Independence Posets

Nathan Williams (UTD)
Joint with Hugh Thomas (UQAM)

BIRS Workshop "Dynamical Algebraic Combinatorics"

October 26, 2020

	Paid Only	Breakout Rooms	Live Captioning	Video	Antitrust
MS Teams					
Zoom					
Google Meet					
US Postal Service					

Properties

Objects

	Paid Only	Breakout Rooms	Live Captioning	Video	Antitrust
MS Teams		✗	✗	✗	✗
Zoom		✗		✗	
Google Meet			✗	✗	✗
US Postal Service	✗				✗

General: a set of "objects" $\mathcal{J}$ and "properties" $\mathcal{M}$, with maps

$$m : \mathcal{J} \to 2^{\mathcal{M}}$$

$$\left(\text{an object} \overset{m}{\longmapsto} \text{its properties}\right)$$

$$j : \mathcal{M} \to 2^{\mathcal{J}}$$

$$\left(\text{a property} \overset{j}{\longmapsto} \text{the objects with that property}\right)$$

Rudolf Wille's
Formal Concept Analysis

$$\text{subsets of objects} \overset{m}{\longmapsto} \text{shared properties}$$

$$\text{subsets of properties} \overset{j}{\mapsto} \text{objects with those properties}$$

	Paid Only	Breakout Rooms	Live Captioning	Video	Antitrust
MS Teams		✗	✗	✗	✗
Zoom		✗		✗	
Google Meet			✗	✗	✗
US Postal Service	✗				✗

The pair $\Big(\{\text{MS Teams}, \text{Google Meet}\}, \{\text{Captioning}, \text{Video}, \text{Antitrust}\}\Big)$ captures the notion *"big corporation video conferencing software"*:

- the common properties of MS Teams, Google Meet are

$$m\Big(\{\text{MS Teams}, \text{Google Meet}\}\Big) = \{\text{Captioning}, \text{Video}, \text{Antitrust}\},$$

- the services with Captioning, Video, Antitrust are

$$j\Big(\{\text{Captioning}, \text{Video}, \text{Antitrust}\}\Big) = \{\text{MS Teams}, \text{Google Meet}\}.$$

General: a *concept* is a pair $(X, Y) \in 2^{\mathcal{J}} \times 2^{\mathcal{M}}$ such that:

$$(i) \qquad m(X) = \bigcap_{x \in X} m(x) = Y$$

$$\left(\text{subset of objects} \overset{m}{\longmapsto} \text{shared properties}\right)$$

$$(ii) \qquad j(Y) = \bigcap_{y \in Y} j(y) = X$$

$$\left(\text{subset of properties} \overset{j}{\longmapsto} \text{common objects}\right)$$

redundant!

	Paid Only	Breakout Rooms	Live Captioning	Video	Antitrust
MS Teams		✗	✗	✗	✗
Zoom		✗		✗	
Google Meet			✗	✗	✗
US Postal Service	✗				✗

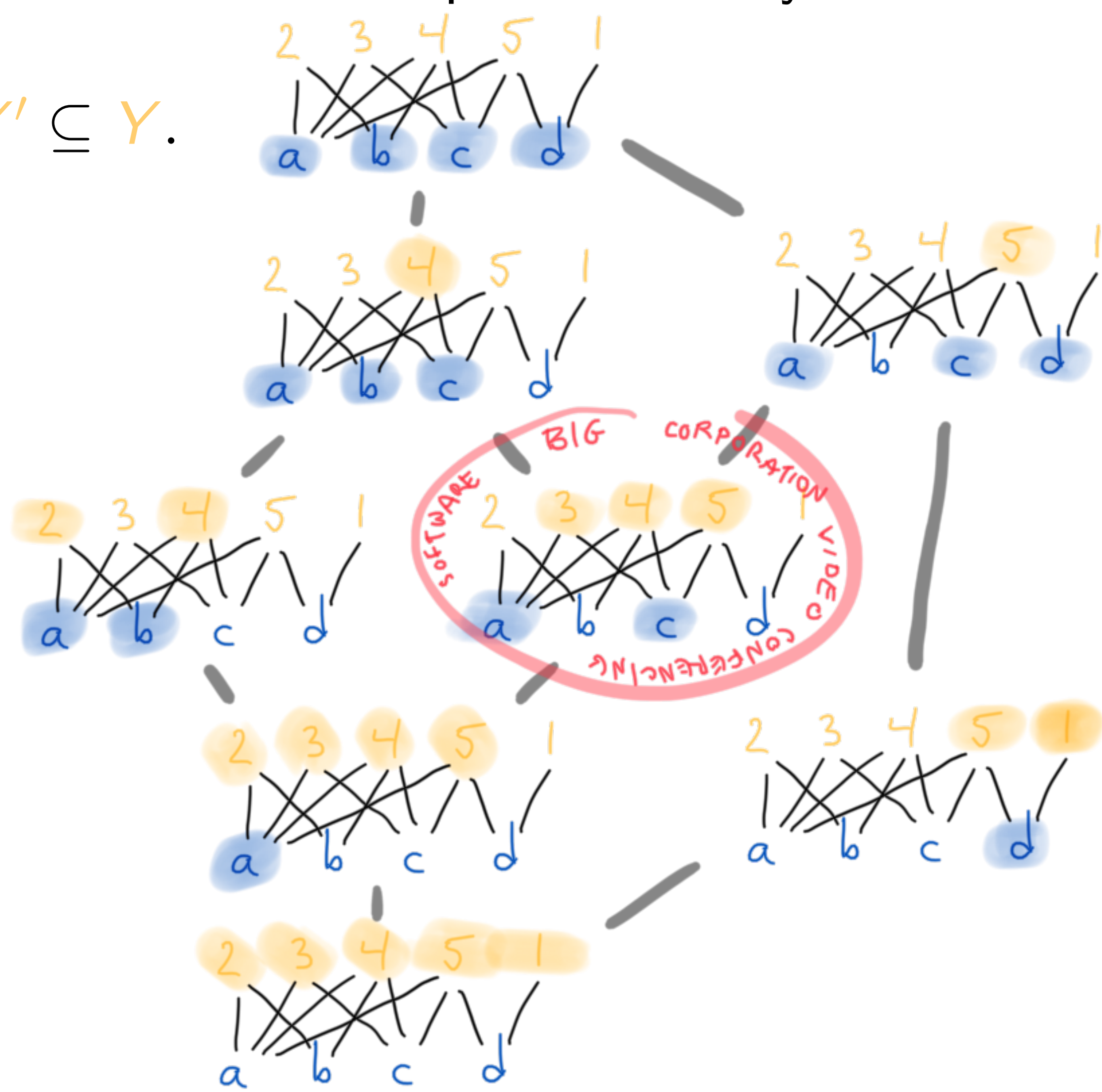
Definition

The *poset of concepts* is the set of all concepts ordered by $(X, Y) \leq (X', Y')$ iff $X \subseteq X'$ or (equivalently) $Y' \subseteq Y$.

redundant!

	1	2	3	4	5
a		✗	✗	✗	✗
b		✗		✗	
c			✗	✗	✗
d	✗				✗

(draw me!)

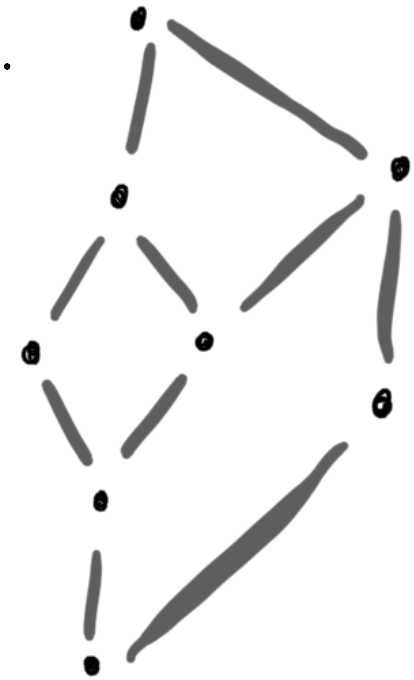


Definition

The *poset of concepts* is the set of concepts ordered by

$$(X, Y) \leq (X', Y') \text{ iff}$$

$$X \subseteq X' \text{ or (equivalently) } Y' \subseteq Y.$$



Theorem (Lattice Representation I)

The poset of concepts is a **lattice**, *and every* **lattice** *arises as a poset of concepts.*

Lattices

"Never in the history of mathematics has a mathematical theory been the object of such vociferous vituperation as lattice theory. Dedekind, Jónsson, Kurosh, Malcev, Ore, von Neumann, Tarski, and most prominently Garrett Birkhoff have contributed a new vision of mathematics, a vision that has been cursed by a conjunction of misunderstandings, resentment, and raw prejudice."

—Gian-Carlo Rota (*The Many Lives of Lattice Theory*)

Definition

A (*finite*) poset L is a *lattice* if each pair $x, y \in L$ have:

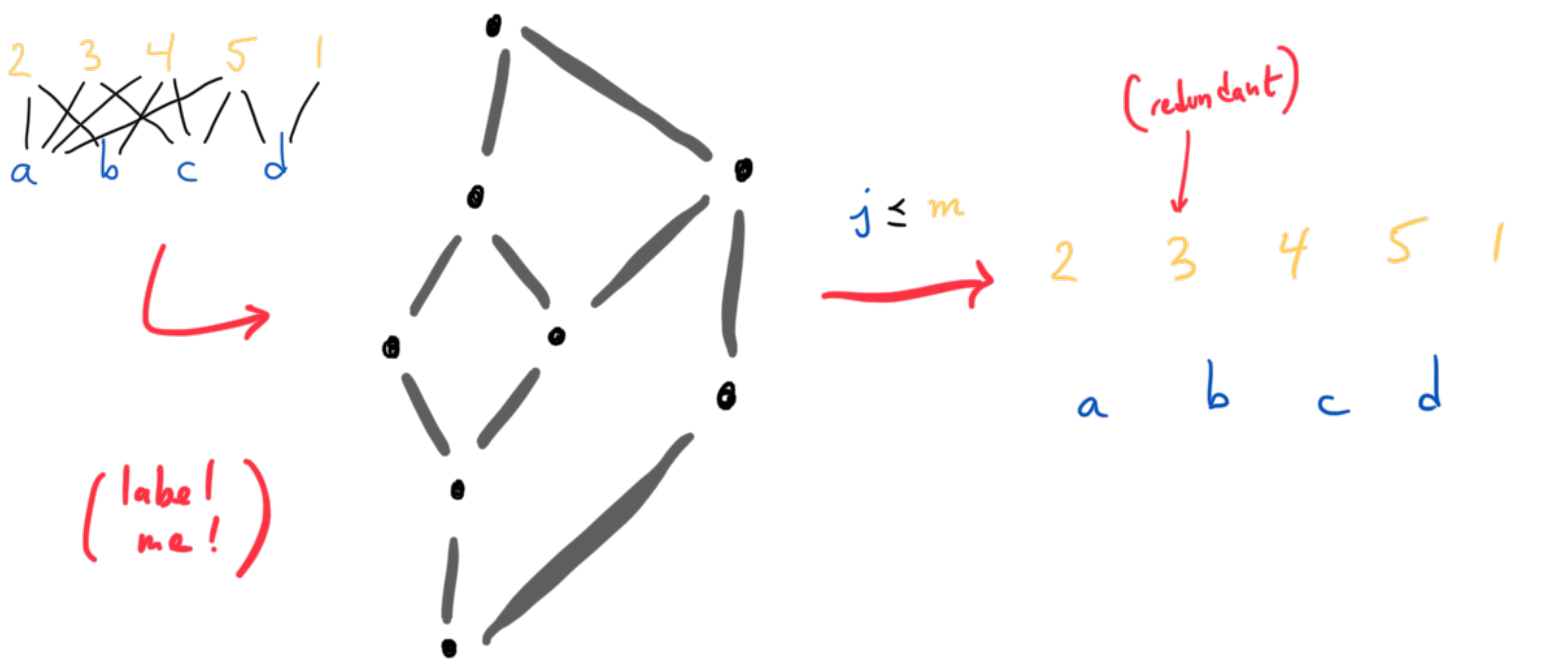
$$\text{a join (least upper bound): } x \vee y, \text{ and}$$
$$\text{a meet (greatest lower bound): } x \wedge y.$$

> *"Like its elder sister group theory, lattice theory is a fruitful source of abstract concepts, common to traditionally unrelated branches of mathematics. Both subjects are based on postulates of an extremely simple and general nature. It was this which convinced me from the first that lattice theory was destined to play—indeed, already did play implicitly—a fundamental role in mathematics. Though its importance will probably never equal that of group theory, I do believe that it will achieve a comparable status."*

—Garrett Birkhoff (*Lattice Theory*, Second Edition).

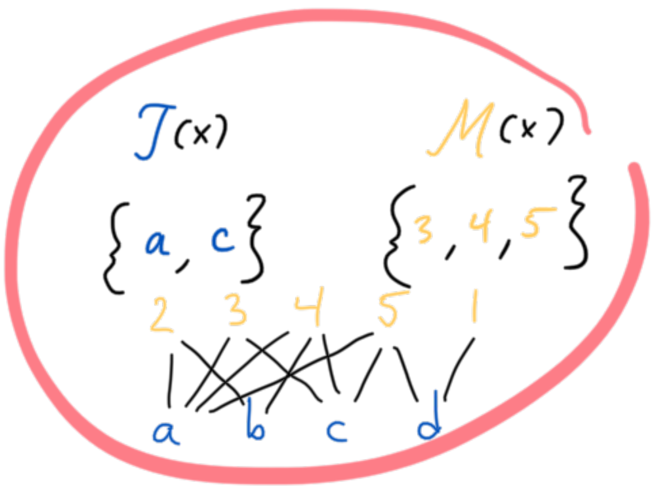
Definition

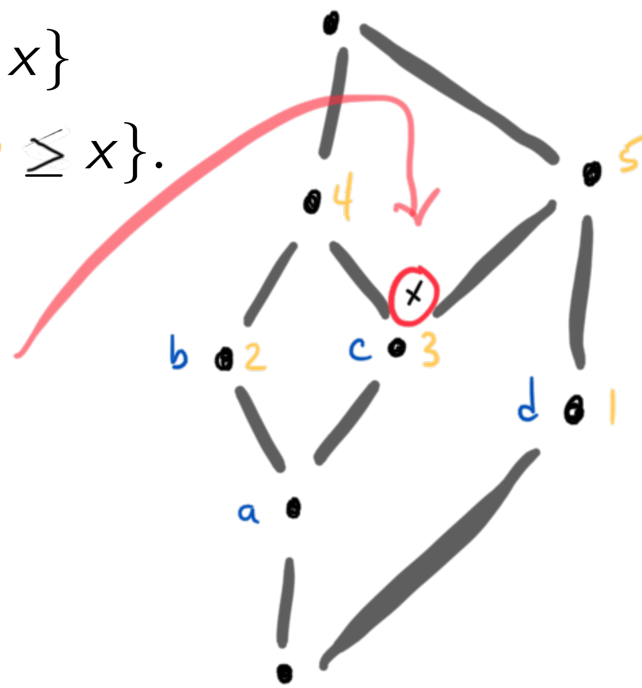
- An element $j \in L$ is join-irreducible if when $j = \bigwedge S$, then $j \in S$. Write $\mathcal{J}$ for the set of all join-irreducibles.
- An element $m \in L$ is meet-irreducible if when $m = \bigvee S$, then $m \in S$. Write $\mathcal{M}$ for the set of all meet-irreducibles.



For $x \in L$, write

$$\mathcal{J}(x) = \{j \in \mathcal{J} : j \leq x\}$$
$$\mathcal{M}(x) = \{m \in \mathcal{M} : m \geq x\}.$$





Since any element is the join of the join-irrs below it and the meet of the meet-irrs above it:

Theorem (Lattice Representation II)

Every finite lattice L is isomorphic to (both)

- $\{\mathcal{J}(x) : x \in L\}$ *under inclusion, and*
- $\{\mathcal{M}(x) : x \in L\}$ *under reverse inclusion.*

Historical Notes

- *Formal concept analysis* was Rudolf Wille's "restructuring" of lattice theory (published 1982).
- Previously appeared in the 1973 thesis of George Markowsky as the *poset of irreducibles*.
- First (?) appeared in 1965 work of Marc Barbut as *l'algèbre des techniques d'analyse hiérarchique*.

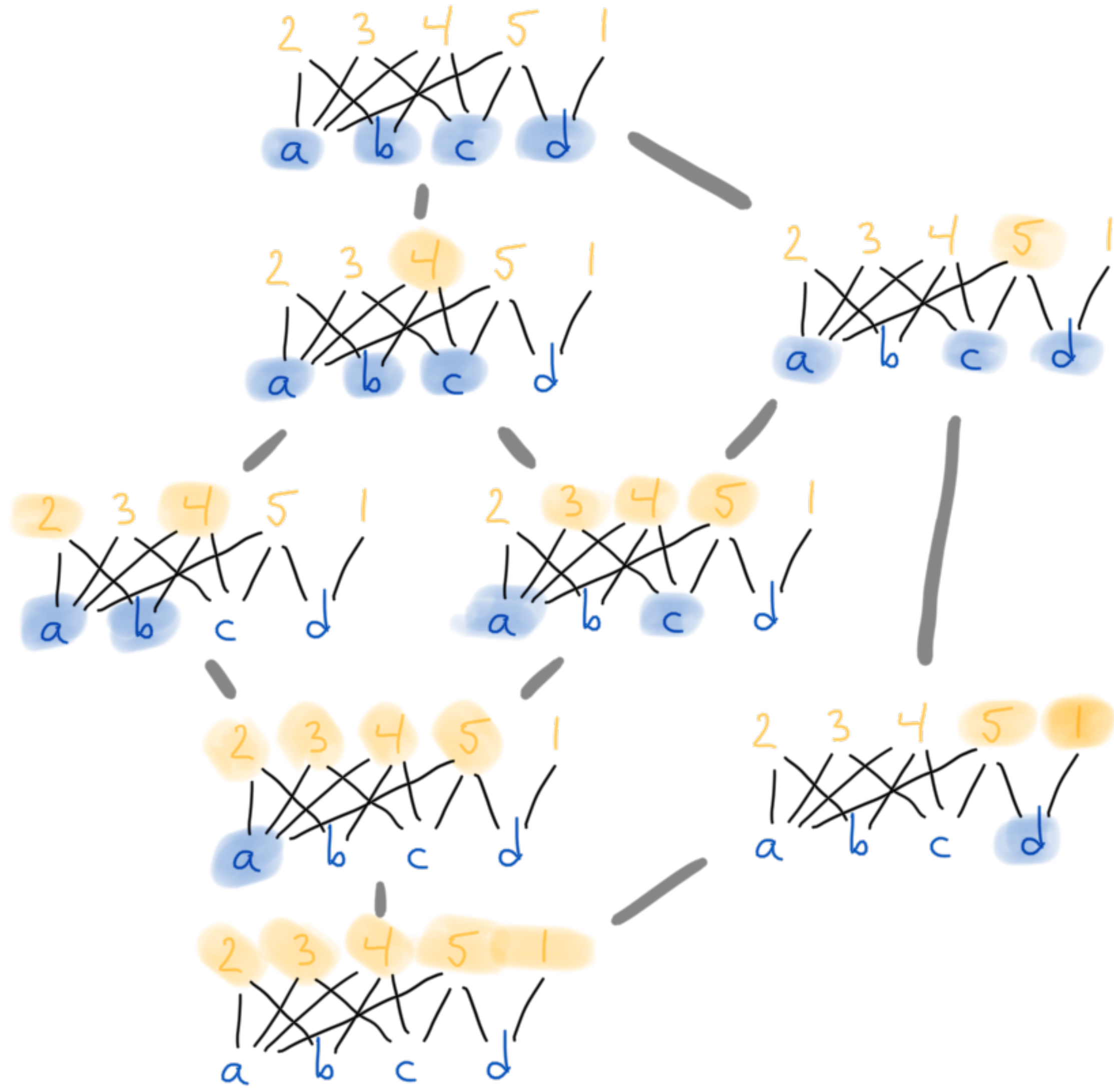
George Markowsky's Extremal Lattices

"When do lattices have a compact representation theorem?"

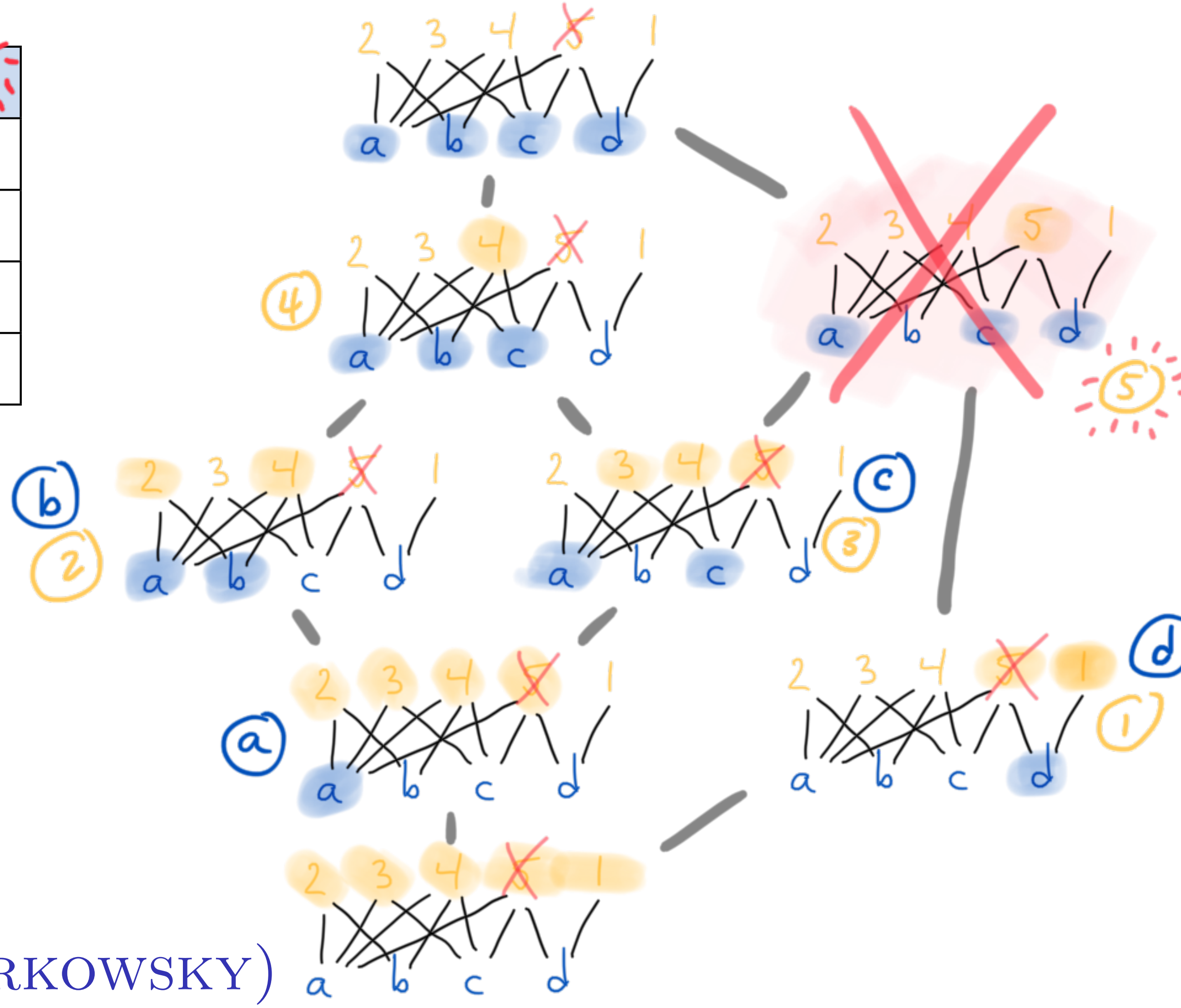
$$|\mathcal{J}| = |\mathcal{M}| = \text{length } L$$

("objects") ("properties") (length of max length chain) too many properties...

	1	2	3	4	5
a		✗	✗	✗	✗
b		✗		✗	
c			✗	✗	✗
d	✗				✗



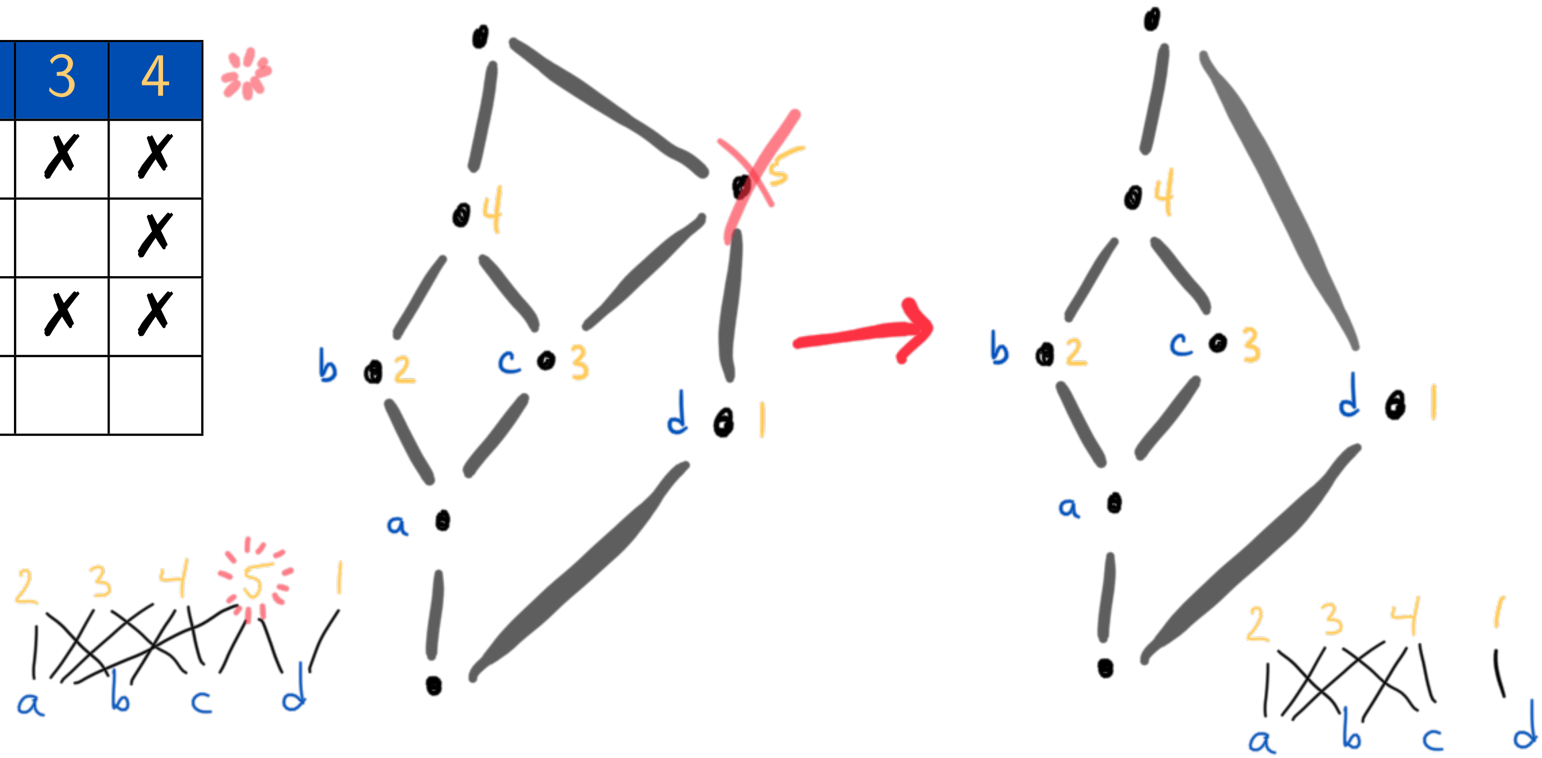
	1	2	3	4	5
a		✗	✗	✗	✗
b		✗		✗	
c			✗	✗	✗
d	✗				✗



Definition (Markowsky)

A lattice is *extremal* if $|\mathcal{J}| = |\mathcal{M}| = \text{length } L$.

	1	2	3	4
a		✗	✗	✗
b		✗		✗
c			✗	✗
d	✗			



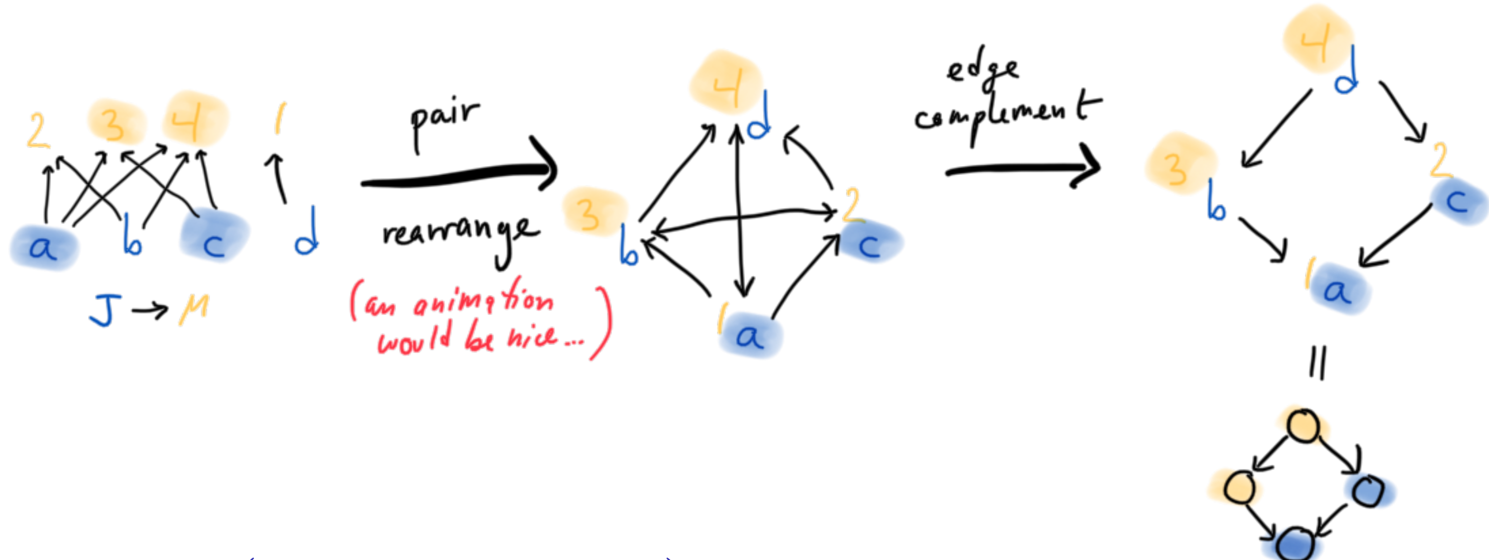
Key point: A chain of maximal length $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ lets us pair up $\mathcal{J}$ and $\mathcal{M}$ by $j_i \leftrightarrow m_i$, where:

$$\begin{aligned} x_i &= j_1 \vee \cdots \vee j_i \\ &= m_{i+1} \wedge \cdots \wedge m_n \end{aligned}$$

(show me!)

A chain of maximal length lets us pair up $\mathcal{J}$ and $\mathcal{M}$:

$$a \leftrightarrow 1,\ b \leftrightarrow 3,\ c \leftrightarrow 2,\ d \leftrightarrow 4$$



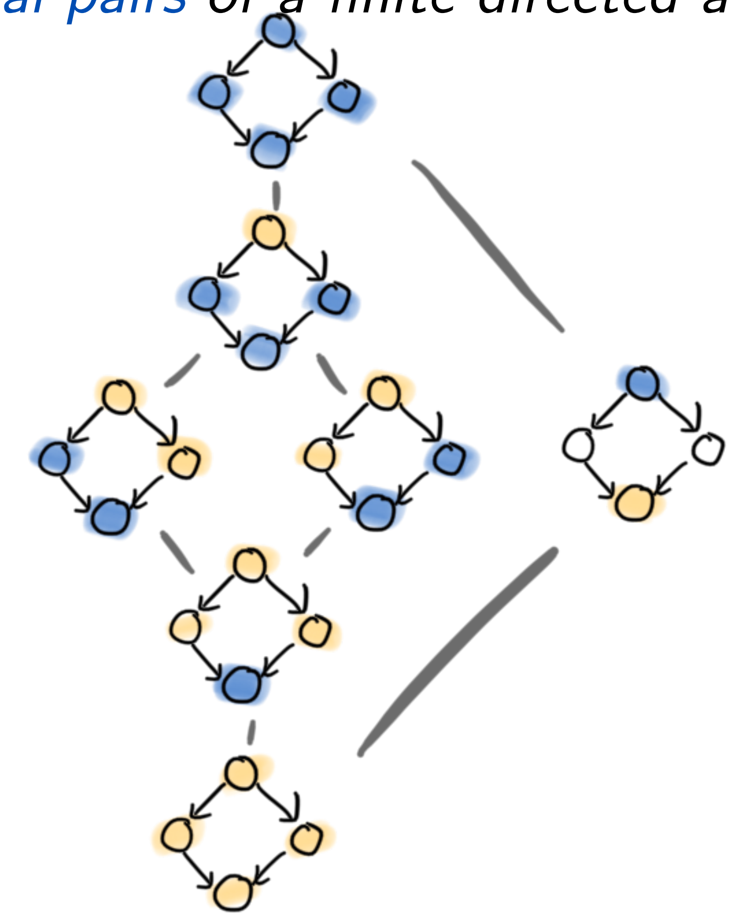
Theorem (Markowsky)

Every finite extremal lattice L is isomorphic to the lattice of maximal orthogonal pairs of a finite directed acyclic graph G.

$$\left\{ (X, Y) : \substack{X \cap Y = \emptyset \\ \text{no arrow } X \to Y \\ X,\, Y \text{ maximal}} \right\} \text{ ordered by } (X, Y) \leq (X', Y') \text{ iff } \substack{X \subseteq X' \\ Y' \subseteq Y} \text{ or}$$

Theorem (Extremal Lattice Representation)

Every finite extremal lattice L is isomorphic to the lattice of maximal orthogonal pairs of a finite directed acyclic graph G.



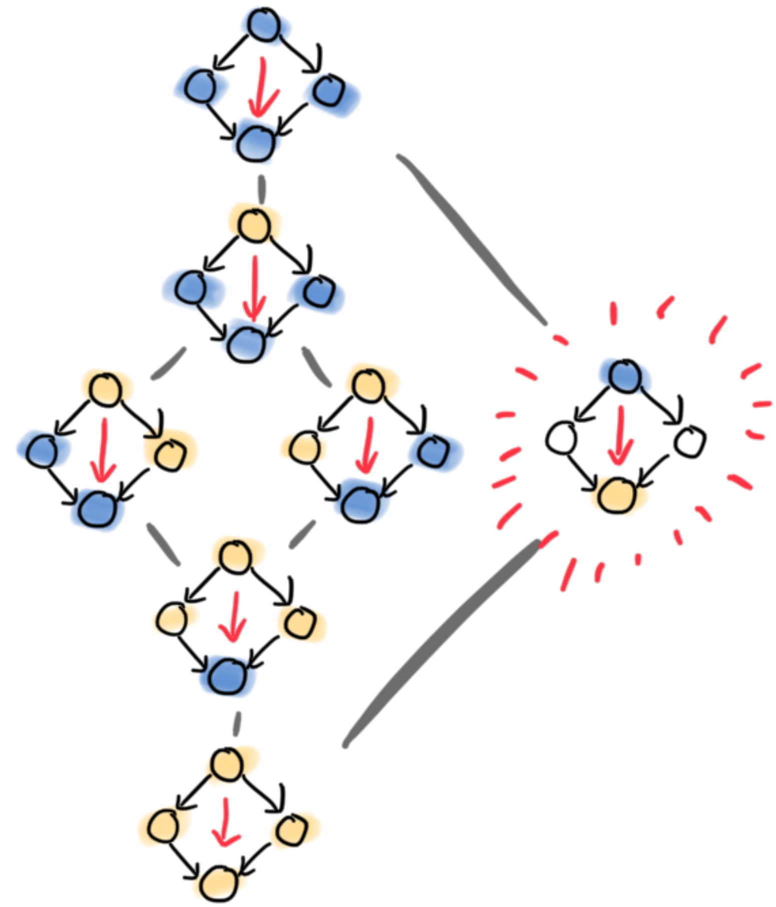
Theorem (Markowsky)

A finite **graded** *extremal lattice L is distributive, in which case G is the comparability graph of a poset.*

(ex: $J(2\times 2)$)

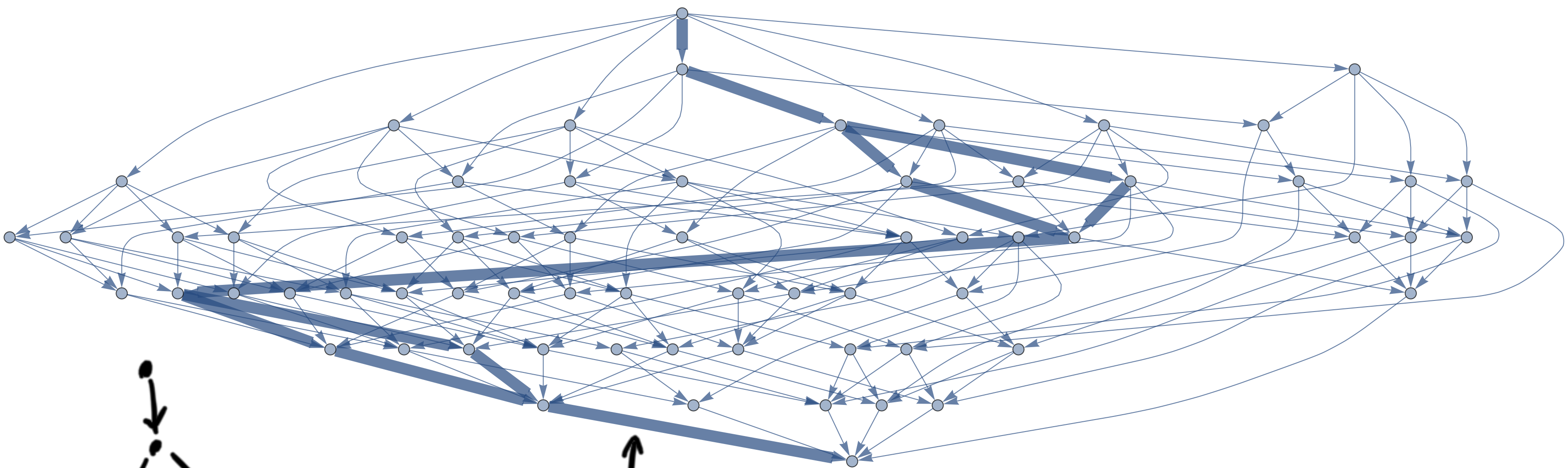
Corollary: *Birkhoff's theorem for distributive lattices.*

	1	2	3	4
a		✗	✗	✗
b		✗		✗
c			✗	✗
d				



Theorem (Markowsky)

A finite graded extremal lattice L is distributive, in which case G is the comparability graph of a poset.



the "skeleton" of an extremal lattice

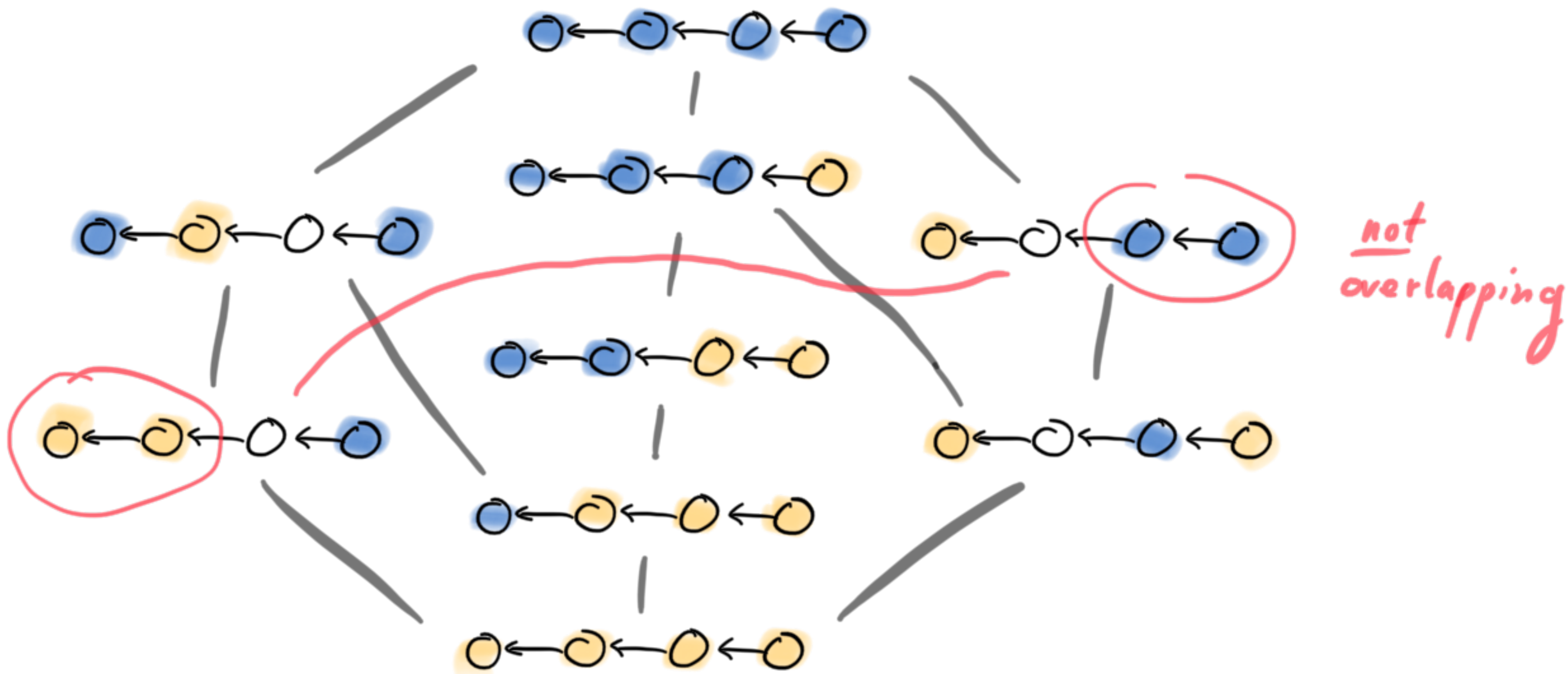
(c-singletons for Cambrian)

Hugh Thomas's Trim Lattices

"What if a distributive lattice weren't graded?"

Extremal, left-modular lattices (every cover gets a label)

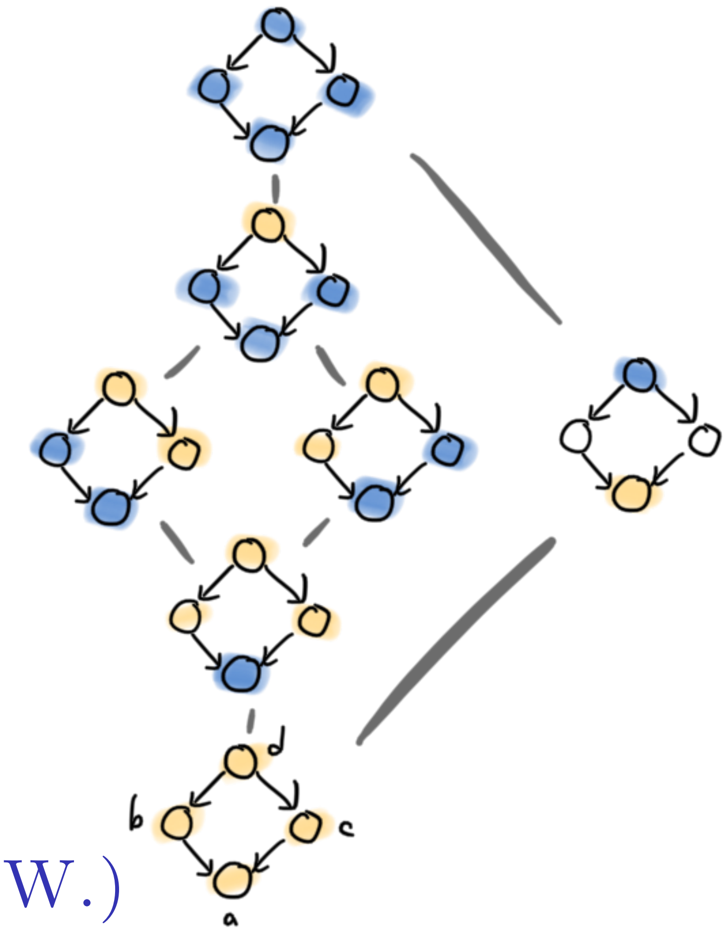
Call a relation $(X, Y) \leq (X', Y')$ in an extremal lattice *overlapping* if $Y \cap X' \neq \emptyset$.



THEOREM (THOMAS-W.)

An extremal lattice is trim iff every relation is overlapping iff every cover relation is overlapping.

Call a relation $(X, Y) \leq (X', Y')$ in an extremal lattice *overlapping* if $Y \cap X' \neq \emptyset$.



Theorem (Thomas-W.)

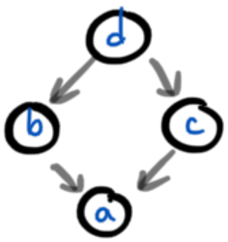
An extremal lattice is trim iff every relation is overlapping iff every cover relation is overlapping.

We get to label ALL edges!!! (Show me!)

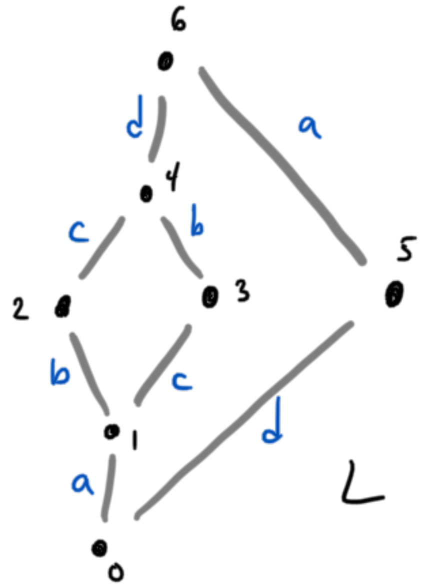
Theorem (Thomas-W.)

$$\mathcal{D} : x \mapsto \{g \in G : y \lessdot x\}$$
$$\mathcal{U} : x \mapsto \{g \in G : x \lessdot y\}$$

are each bijections from the elements of a trim lattice L to the independent sets of G.



G

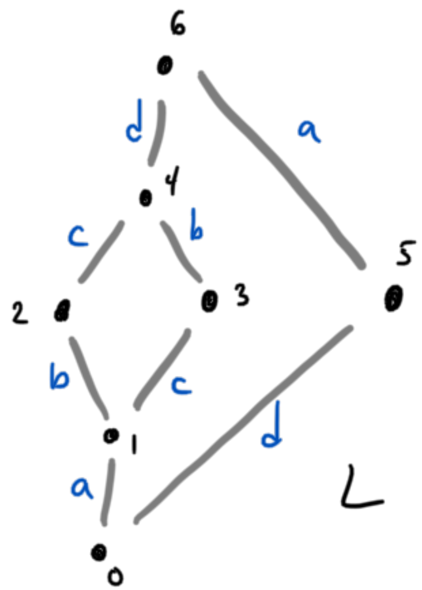


Define *global rowmotion* by the unique element $\mathrm{row}(x)$ with

$$\mathcal{D}(x) = \mathcal{U}(\mathrm{Row}(x)).$$

(Show me!)

Define *local rowmotion* by "flipping" in order of max-len chain.



THEOREM (T-W. "ROWMOTION IN SLOW MOTION")

Both definitions of rowmotion agree. (Show me!)

Independence Posets

"What if a distributive lattice weren't a lattice?'

Click for interactive version (presented by Hugh last Friday)

Recall: For trim lattices L,

- Every cover relation is overlapping ($Y \cap X' \neq \emptyset$) and
- There are bijections

$$\mathcal{D} : x \mapsto \{g \in G : y \lessdot x\}$$
$$\mathcal{U} : x \mapsto \{g \in G : x \lessdot y\}$$

from L to the independent sets of G.

Questions:

- How do $\mathcal{D}$ and $\mathcal{U}$ fit together?
- Can we express cover relations on $\mathcal{D}$ and $\mathcal{U}$ in a simple way?

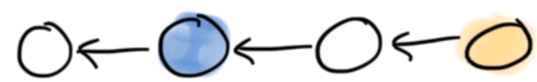
Fix G a finite directed acyclic graph.

Definition (Thomas-W.)

A pair $(\mathcal{D}, \mathcal{U})$ of independent sets of G is *orthogonal* if $\mathcal{D} \cap \mathcal{U} = \emptyset$ and there is no arrow $\mathcal{D} \to \mathcal{U}$. An orthogonal pair is *tight* if whenever

- an element of $\mathcal{D}$ is increased

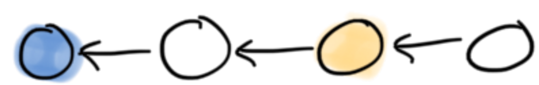
  move ○ back along an arrow



OR

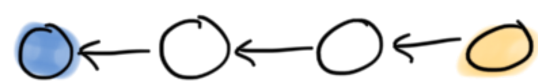
- an element of $\mathcal{U}$ is decreased

  move ○ forward along an arrow



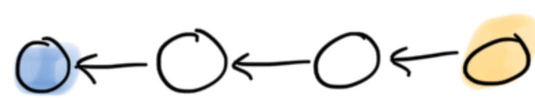
OR

- an element is added to $\mathcal{D}$



OR

- an element is added to $\mathcal{U}$

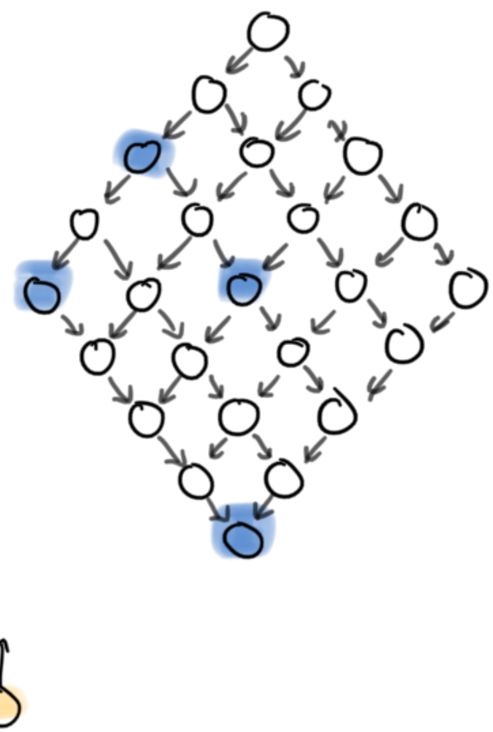


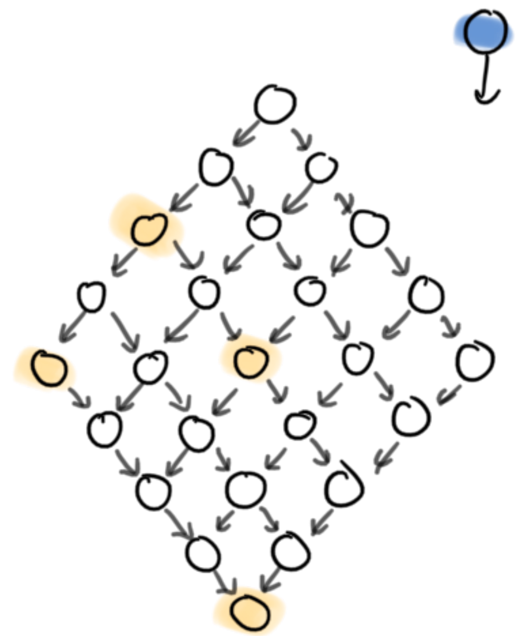
the result isn't orthogonal.

Theorem (Thomas-W.)

Any independent set $\mathcal{I}$ can be uniquely completed to a top $(\mathcal{I}, \mathcal{U})$ and a top $(\mathcal{D}, \mathcal{I})$.

(Show me!)





Define *global rowmotion* by $\mathrm{row}(\mathcal{I}, \mathcal{U}) = (\mathcal{D}, \mathcal{I})$.

Definition ("Flips")

If $g \in \mathcal{D}$ or $g \in \mathcal{U}$, $\mathrm{flip}_g(\mathcal{D},\mathcal{U})$ is defined by:

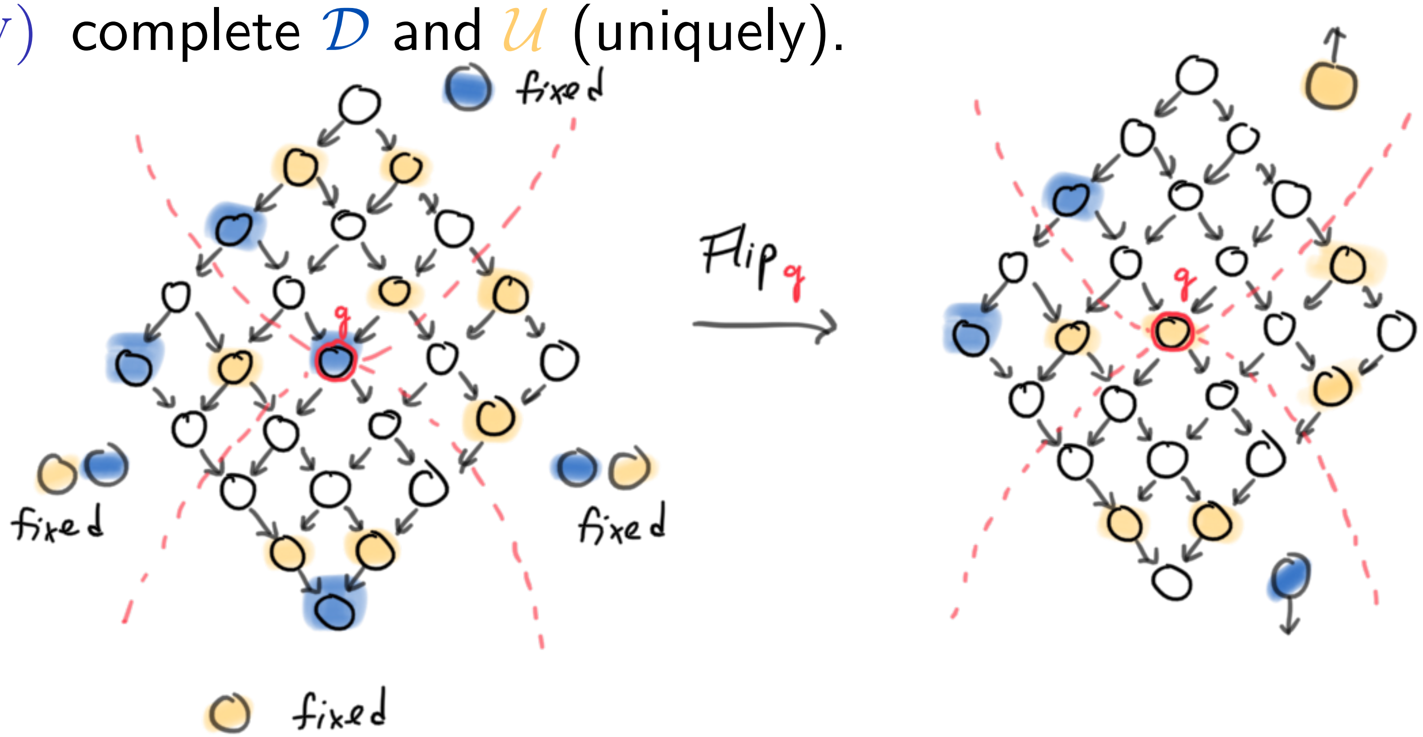
(I) fix all elements of $\mathcal{D}$ not $< g$

(II) fix all elements of $\mathcal{U}$ not $> g$

(III) swap g from $\mathcal{D}$ to $\mathcal{U}$ or vice-versa

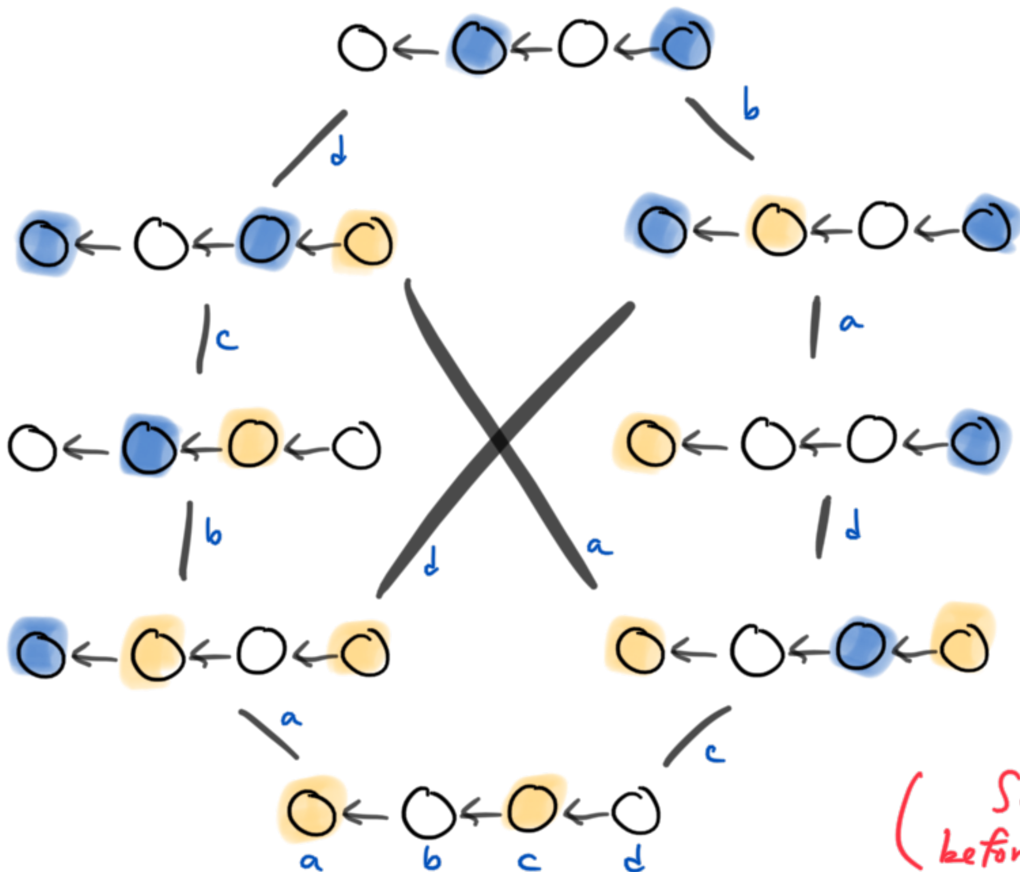
(IV) complete $\mathcal{D}$ and $\mathcal{U}$ (uniquely).

(Um... show me?)



Theorem (Thomas-W.)

The independence poset of G is the poset with cover relations $(\mathcal{D},\mathcal{U}) \lessdot \mathrm{flip}_g(\mathcal{D},\mathcal{U})$. *Rowmotion can be computed in slow motion.*



(Same digraph G as before; different structure!)

p.23

Theorem (Thomas-W.)

If an independence poset is a lattice, it is a trim lattice. If it is a graded lattice, it is a distributive lattice.

“What if a distributive lattice weren’t gr~~ade~~d?”

a lattice

Thank You!

Future work:

- Many(!) combinatorial objects can be encoded as independent sets. What new and old structures arise using independence posets?
- With generalized rowmotion, new examples for DAC?
- Can define posets on integer points in dilations of "independence polytopes". And?
- Random sampling (CFTP)?