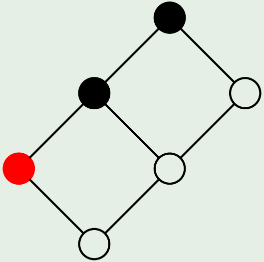

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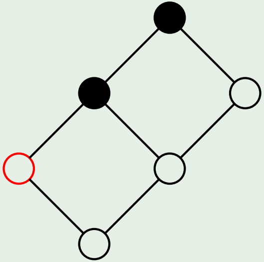

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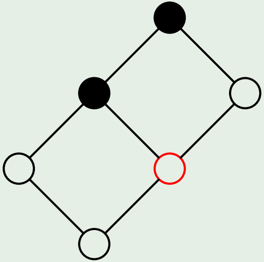

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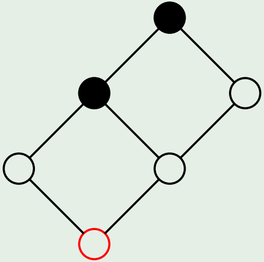

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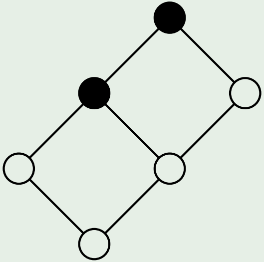
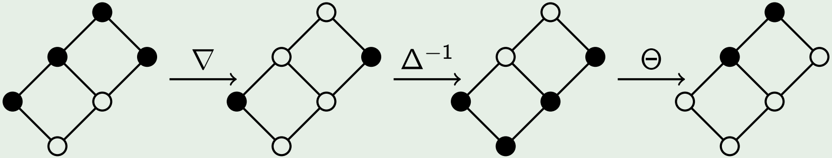


This step-by-step toggling process gives the same result as the three-step one mentioned earlier:

Start with an order filter, and

1. ∇: Take the minimal elements (giving an antichain)
2. Δ^{-1}: Saturate downward (giving a order ideal)
3. Θ: Take the complement (giving an order filter)

Example



Antichain toggling and rowmotion

Striker has generalized the notion of toggles relative to any class of "allowed" subsets, not necessarily order filters.

Definition

Let $e \in P$. Then the **antichain toggle** corresponding to e is the map $\tau_e : \mathcal{A}(P) \to \mathcal{A}(P)$ defined by

$$\tau_e(A) = \begin{cases} A \cup \{e\} & \text{if } e \notin A \text{ and } A \cup \{e\} \in \mathcal{A}(P), \\ A \setminus \{e\} & \text{if } e \in A, \\ A & \text{otherwise.} \end{cases}$$

Let $\mathrm{Tog}_{\mathcal{A}}(P)$ denote the **toggle group** of $\mathcal{A}(P)$ generated by the toggles $\{\tau_e \mid e \in P\}$.

Theorem (Joseph 2017)

Applying the antichain toggles τ_e from bottom to top along a linear extension of P gives $\rho_{\mathcal{A}}$, rowmotion on antichains of P.

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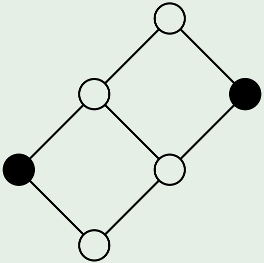

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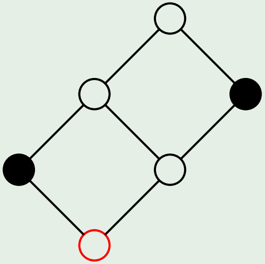

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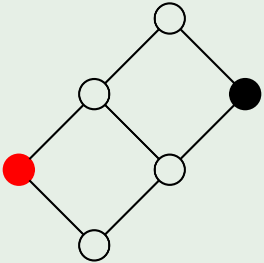

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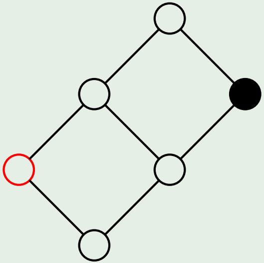

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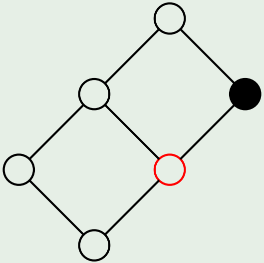

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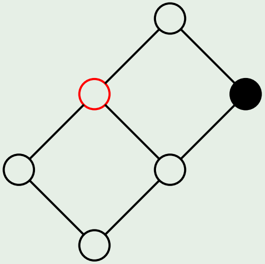

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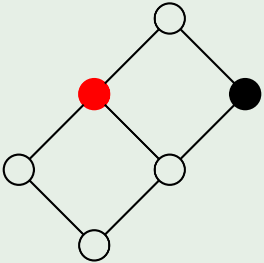

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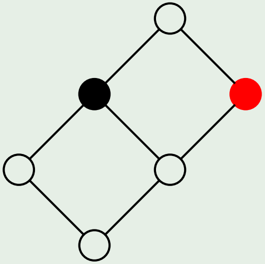
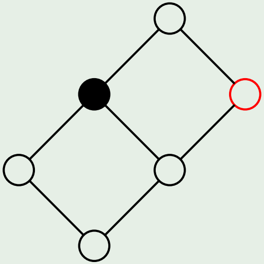


Antichain toggling and rowmotion

Theorem (Joseph 2017)

Applying the antichain toggles τ_e from bottom to top on P gives $\rho_{\mathcal{A}}$, rowmotion on antichains of P.

Example



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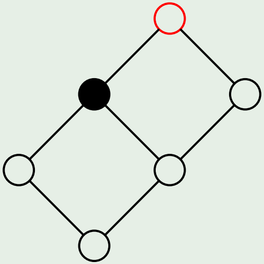

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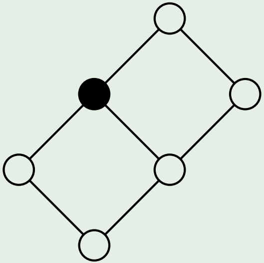
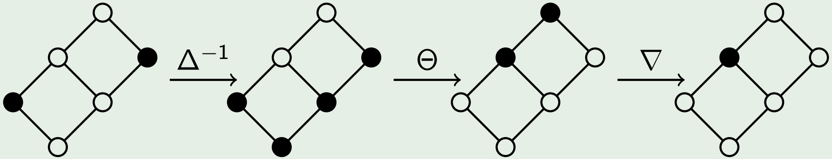


This gives the same result as the 3-step process

1. Δ^{-1}: Saturate downward (giving a order ideal)
2. Θ: Take the complement (giving an order filter)
3. ∇: Take the minimal elements (giving an antichain)

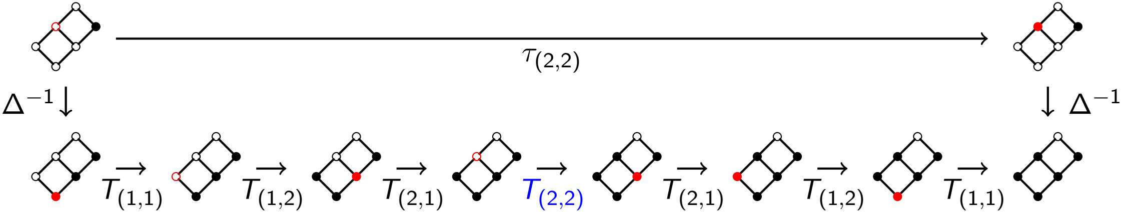
Example



Toggle Group Isomorphisms

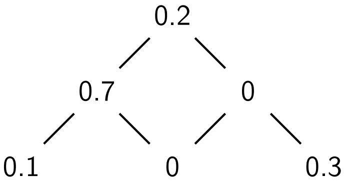
Let $\mathrm{Tog}_{\mathcal{J}}(P) := \langle T_v : v \in P \rangle$, the **order toggle group**.
Let $\mathrm{Tog}_{\mathcal{A}}(P) := \langle \tau_v : v \in P \rangle$, the **antichain toggle group**.
M. Joseph constructed an explicit isomorphism between these: Set $\eta_e := T_{x_1} T_{x_2} \cdots T_{x_k}$, where $(x_1, x_2, \ldots, x_k)$ is a linear extension of the subposet $\{x \in P : x < e\}$. Then $\tau_e^* := \eta_e T_e \eta_e^{-1}$ mimics the action of τ_e.

$$\begin{array}{ccc} \mathcal{A}(P) & \xrightarrow{\tau_e} & \mathcal{A}(P) \\ \Delta^{-1}\downarrow & & \downarrow\Delta^{-1} \\ \mathcal{J}(P) & \xrightarrow[\tau_e^*]{} & \mathcal{J}(P) \end{array}$$

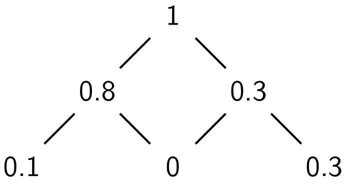


Stanley defined some polytopes associated with posets [Sta86].

- $\mathcal{C}(P)$ is the **chain polytope** of P, the set of $f \in [0,1]^P$ such that $\sum_{i=1}^{n} f(x_i) \leq 1$ for all chains $x_1 < x_2 < \cdots < x_n$.
- $\mathcal{O}(P)$ is the **order polytope** of P, the set of all order-preserving labelings $f \in [0,1]^P$. Saying f is order-preserving means $f(x) \leq f(y)$ when $x \leq y$ in P.



$\in \mathcal{C}(P)$;



$\in \mathcal{O}(P)$

- In particular, $\{0,1\}$-labelings in $\mathcal{C}(P) \longleftrightarrow \mathcal{A}(P)$ (the vertices of $\mathcal{C}(P)$), and $\{0,1\}$-labelings in $\mathcal{O}(P) \longleftrightarrow \mathcal{U}(P)$ (the vertices of $\mathcal{O}(P)$).

Generalizing toggling to the piecewise-linear realm

Definition (Einstein–Propp)

Set $\widehat{P} := P \cup \{\widehat{0}, \widehat{1}\}$. The **piecewise-linear order toggle** $T_v : \mathcal{O}(P) \to \mathcal{O}(P)$ is

$$(T_v(f))(x) = \begin{cases} f(x) & \text{if } x \neq v \\ \max\limits_{y \lessdot v} f(y) + \min\limits_{y \gtrdot v} f(y) - f(v) & \text{if } x = v \end{cases} \quad \text{with } f(\widehat{0}) = 0 \text{ and } f(\widehat{1}) = 1.$$

"Midpoint reflection of $f(v)$ in allowable interval $\left[\max\limits_{y \lessdot v} f(y), \min\limits_{y \gtrdot v} f(y)\right]$."

Definition (M. Joseph)

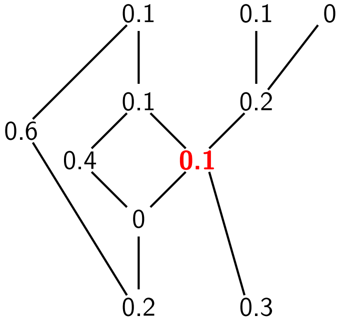
For $v \in P$, let $\mathrm{MC}_v(P)$ denote the set of all maximal chains of P through v. The **piecewise-linear antichain toggle** (or **chain polytope toggle**) $\tau_v : \mathcal{C}(P) \to \mathcal{C}(P)$ is

$$(\tau_v(g))(x) = \begin{cases} 1 - \max\left\{ \sum\limits_{i=1}^{k} g(y_i) \,\middle|\, (y_1, \ldots, y_k) \in \mathrm{MC}_v(P) \right\} & \text{if } x = v \\ g(x) & \text{if } x \neq v \end{cases}.$$

Toggles on the chain polytope $\mathcal{C}(P)$

As usual, To define $\tau_e : \mathcal{C}(P) \to \mathcal{C}(P)$, given $g \in \mathcal{C}(P)$ and $e \in P$, $\tau_e(g)$ can only differ from g at the value of e.

$$\big(\tau_e(g)\big)(e) = 1 - \max\left\{ \sum_{i=1}^{k} g(y_i) \;\middle|\; \begin{array}{c} (y_1, \ldots, y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } e \end{array} \right\}$$



Toggles on the chain polytope $\mathcal{C}(P)$

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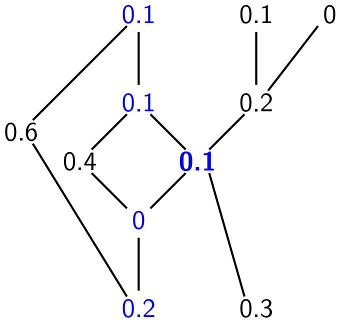

$$0.2 + 0 + 0.1 + 0.1 + 0.1 = 0.5$$

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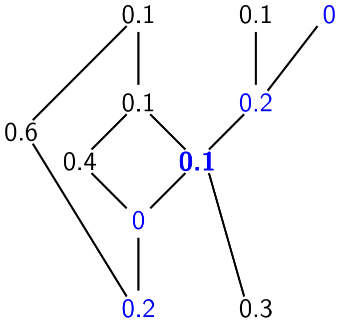

$$0.2 + 0 + 0.1 + 0.2 + 0 = 0.5$$

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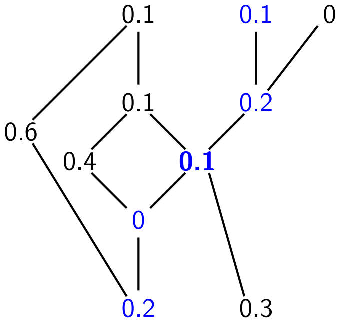

$$0.2 + 0 + 0.1 + 0.2 + 0.1 = 0.6$$

Toggles on the chain polytope $\mathcal{C}(P)$

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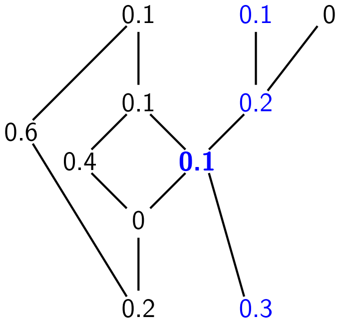

$$0.3 + 0.1 + 0.2 + 0.1 = 0.7$$

Toggles on the chain polytope $\mathcal{C}(P)$

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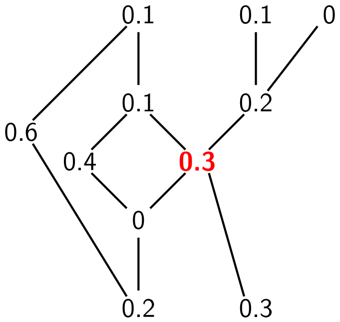
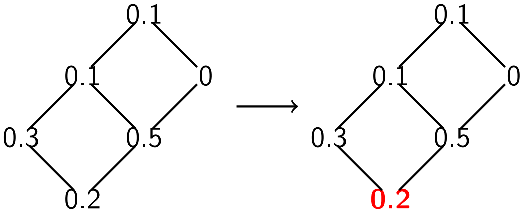


0.7 is max and $1 - 0.7 = 0.3$

Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



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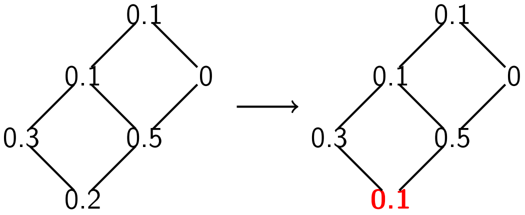

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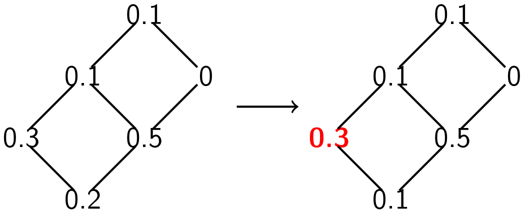

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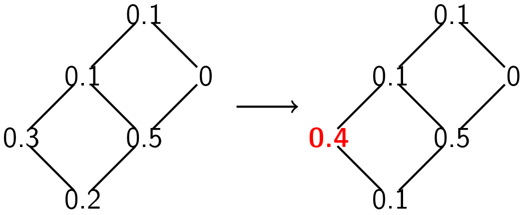

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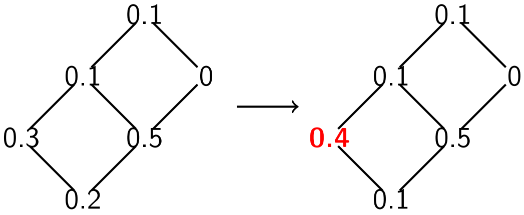

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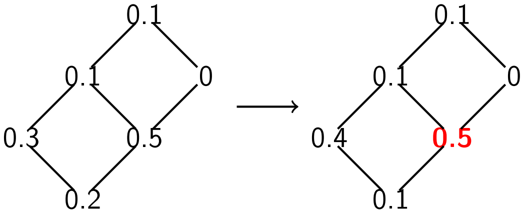

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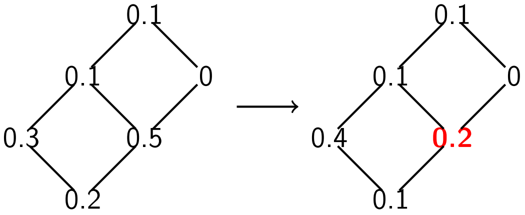

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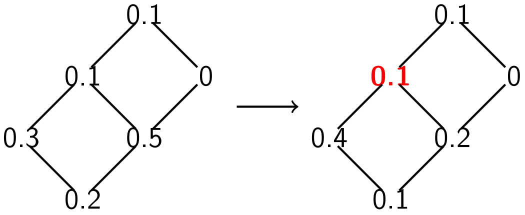

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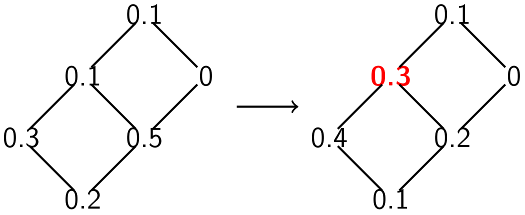

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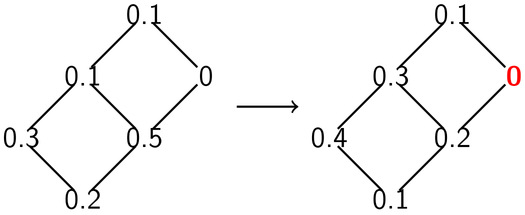

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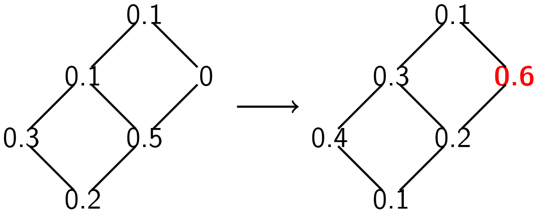

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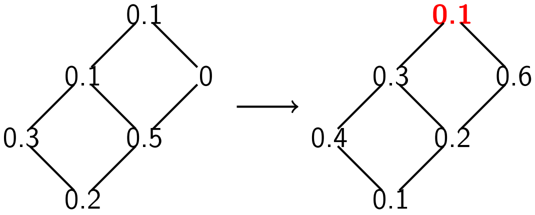

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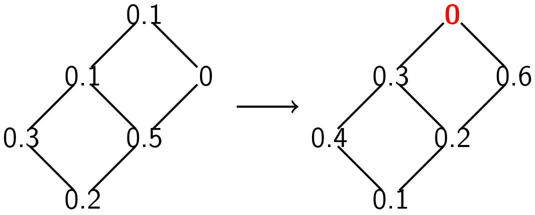

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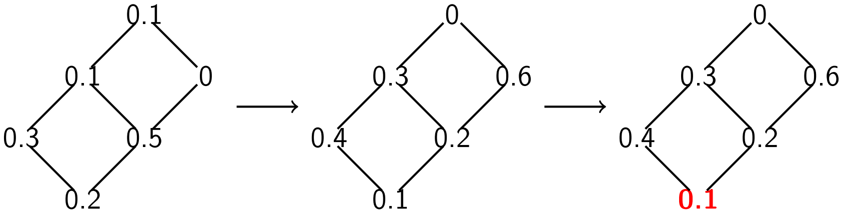

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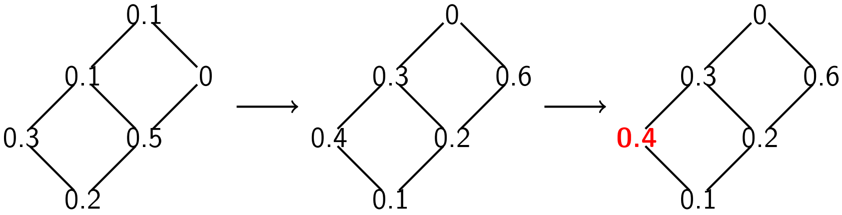

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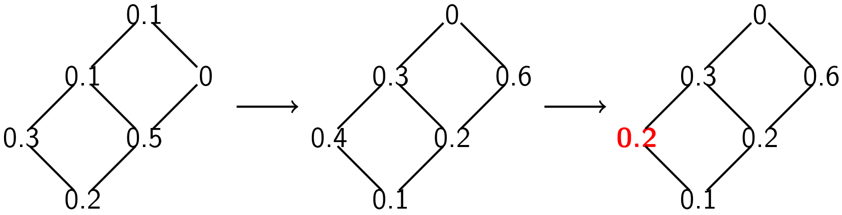

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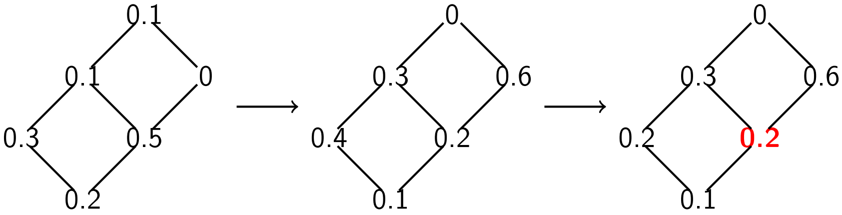

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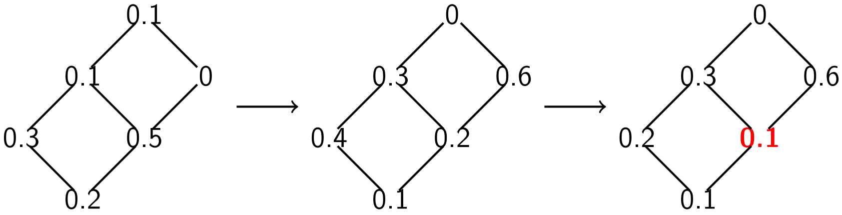

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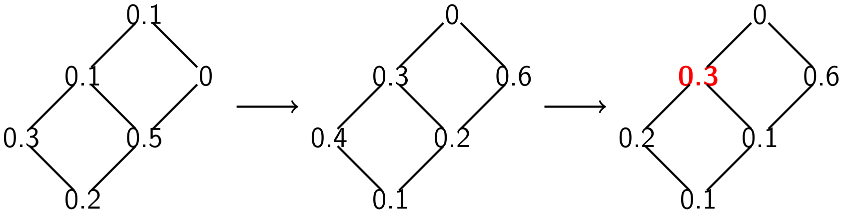

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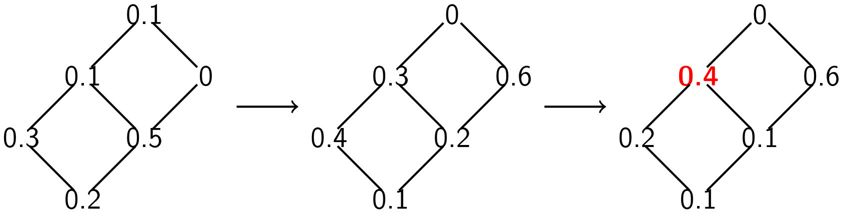

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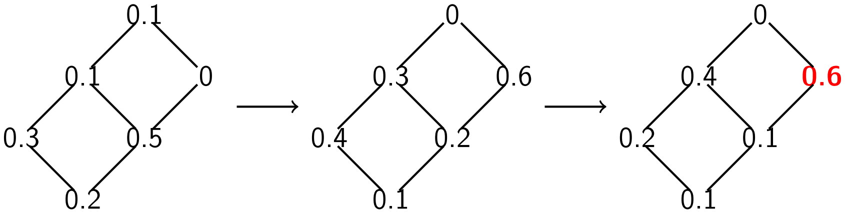

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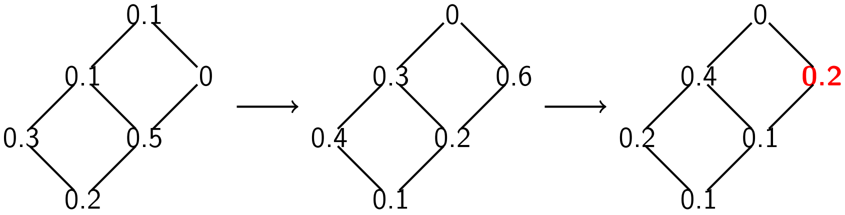

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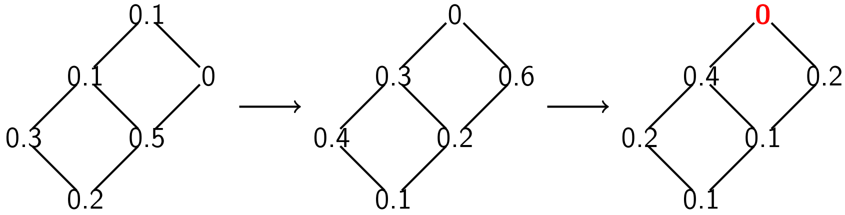

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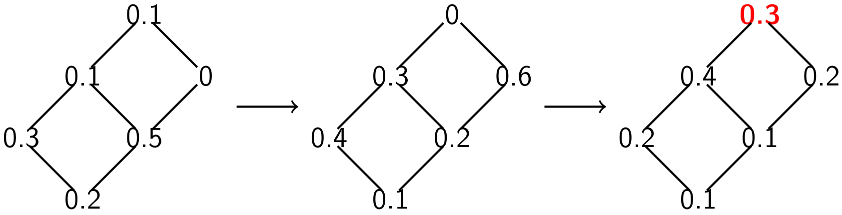

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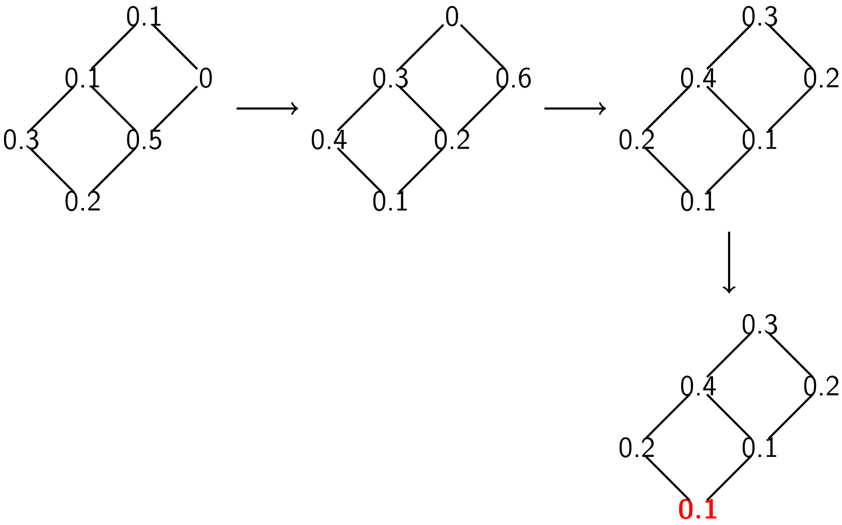

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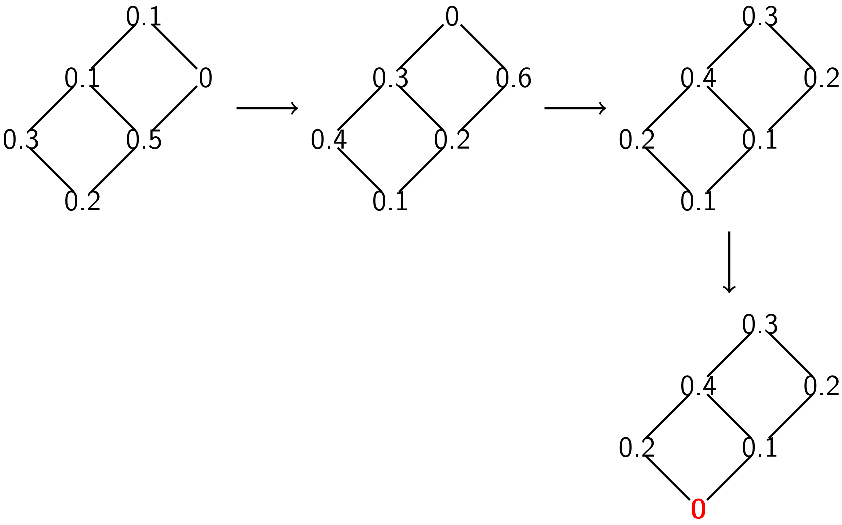

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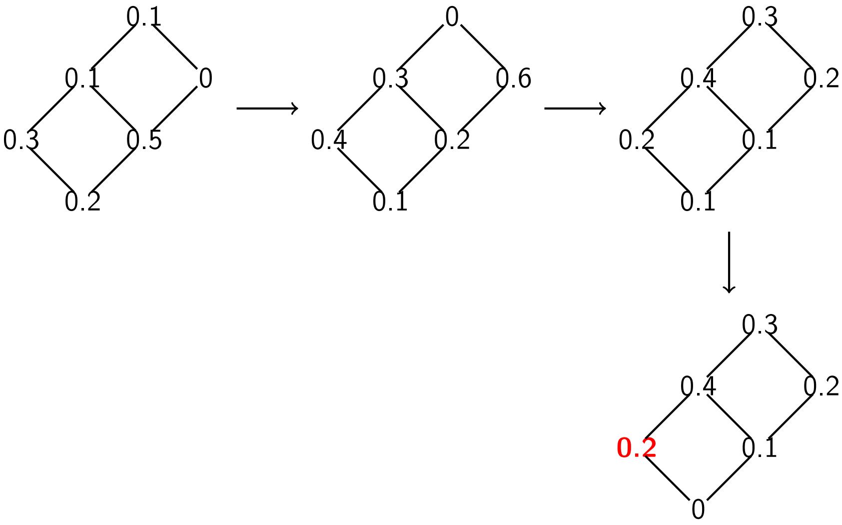

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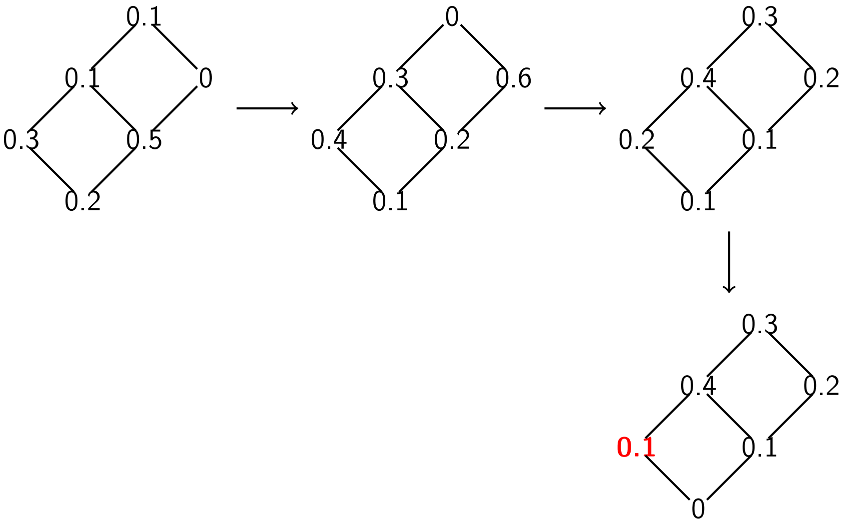

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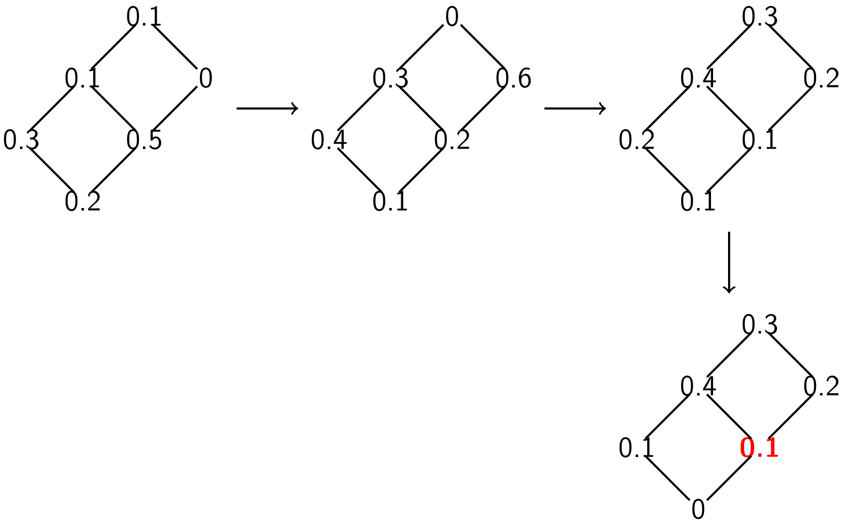

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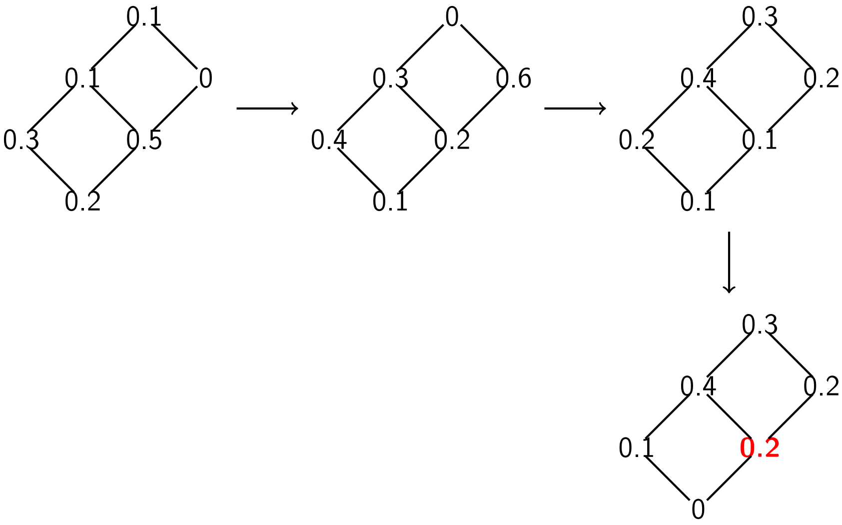

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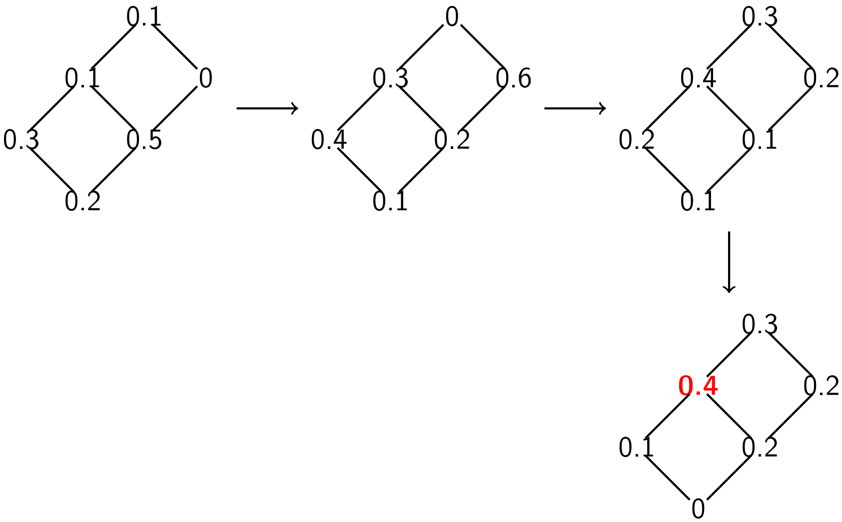
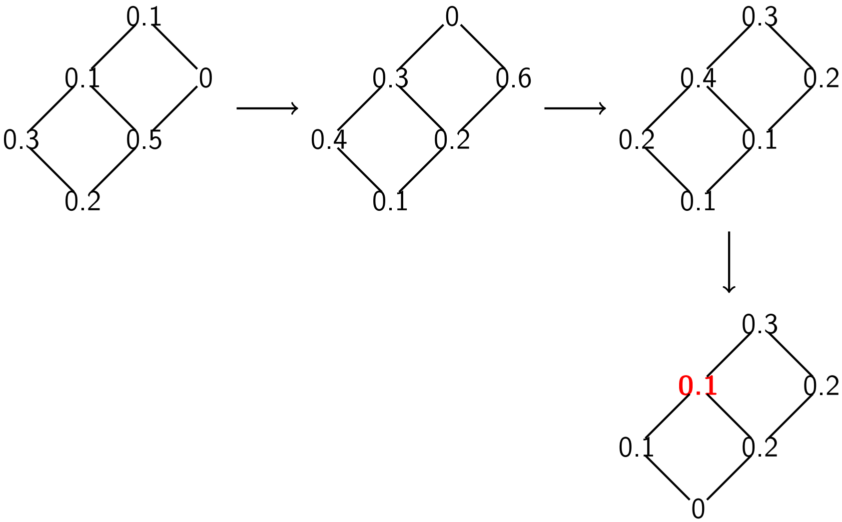
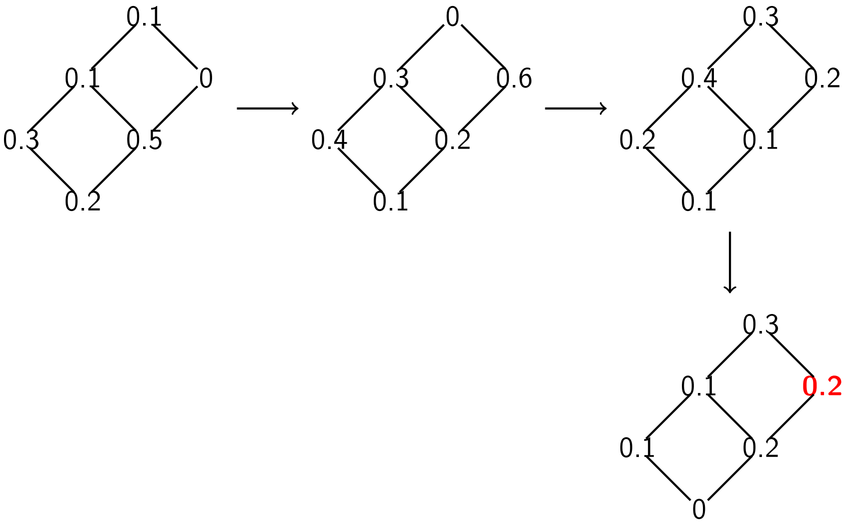


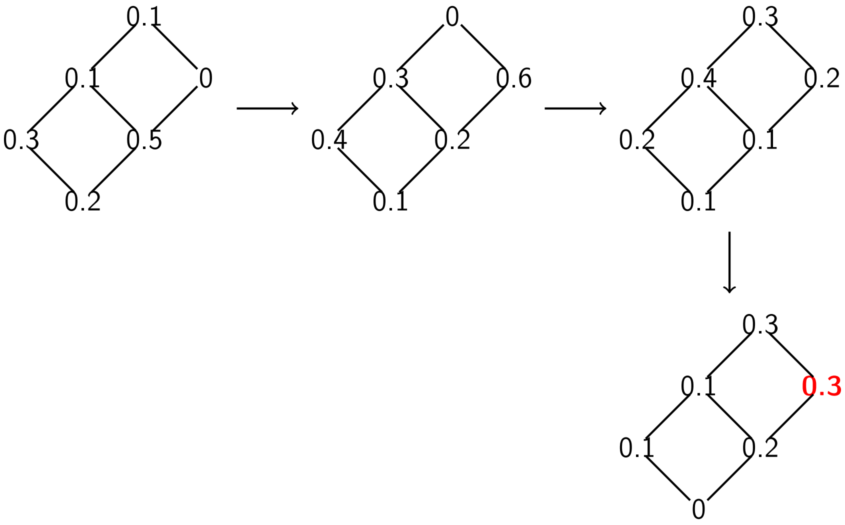
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



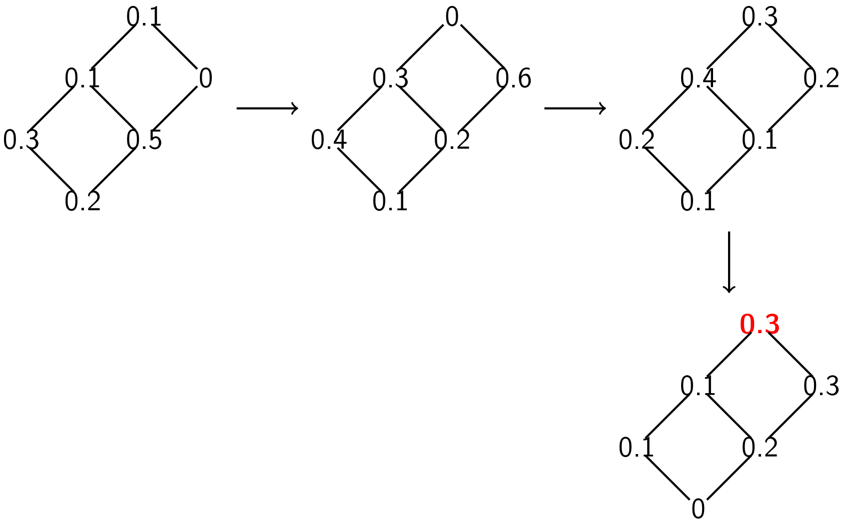
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



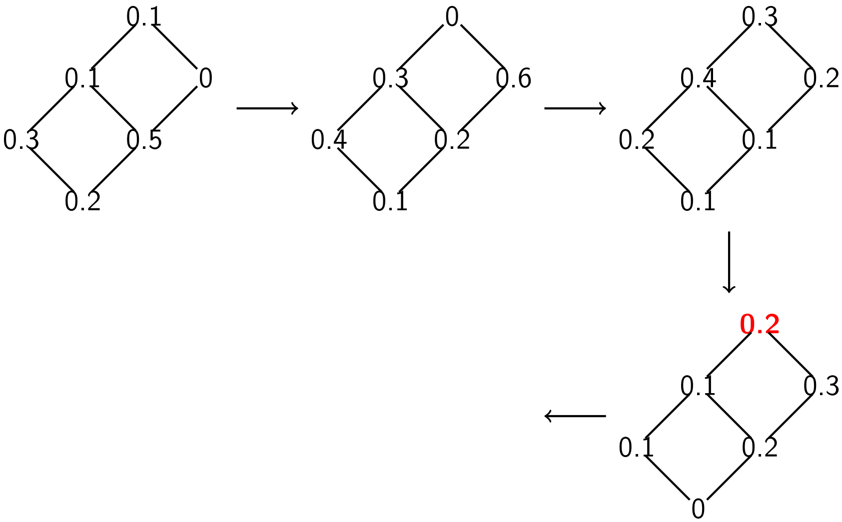
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



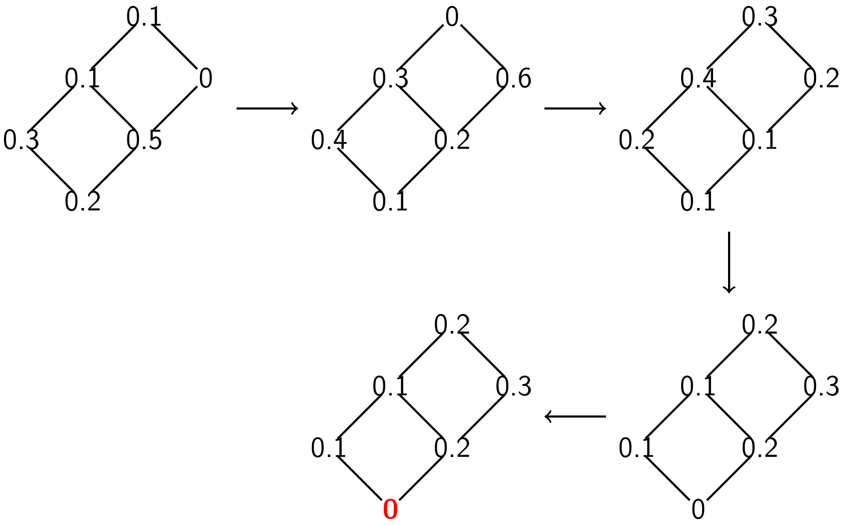
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



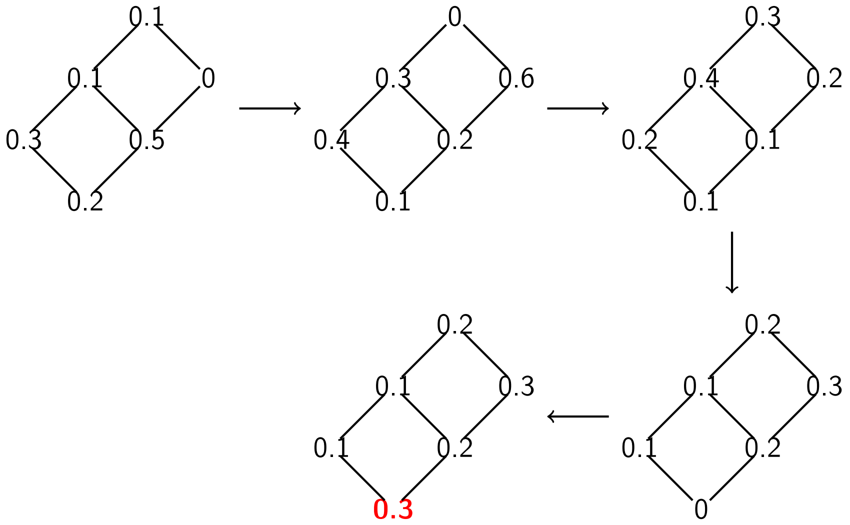
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



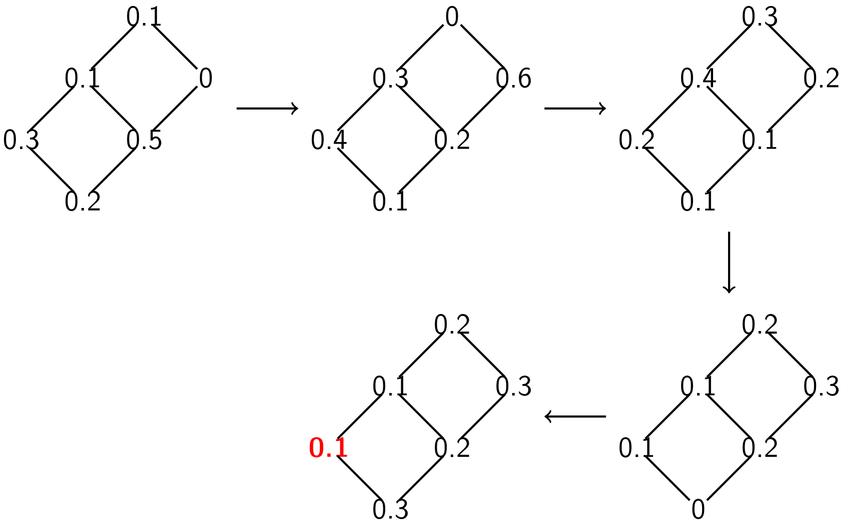
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



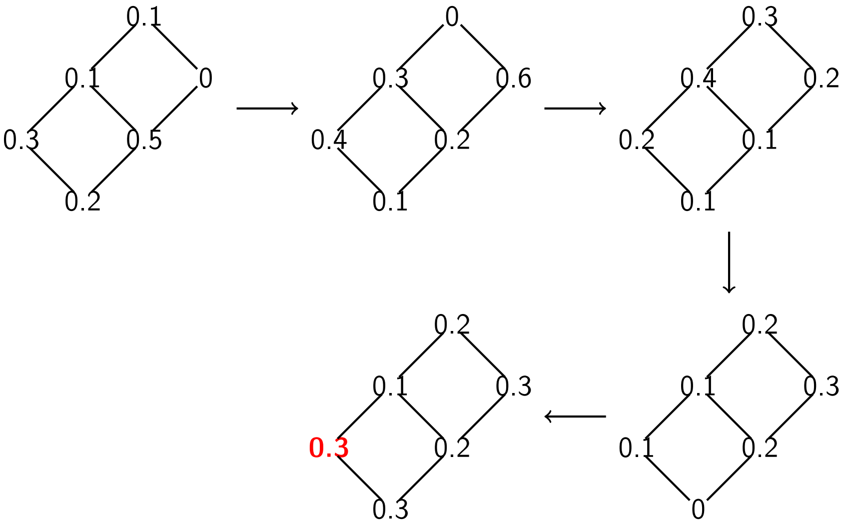
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



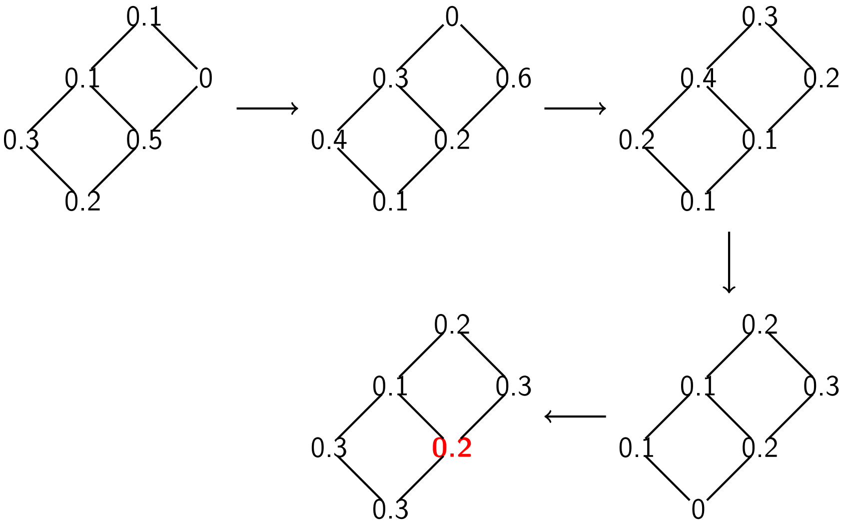
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



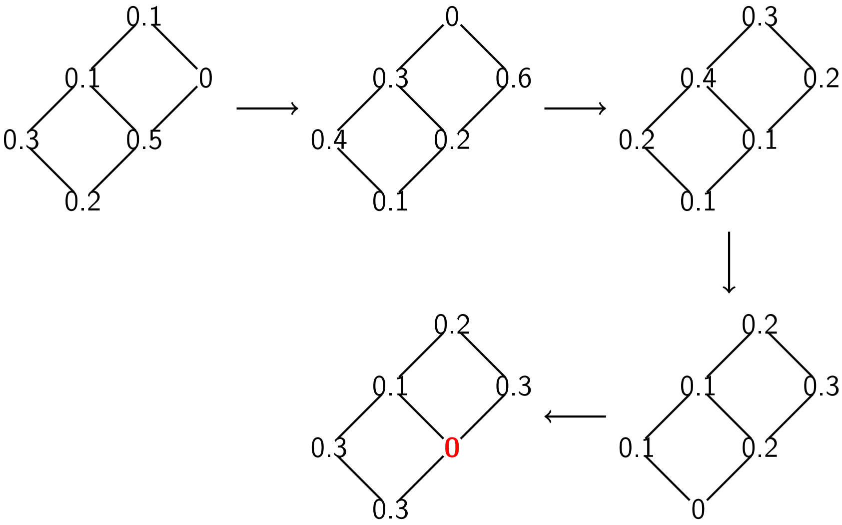
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



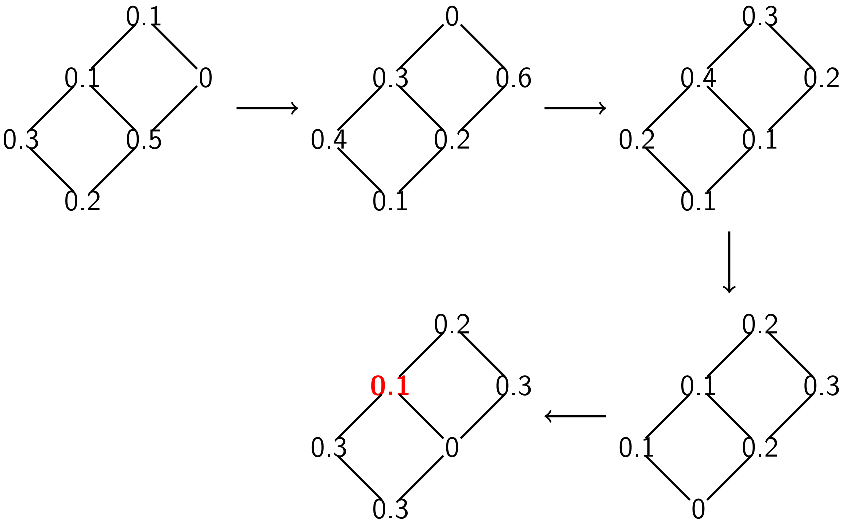
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



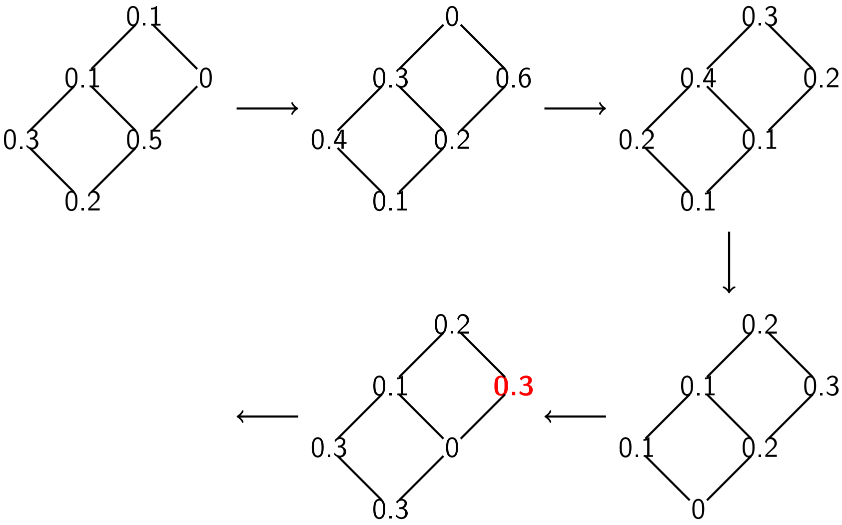
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



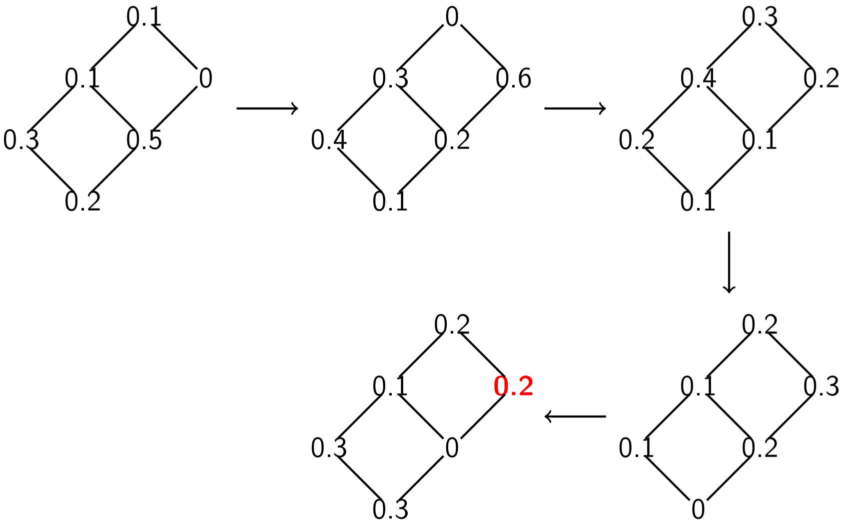
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



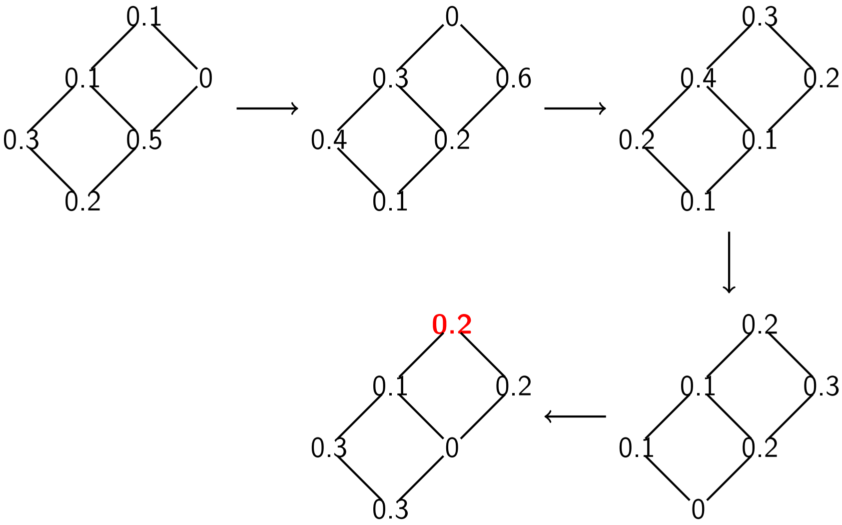
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



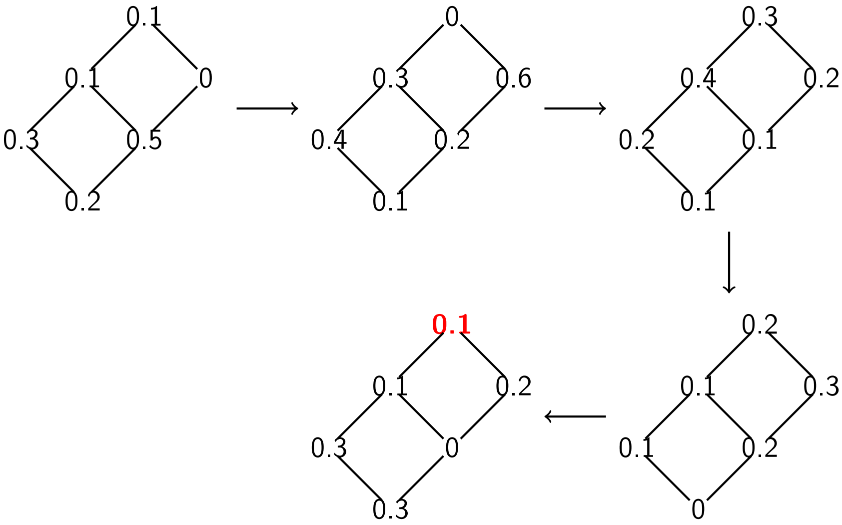
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



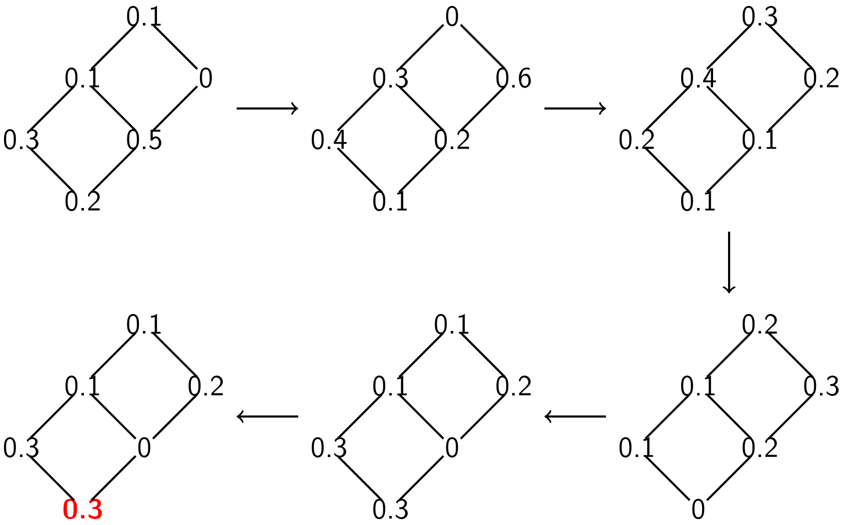
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



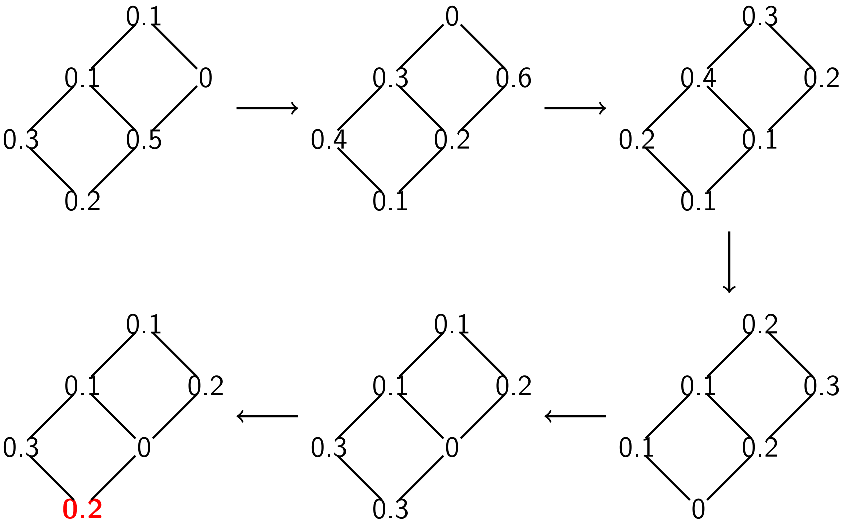
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



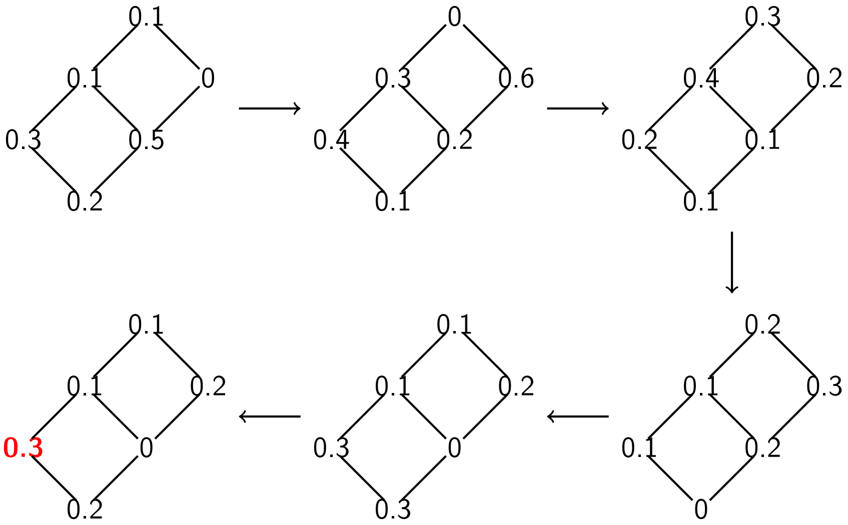
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



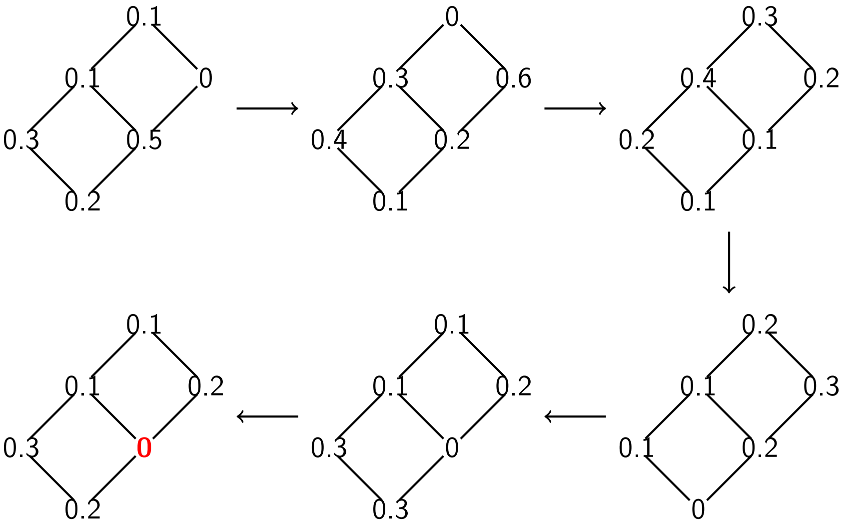
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



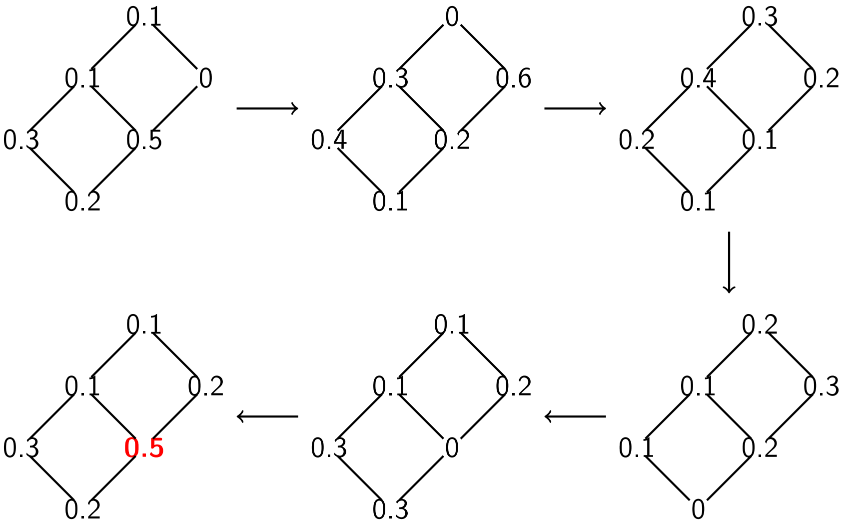
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



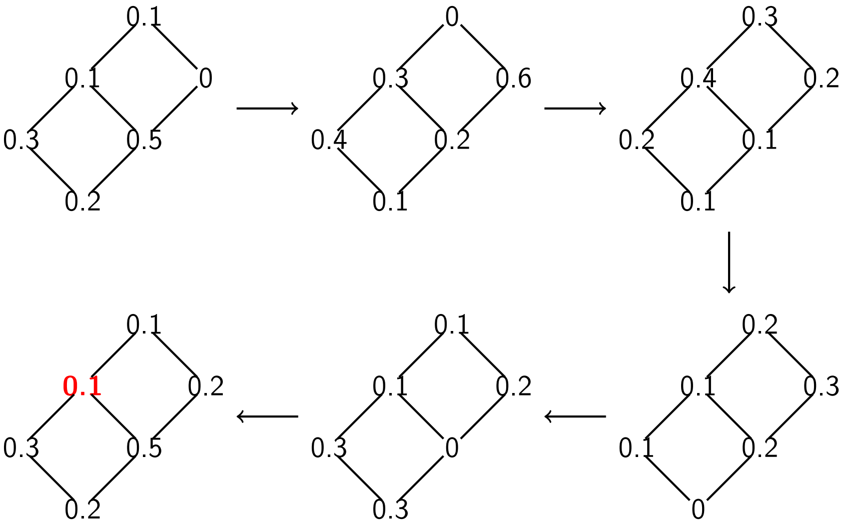
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



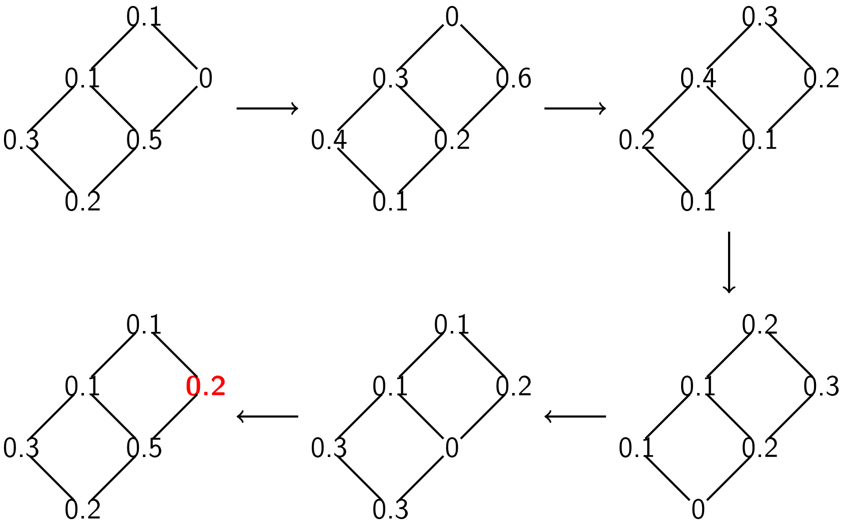
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



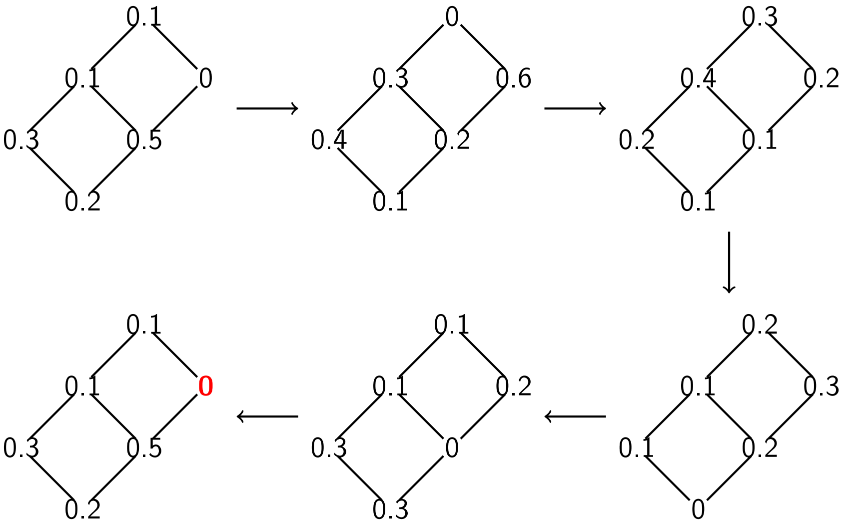
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



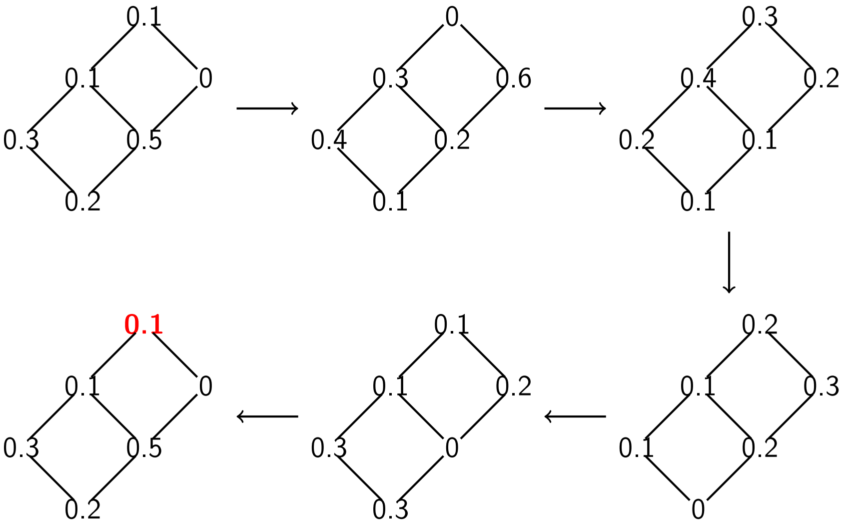
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



Detropicalizing from the piecewise-linear realm to the birational realm

Einstein and Propp showed how to lift of order-ideal toggling and rowmotion on $\mathcal{O}(P)$ to the birational realm [EiPr13+]. To do this, we replace max with + and + with multiplication. Under this dictionary

$$(\tau_v(g))(v) = 1 - \max\left\{ \sum_{i=1}^{k} g(y_i) \;\middle|\; \begin{array}{c} (y_1, \ldots, y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } v \end{array} \right\}$$

becomes

$$(\tau_v(g))(v) = \frac{C}{\sum \left\{ \prod_{i=1}^{k} g(y_i) \;\middle|\; \begin{array}{c} (y_1, \ldots, y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } v \end{array} \right\}}$$

whereas

$$(T_v(g))(v) = \max_{y \lessdot v} f(y) + \min_{y \gtrdot v} f(y) - f(v)$$

becomes

$$\frac{\sum\limits_{y \in \widehat{P}, y \lessdot v} f(y)}{f(v) \sum\limits_{y \in \widehat{P}, y \gtrdot v} \frac{1}{f(y)}}$$

Birational Antichain Rowmotion (BAR-motion)

Now we'll define the **birational antichain toggle** corresponding to $e \in P$.

Definition

For $e \in P$, and field $\mathbb{K}$, let $\tau_e : \mathbb{K}^P \to \mathbb{K}^P$ be defined as the birational map that only changes the value at e in the following way.

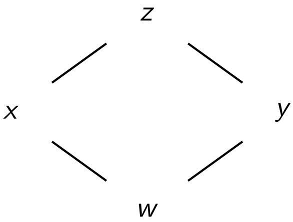
$$(\tau_e(g))(e) = \frac{C}{\sum \left\{ \prod_{i=1}^{k} g(y_i) \,\middle|\, \begin{array}{c} (y_1, \ldots, y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } e \end{array} \right\}}$$

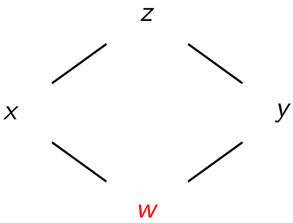
Definition

BAR-motion (birational antichain rowmotion) is the birational map obtained by applying the birational antichain toggles from the bottom to the top.

BAR-motion on $[2] \times [2]$

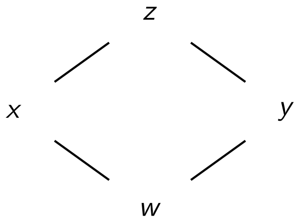
$g =$

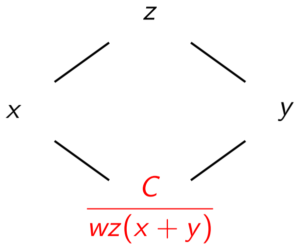




BAR-motion on $[2] \times [2]$

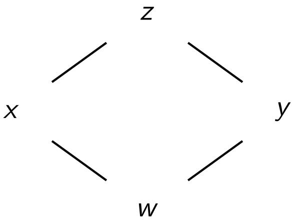
$g =$

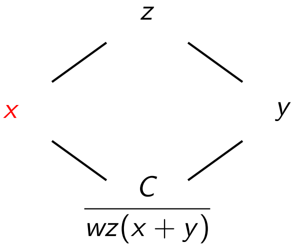




BAR-motion on $[2] \times [2]$

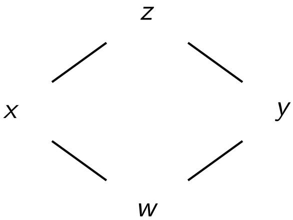
$g =$

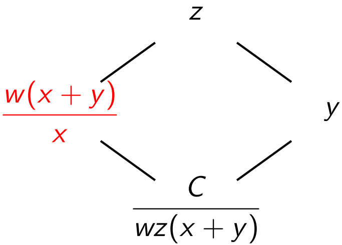




BAR-motion on $[2] \times [2]$

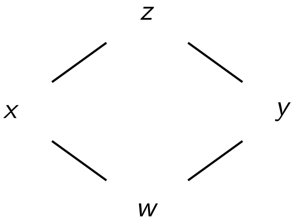
$g =$

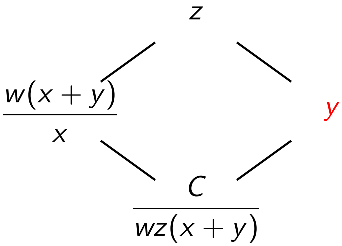




BAR-motion on $[2] \times [2]$

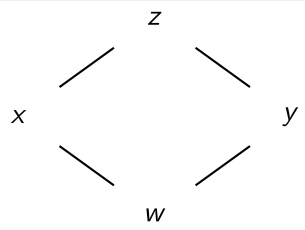
$g =$

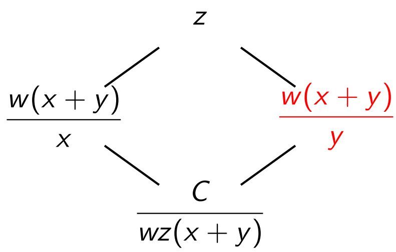




BAR-motion on $[2] \times [2]$

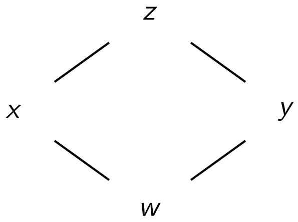
$g =$

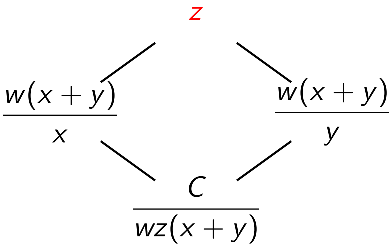




BAR-motion on $[2] \times [2]$

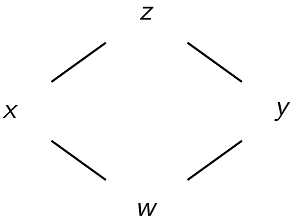
$g =$



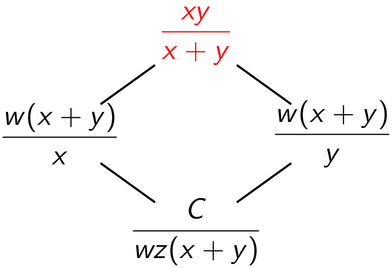


BAR-motion on $[2] \times [2]$

$g =$

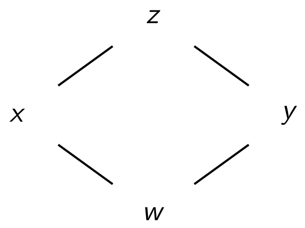


$\mathrm{BAR}(g) =$

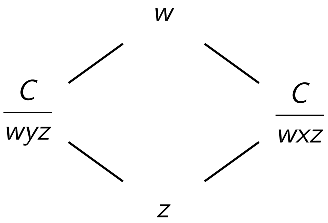


BAR-motion on $[2] \times [2]$

$g =$

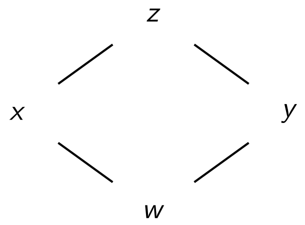


$\mathrm{BAR}^2(g) =$

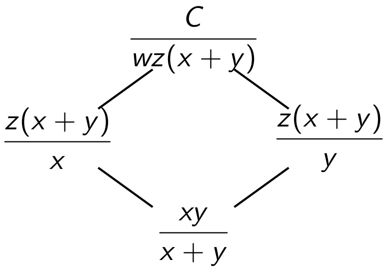


BAR-motion on $[2] \times [2]$

$g =$

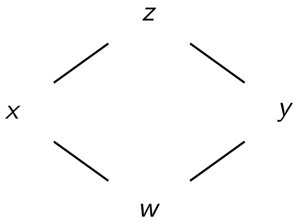


$\mathrm{BAR}^3(g) =$

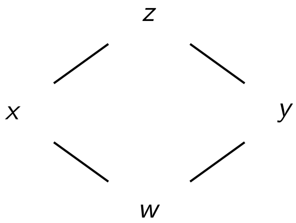


BAR-motion on $[2] \times [2]$

$g =$



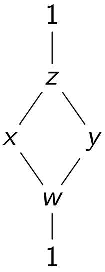
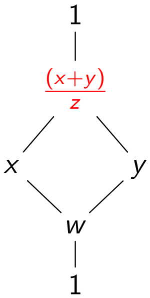
$\mathrm{BAR}^4(g) =$



BOR-motion on $[2] \times [2]$

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

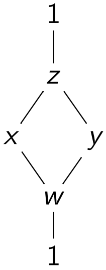
original labelling f	labelling $T_{(1,1)} f$
	

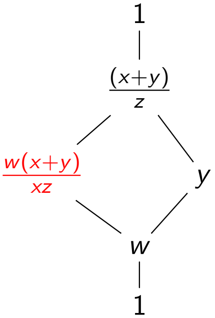
We are using $\mathrm{BOR} = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(1,0)} T_{(1,1)} f$



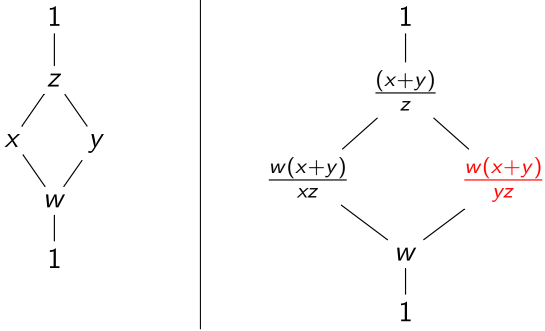


We are using $\mathrm{BOR} = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

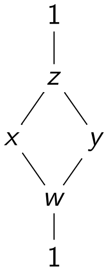
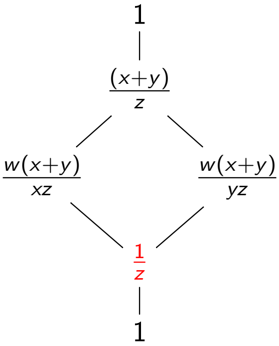
original labelling f	labelling $T_{(0,1)}\, T_{(1,0)}\, T_{(1,1)} f$



We are using $\mathrm{BOR} = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,0)} T_{(0,1)} T_{(1,0)} T_{(1,1)} f = \mathrm{BAR}\, f$
	

We are using $\mathrm{BOR} = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

BOR-motion orbit on a product of chains

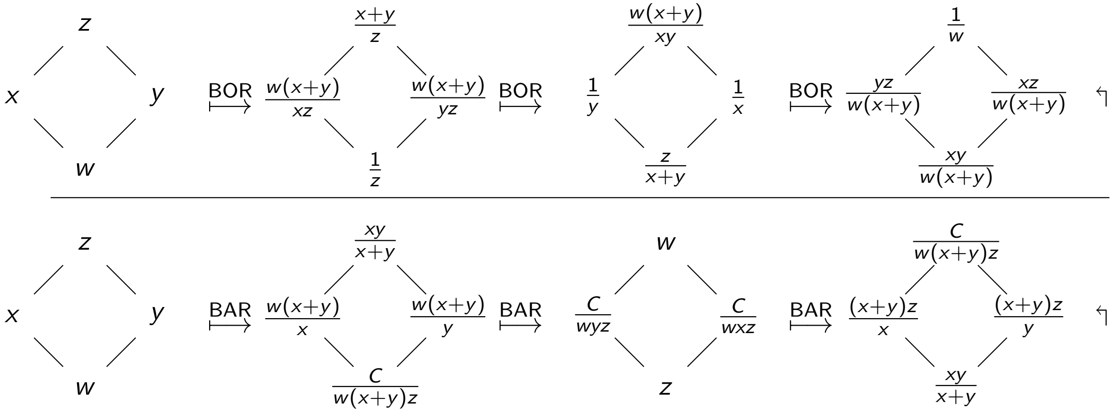
Example: Iterating this procedure we get

$$\mathrm{BOR}\, f = \begin{array}{ccc} & \frac{(x+y)}{z} & \\ \diagup & & \diagdown \\ \frac{(x+y)w}{xz} & & \frac{(x+y)w}{yz} \\ \diagdown & & \diagup \\ & \frac{1}{z} & \end{array} , \qquad \mathrm{BOR}^2 f = \begin{array}{ccc} & \frac{(x+y)w}{xy} & \\ \diagup & & \diagdown \\ \frac{1}{y} & & \frac{1}{x} \\ \diagdown & & \diagup \\ & \frac{z}{x+y} & \end{array} ,$$

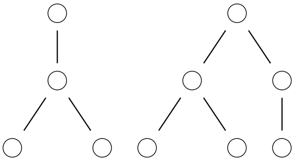
$$\mathrm{BOR}^3 f = \begin{array}{ccc} & \frac{1}{w} & \\ \diagup & & \diagdown \\ \frac{yz}{(x+y)w} & & \frac{xz}{(x+y)w} \\ \diagdown & & \diagup \\ & \frac{xy}{(x+y)w} & \end{array} , \qquad \mathrm{BOR}^4 f = \begin{array}{ccc} & z & \\ \diagup & & \diagdown \\ x & & y \\ \diagdown & & \diagup \\ & w & \end{array} .$$

Orbits for BOR-motion and BAR-motion on $[2] \times [2]$

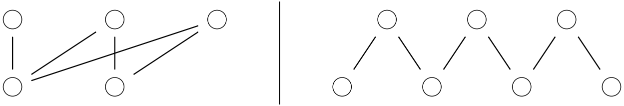
Here are the full orbits of BOR and BAR on a generic labeling for $P = [2] \times [2]$:



- The order of BOR on $[a] \times [b]$ is $a + b$ [GrRo15, Thm. 30]
- The order of BOR on "graded rooted forests" with all leaves on level n (indexed from 1) is finite and satisfies $\mathrm{ord}(\mathrm{BOR}) = \mathrm{ord}(\rho_{\mathcal{J}}) \mid \mathrm{LCM}(1, 2, \ldots, n+1)$ [GrRo16].
  Example: For P as shown, $\mathrm{ord}(\mathrm{BOR}) = \mathrm{ord}(\rho_{\mathcal{J}}) \mid \mathrm{LCM}(1, 2, 3, 4) = 12$.

$P =$ 

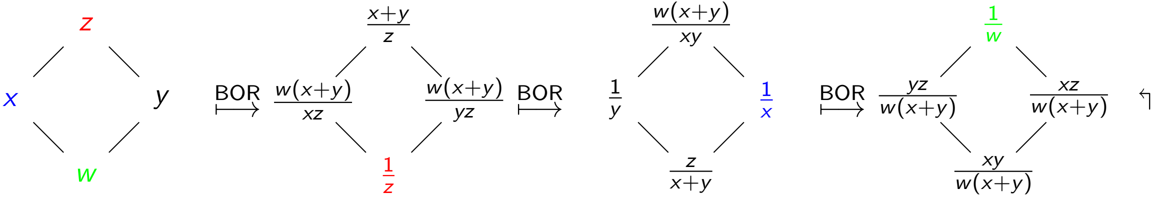
- **NB:** Most posets have $\mathrm{ord}(\mathrm{BOR}) = \infty$, e.g., the Boolean lattices B_3 OR the two below:



Antipodal Homomesy for BOR-motion on rectangular posets

- **Antipodal reciprocity:** [GrRo15, Thm. 32] Antipodal points in $P = [a] \times [b]$ satisfy:

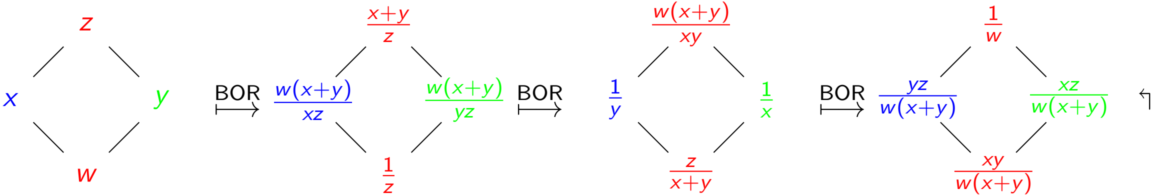
$$f\left(a+1-i, b+1-k\right) = \frac{1}{\left(\mathrm{BOR}^{i+k-1} f\right)(i,k)}.$$



File Homomesy for BOR-motion

Musiker–R gave a formula for iterates of birational rowmotion in terms of ratios of families of non-intersecting lattice paths (NILPs). This allowed them to reprove the periodicity and antipodal homomesy results, as well as the following refined homomesy, which lifts a known one for $\rho_{\mathcal{J}}$ [MuRo19].

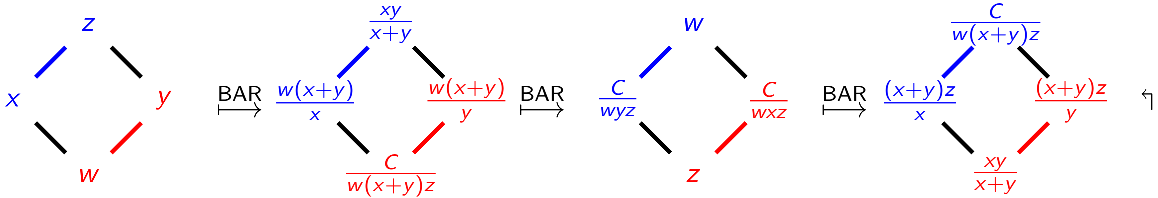
Given a file F in $[a] \times [b]$, $\displaystyle\prod_{k=1}^{a+b} \prod_{(i,j)\in F} \left(\mathrm{BOR}^k\, f\right)(i,j) = 1$. i.e., the statistic $\prod_{(i,j)\in F} \widetilde{\mathbb{1}}_{(i,j)}$ is birationally homomesic under BOR.



Properties of BAR-motion

- The order of BAR on $[a] \times [b]$ is $a + b$. This follows from [G–R] via our equivariant toggle-group isomorphisms.
- The homomesy results for antichain cardinality in the combinatorial $\rho_{\mathcal{A}}$ setting lift to this setting. Because...
- We can lift the *Stanley–Thomas* word to this setting as an equivariant *surjection*, cyclically rotating with *BAR*. It proves homomesy, but not periodicity [JoRo20+].

Here is the full orbit of BAR on a generic labeling for $P = [2] \times [2]$, with ST-words.



$\left(wy, xz, \frac{C}{wx}, \frac{C}{yz}\right)$ $\left(\frac{C}{yz}, wy, xz, \frac{C}{wx}\right)$ $\left(\frac{C}{wx}, \frac{C}{yz}, wy, xz\right)$ $\left(xz, \frac{C}{wx}, \frac{C}{yz}, wy\right)$

Lifting to NC toggles and NC Order rowmotion

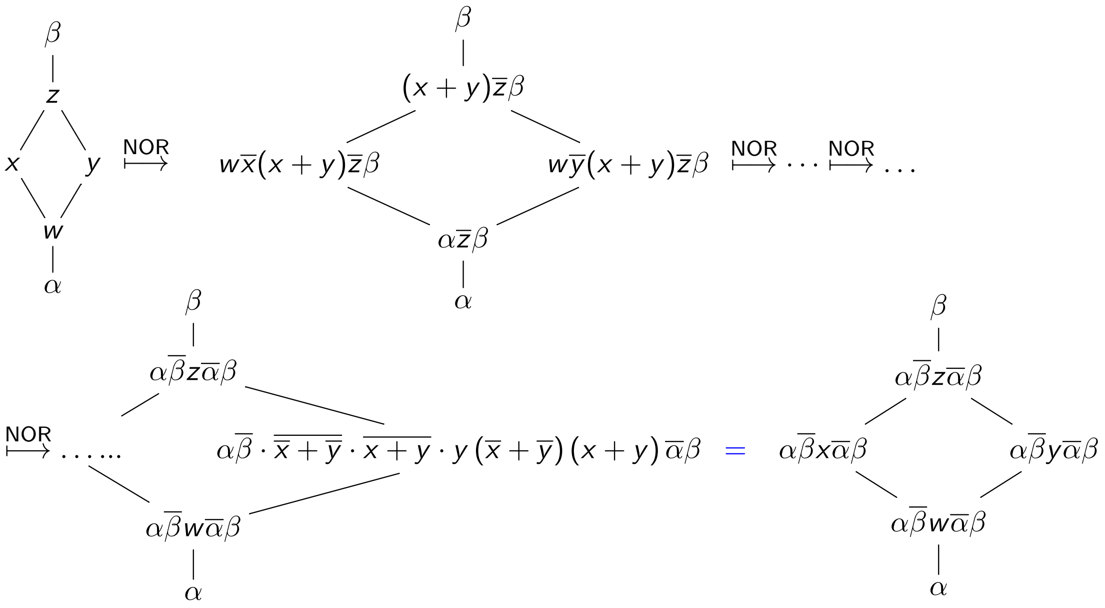
Darij Grinberg lifted birational toggling to work over a skew field $\mathbb{S}$; write $\overline{m}$ for m^{-1}. Set

$$(T_v(f))(v) = \left(\sum_{u \in \widehat{P}, u \lessdot v} f(u) \right) \overline{f(v)} \left(\sum^{\#}_{u \in \widehat{P}, u \gtrdot v} f(u) \right), \text{ where}$$

$$\sum^{\#}_{u \in \widehat{P}, u \gtrdot v} f(u) = \overline{\sum_{u \in \widehat{P}, u \gtrdot v} \overline{f(u)}}.$$

- These "toggles" are no longer involutions, but we can define their inverses, called "elggots" E_v. Toggles and Elggots for elements which do not cover each other commute (among themselves and with each other).
- As usual, we define Noncommutative Order Rowmotion by $\mathrm{NOR} := T_{x_1} T_{x_2} \ldots T_{x_n}$, where $(x_1, \ldots, x_n)$ is a linear extension of P.
- To spice things up, we can also fix $f(\widehat{0}) = \alpha$ and $f(\widehat{1}) = \beta$ to see what happens.

NOR-motion example

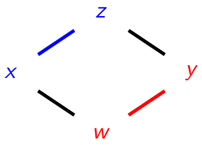


NAR-motion

- Joseph-R. [JoRo20+] lifted birational antichain toggles to the noncommutative setting, and proved that the bijection between the NC order toggle group and the NC antichain toggle lifts as well (again with toggles and elggots).
- We define NAR as usual, and show that NAR and NOR have the same order.
- The Stanley-Thomas word lifts even to this setting, as a tuple that cyclically rotates with the action of NAR.

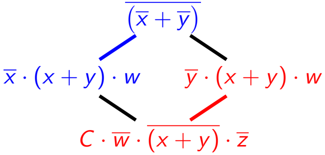
NAR-motion and NC-Stanley–Thomas Word

The NAR-orbit for a generic labeling on $P = [2] \times [2]$ and Stanley–Thomas words



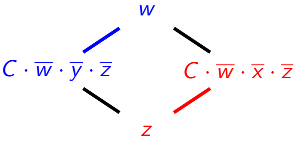
$g = \mathrm{NAR}^4(g)$

$\mathrm{ST}_g = (yw, zx, C \cdot \overline{w} \cdot \overline{x}, C \cdot \overline{y} \cdot \overline{z})$



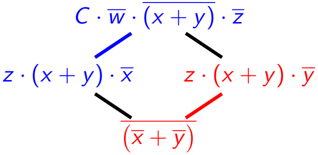
$\mathrm{NAR}(g)$

$\mathrm{ST}_{\mathrm{NAR}(g)} = (C \cdot \overline{y} \cdot \overline{z}, yw, zx, C \cdot \overline{w} \cdot \overline{x})$



$\mathrm{NAR}^2(g)$

$\mathrm{ST}_{\mathrm{NAR}^2(g)} = (C \cdot \overline{w} \cdot \overline{x}, C \cdot \overline{y} \cdot \overline{z}, yw, zx)$



$\mathrm{NAR}^3(g)$

$\mathrm{ST}_{\mathrm{NAR}^3(g)} = (zx, C \cdot \overline{w} \cdot \overline{x}, C \cdot \overline{y} \cdot \overline{z}, yw)$

Open Problems and Questions

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We've had trouble even with how to handle technicalities:

Q: What is the analogue of a rational map over a skew field?

Q: Is there any analogue of Zariski topology in this setting? (or other workaround?)

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Q: What other operations can we birationalize? (See talks by Hopkins and Joseph on Wednesday!)

Summary and Take Aways

- Studying dynamics on objects in algebraic combinatorics is interesting at a variety of levels: combinatorial, piecewise-linear, birational, and noncommutative.

- All of our themes apply at all levels:
1) *Periodicity/order, orbit structure*; 2) *Homomesy*; and 3) *Equivariant bijections*.

- Examples of cyclic sieving are also ripe for homomesy hunting.

- Maps which can be built out of toggling involutions seem particularly fruitful.

- Combinatorial objects are often discrete "shadows" of continuous PL objects, which in turn reflect algebraic dynamics. But combinatorial tools are still frequently useful, even at higher level.

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Thanks very much for coming to this talk!

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