INFINITE FRIEZES & BRACELETS

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(This talk is being recorded)

INFINITE FRIEZES & BRACELETS

1. Conway-Coxeter (finite) friezes

Row of 0's 0 0 0 0 0 0 0 0 0 0 0

Row of 1's 1 1 1 1 1 1 1 1 1 1 1 1

Positive integers 1 4 1 2 2 2 1 4 1 2 2 2 . . .

SL_2-rule . . . 3 3 1 3 3 1 3 3 1 3 3 1

2 2 1 4 1 2 2 2 1 4 1 2

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc = 1$ 1 1 1 1 1 1 1 1 1 1 1 1

bounded below by a row of 1s

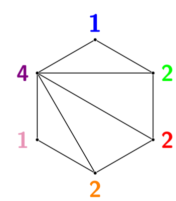
Thm (Conway & Coxeter 1970s)

$\left\{\begin{array}{l}\text{Conway-Coxeter friezes} \\ \text{with } n \text{ nontrivial rows}\end{array}\right\} \longleftrightarrow \{\text{triangulations of } (n+3)\text{-gon}\}$

Ex. $n=3$

1 4 1 2 2 2

quiddity sequence



6-gon

Thm (Propp + REACH (REU) 2001, Caldero & Chapoton 2004)

Finite friezes $\Rightarrow$ cluster algebras type A_n

Positive integer entries $\longrightarrow$ # terms in Laurent polynomial expansions of cluster variables

$$\begin{array}{ccccccccccc}
1 & & 1 & & & 1 & & 1 & & & \\
& x_3 & & \frac{x_1x_3+1+x_2}{x_2x_3} & & & \frac{x_2+1}{x_1} & & & x_1 & \\
x_2 & & \frac{x_1x_3+1}{x_2} & & & \frac{x_2^2+2x_2+1+x_1x_3}{x_1x_2x_3} & & & x_2 & & \\
& x_1 & & \frac{x_1x_3+1+x_2}{x_1x_2} & & & \frac{x_2+1}{x_3} & & & x_3 & \\
1 & & 1 & & & 1 & & 1 & & &
\end{array}$$

$$\begin{array}{ccccccccc}
1 & & 1 & & 1 & & 1 & & \\
& 1 & & 3 & & 2 & & 1 & \\
1 & & 2 & & 5 & & 1 & & \\
& 1 & & 3 & & 2 & & 1 & \\
1 & & 1 & & 1 & & 1 & &
\end{array}$$

2. Infinite friezes

Same rule, but remove the bottom boundary condition (Tschabold 2015)

Ex: periodic with $n=5$

Row of 0's

Row of 1's

Positive integers

SL_2-rule

$$\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix}$$

$ad-bc=1$

```
   0   0   0   0   0   0   0   0   0   0   0
 1   1   1   1   1   1   1   1   1   1   1   1  ...
   4   1   2   3   2   4   1   2   3   2   4   1
     3   1   5   5   7   3   1   5   5   7   3   1
       2   2   8  17   5   2   2   8  17   5   2   2
...      3   3  27  12   3   3   3  27  12   3   3
           4  10  19   7   4   4  10  19   7   4   4
             13   7  11   9   5  13   7  11   9   5
                9   4  14  11  16   9   4  14  11  16
                  5   5  17  35  11   5   5  17  35
                    6   6  54  24   6   6  54  24   6
                              ⋮
```

Classification of Infinite Friezes (Baur, Parsons, Tschabold 2015)

Every infinite periodic frieze corresponds to a triangulation of an annulus or once-punctured disk

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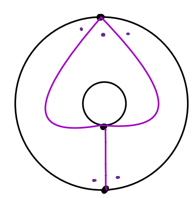
Ex 1: annulus

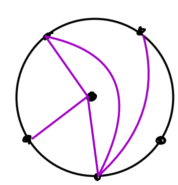
```
 1    1    1    1    1    1
   2    3    2    3    2    3
     5    5    5    5    5    5
       8   12    8   12    8   12
        19   19   19   19   19   19
          30   45   30   45   30   45
            71   71   71   71   71   71
             112  168  112  168  112  168
               265  265  265  265  265  265
```



quiddity sequence: 2 3

Ex 2: punctured disk

```
 1   1   1   1   1   1   1   1   1   1   1   1
   4   1   2   3   2   4   1   2   3   2   4   1
     3   1   5   5   7   3   1   5   5   7   3   1
       2   2   8  17   5   2   2   8  17   5   2   2
         3   3  27  12   3   3   3  27  12   3   3
           4  10  19   7   4   4  10  19   7   4   4
             13   7  11   9   5  13   7  11   9   5
                9   4  14  11  16   9   4  14  11  16
                  5   5  17  35  11   5   5  17  35
                    6   6  54  24   6   6  54  24   6
```



quiddity sequence: 41232

Later: infinite frieze with cluster algebra elements

3. Cluster algebras (Fomin – Zelevinsky, 2001)

Idea: A cluster algebra is a subring $\mathcal{A}$ of $\mathbb{Q}(x_1, \dots, x_n)$

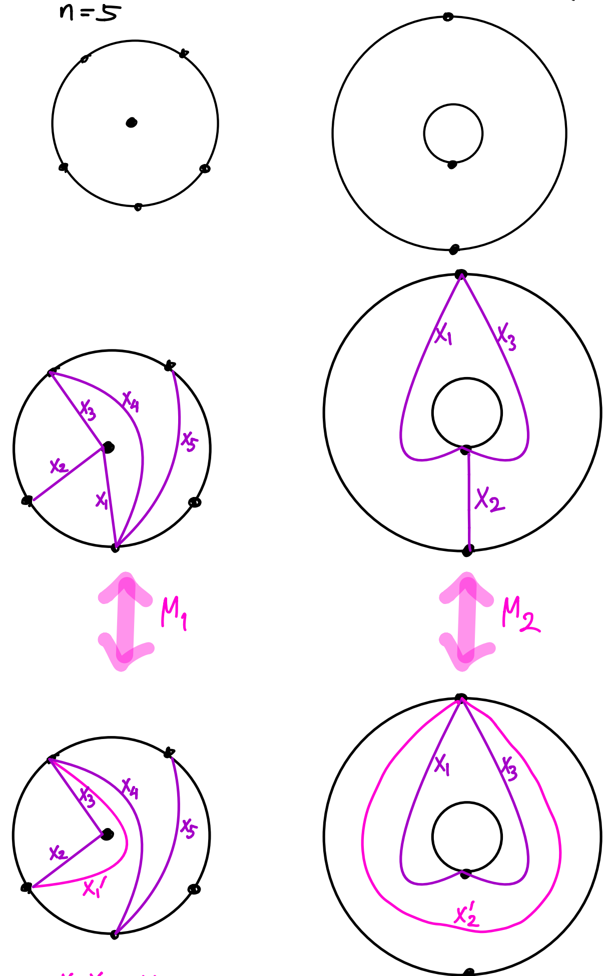
- generated by cluster variables.
- Start with n initial cluster variables + some data, then compute all cluster variables iteratively

Cluster algebras from surfaces (Fomin-Shapiro-Thurston, 2006)

D_n — once-punctured disk

$\tilde{A}_{p,q}$ — annulus w/ $p+q$ marked points on the boundary

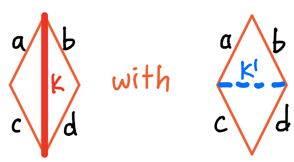
$n=5$ $p,q = 1,2$



An arc is an internal curve between marked points

A triangulation is a maximal collection of non-crossing arcs

A flip μ_k replaces



Ptolemy rule
$k'k = ad + bc$

$$x_1' = \frac{x_2 x_4 + x_3}{x_1}$$

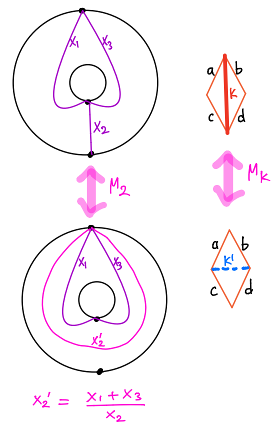
$$x_2' = \frac{x_1 + x_3}{x_2}$$

How to construct cluster variables

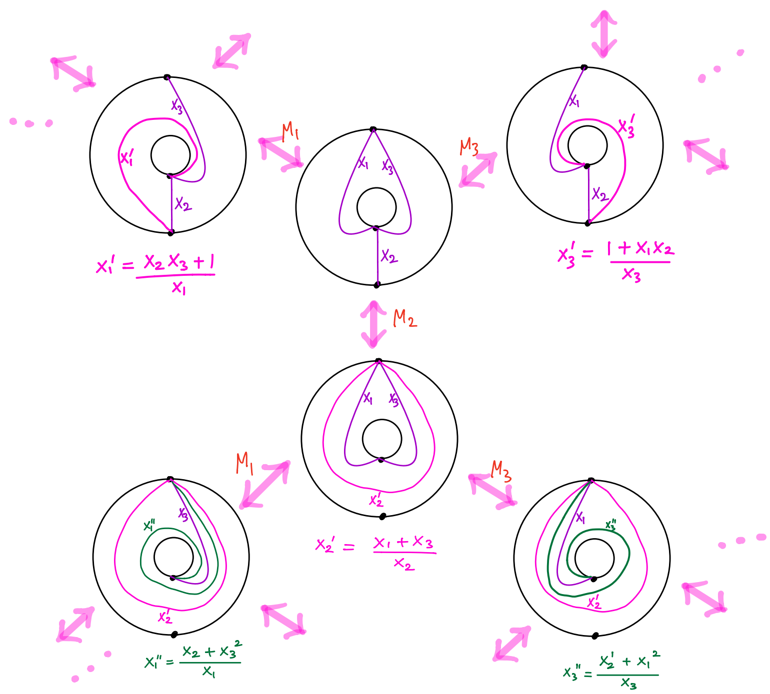
initial cluster $\{x_1, x_2, x_3\}$

Mutation at x_2

a new cluster $\left\{x_1, \frac{x_1+x_3}{x_2}, x_3\right\}$



Repeat this mutation process to produce all clusters



$$\{\text{cluster variables}\} = \bigcup_{\text{all clusters } \mathbf{x}} \{\text{elements of } \mathbf{x}\}$$

Laurent phenomenon & positivity:

Every cluster variable is a Laurent polynomial [Fomin-Zelevinsky 2001]

i.e $\frac{\text{polynomial}}{\text{monomial}}$

with positive coefficients in the initial cluster
[Lee-Schiffler 2013]

4. Infinite frieze of positive Laurent polynomials

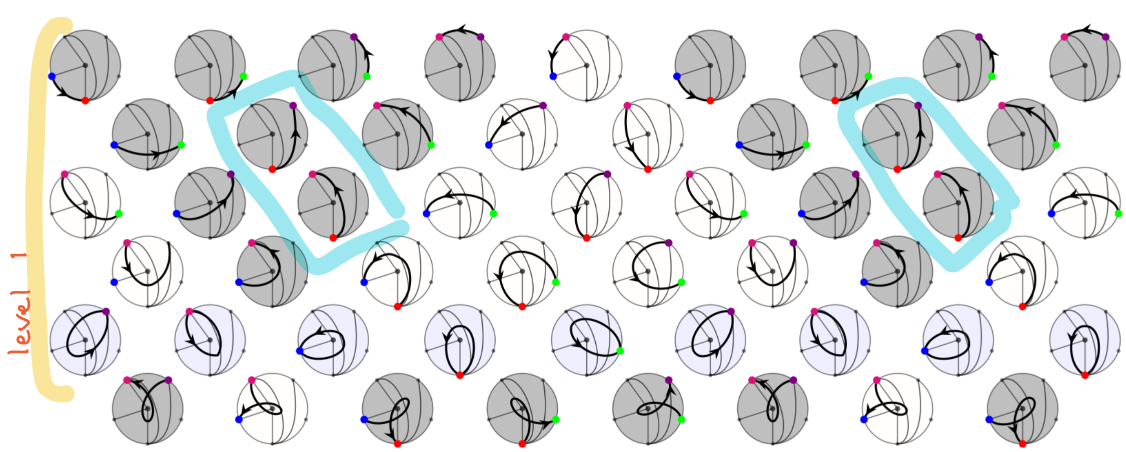
Idea For cluster algebras from surfaces

arcs $\longleftrightarrow$ cluster variables

generalized arcs (self-crossing is allowed) $\longrightarrow$ cluster algebra elements which are Laurent polynomials with positive coefficients

[G., Musiker, Vogel 2016]

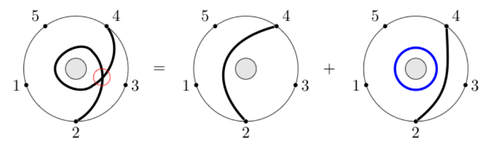
The cluster algebra elements corresponding to generalized arcs between the same boundary of an annulus or a punctured disk form an infinite frieze.



An infinite frieze of elements of the cluster algebra corresponding to peripheral curves in a punctured disk.

Why? The self-intersecting arcs correspond to elements of $\mathcal{A}$ via skein relation (Musiker – Williams 2011)

+



Example: Resolving a self-crossing.

- When the variables are specialized to 1, we recover the integer frieze pattern. When specialized to nonzero numbers, we get an infinite frieze pattern with nonzero entries.

1 1 1 1 1 1 1 1 1 1 1

4 **1** 2 3 2 4 **1** 2 3 2 4

3 **1** 5 5 7 3 **1** 5 5 7

2 2 8 17 5 2 2 8 17 5

3 3 27 12 3 3 3 27 12

- -

4 10 19 7 4 4 10 19 7

13 7 11 9 5 13 7 11

9 4 14 11 16 9 4 14

5 5 17 35 11 5 5

6 6 54 24 6 6 6

Divide rows into levels

Level 1 consists of curves with 0 self-crossings

Level 2 consists of curves with 1 self-crossing

⋮

Level k consists of curves with $k-1$ self-crossings

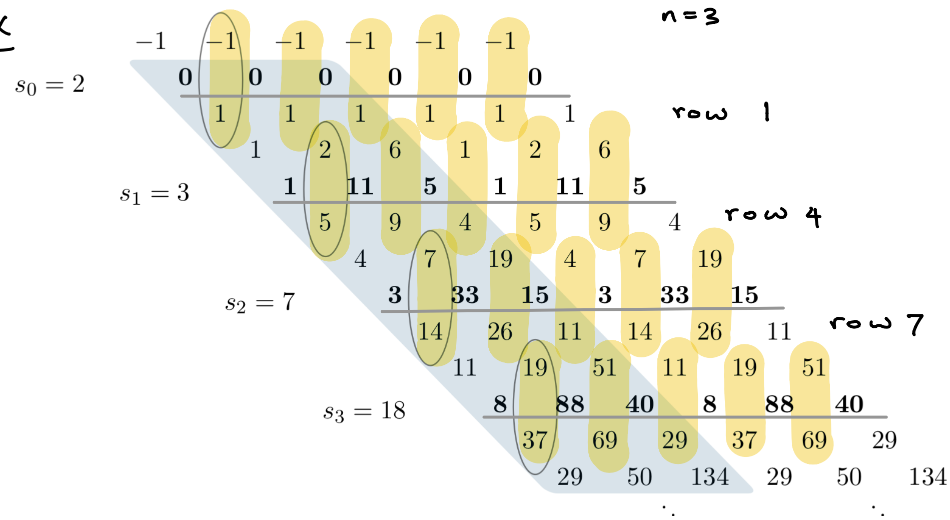
An infinite frieze pattern from $\odot$, periodic with $n=5$

```
        1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
          4   1   2   3   2   4   1   2   3   2   4   1   2   3   2
            3   1   5   5   7   3   1   5   5   7   3   1   5   5   7
              2   2   8  17   5   2   2   8  17   5   2   2   8  17
      Level 1   3   3  27  12   3   3   3  27  12   3   3   3  27  12
                -----------------------------------------------------
                  4  10  19   7   4   4  10  19   7   4   4  10  19
                   13   7  11   9   5  13   7  11   9   5  13   7  11
                      9   4  14  11  16   9   4  14  11  16   9   4
      Level 2           5   5  17  35  11   5   5  17  35  11   5   5
                          6   6  54  24   6   6   6  54  24   6   6
                          -------------------------------------------
                            7  19  37  13   7   7  19  37  13   7   7
      Level 3                22  13  20  15   8  22  13  20  15   8
                               15   7  23  17  25  15   7  23  17  25
                                  8   8  26  53  17   8   8  26  53
                                    9   9  81  36   9   9   9  81  36
                                    ---------------------------------
                                     10  28  55  19  10  10  28  55
      Level 4                          31  19  29  21  11  31  19  29
                                         21  10  32  23  34  21
```

each level has $n=5$ rows

5. Growth coefficients

Ex



[Baur, Fellner, Parsons, Tschabold 2016]

- In an n-periodic infinite frieze of positive integers, the difference between the entries in rows $(nk+1)$ & $(nk-1)$ (on the same column) is constant.

- The constants s_k's satisfy the normalized Chebyshev polynomials:

$$s_k = T_k(s_1) \quad \forall\ k$$

where the normalized Chebyshev polynomial is defined by

$$T_0(x) := 2$$

$$T_1(x) := x$$

$$T_k(x) := x\,T_{k-1}(x) - T_{k-2}(x)$$

6. Growth coefficients and bracelets

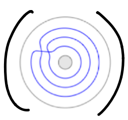
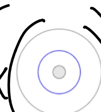
[Musiker-Schiffler-Williams 2011]

The element $x(\mathrm{Brac}_k) \in \mathcal{A}$ associated to a bracelet which crosses itself $k-1$ times is an important element.

Bracelets $Brac_1$, $Brac_2$, and $Brac_3$.

- Certain products of the cluster variables and the bracelets form a nice basis of $\mathcal{A}$ called the bracelets basis.
- The elements $x(\mathrm{Brac}_k)$ satisfy the normalized Chebyshev polynomials

$$x(\mathrm{Brac}_k) = T_k(x(\mathrm{Brac}_1))$$

Ex. $x\Big($  $\Big) = T_3\Big(x\Big($  $\Big)\Big)$

[G., Musiker, Vogel 2016]

In the frieze of Laurent polynomials, the "jump" between level k & $k+1$ is the cluster algebra element which corresponds to the bracelet which crosses itself $k-1$ times.

The constant s_k = # terms in the Laurent expansion of $x(\mathrm{Brac}_k)$

In punctured disk case, this growth factor is always 2.

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

4 1 2 3 2 4 1 2 3 2 4 1 2 3 2 4 1 2 3

3 1 5 5 7 3 1 5 5 7 3 1 5 5 7 3 1 5

2 2 8 17 5 2 2 8 17 5 2 2 8 17 5 2 2 8

3 3 27 12 3 3 3 27 12 3 3 3 27 12 3 3 3 2

4 10 19 7 4 4 10 19 7 4 4 10 19 7 4 4 10

13 7 11 9 5 13 7 11 9 5 13 7 11 9 5 13

9 4 14 11 16 9 4 14 11 16 9 4 14 11 16 9

5 5 17 35 11 5 5 17 35 11 5 5 17 35 11

6 6 54 24 6 6 6 54 24 6 6 6 54 24 6

7 19 37 13 7 7 19 37 13 7 7 19 37 13

22 13 20 15 8 22 13 20 15 8 22 13 20 15

15 7 23 17 25 15 7 23 17 25 15 7 23 1

8 8 26 53 17 8 8 26 53 17 8 8 26

9 9 81 36 9 9 9 81 36 9 9 9 8

10 28 55 19 10 10 28 55 19 10 10 28

31 19 29 21 11 31 19 29 21 11 31 1

21 10 32 23 34 21 10 32 23 34 21

11 11 35 71 23 11 11 35 71 23 1

12 12 108 48 12 12 12 108 48 12

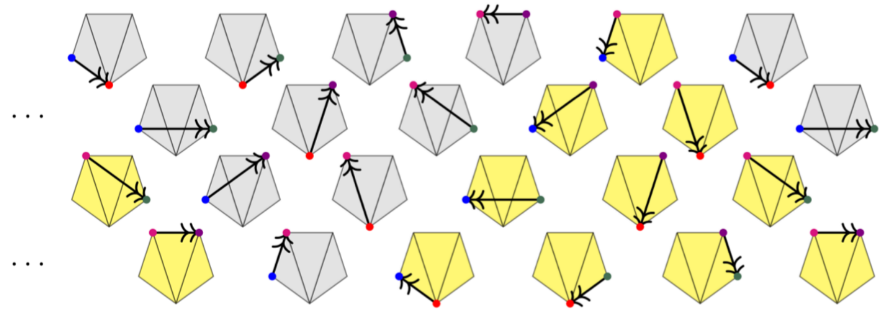
13 37 73 25 13 13 37 73 25 1

40 25 38 27 14 40 25 38

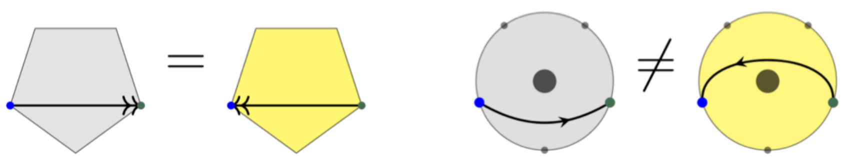
For the punctured disk, every bracelet corresponds to the integer 2

7. Complement symmetry

A Conway-Coxeter frieze is invariant under a glide reflection



In a polygon vs a punctured disk/annulus



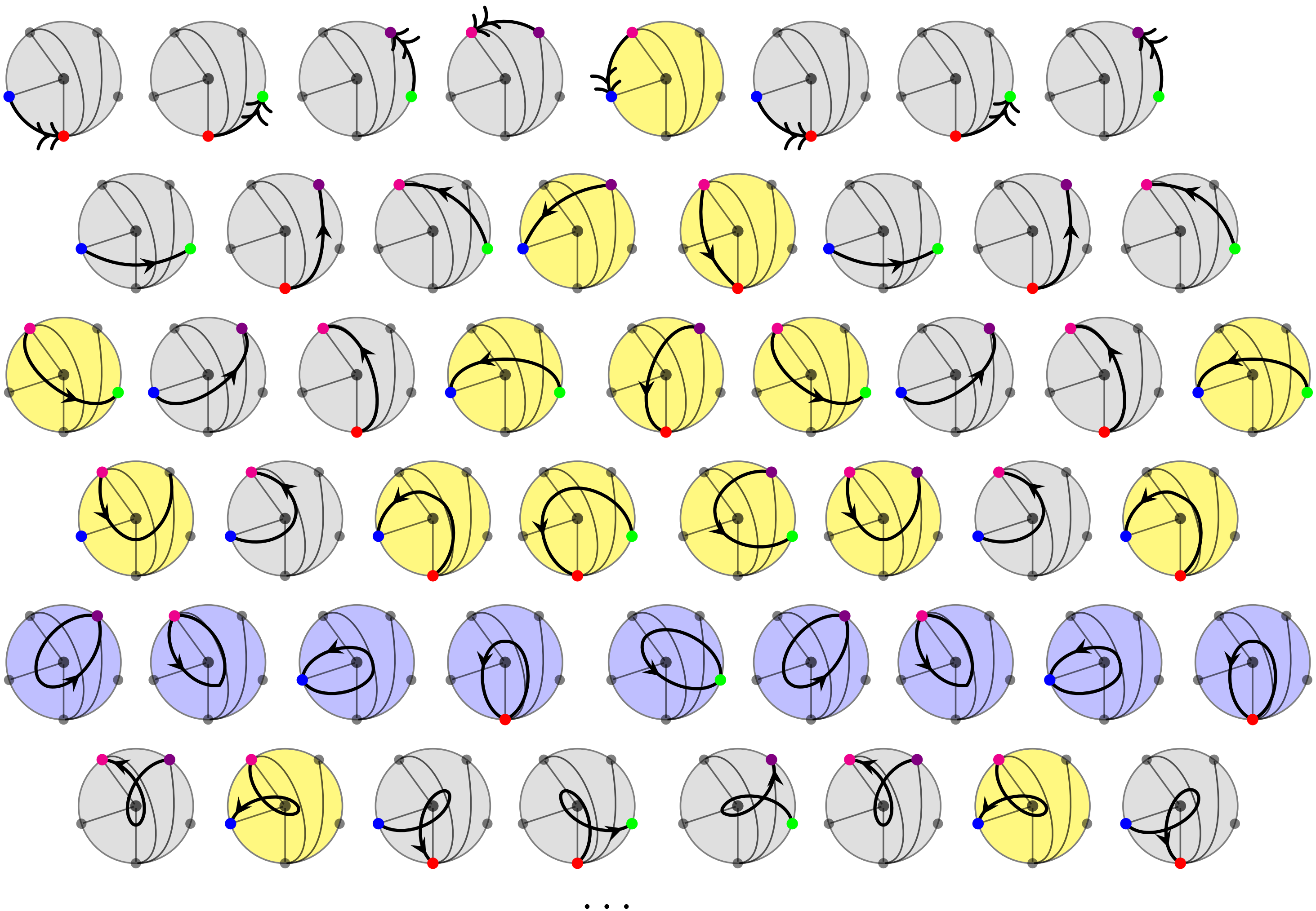
No glide reflection symmetry for infinite friezes,

but there is a "complement symmetry"

Def Let $i<j$ and let γ_k be the arc from i to j with $k-1$ self-crossings. The complementary arc γ_k^C of γ_k is the arc from j to i with $k-1$ self-crossings

γ_1 γ_1^C γ_3 γ_3^C

Complementary arcs in infinite friezes



Arithmetic progressions in frieze patterns from punctured disks (Tschabold)

```
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
  4   1   2   3   2   4   1   2   3   2   4   1   2   3   2
    3   1   5   5   7   3   1   5   5   7   3   1   5   5   7
      2   2   8  17   5   2   2   8  17   5   2   2   8  17
        3   3  27  12   3   3   3  27  12   3   3   3  27  12
        --------------------------------------------------------
          4  10  19   7   4   4  10  19   7   4   4  10  19
           13   7  11   9   5  13   7  11   9   5  13   7  11
              9   4  14  11  16   9   4  14  11  16   9   4
                5   5  17  35  11   5   5  17  35  11   5   5
                  6   6  54  24   6   6   6  54  24   6   6
                  ----------------------------------------------
                    7  19  37  13   7   7  19  37  13   7   7
                     22  13  20  15   8  22  13  20  15   8
                       15   7  23  17  25  15   7  23  17  25
                          8   8  26  53  17   8   8  26  53
                            9   9  81  36   9   9   9  81  36
                            ------------------------------------
                             10  28  55  19  10  10  28  55
                               31  19  29  21  11  31  19
```

Geometric interpretation of the arithmetic progression

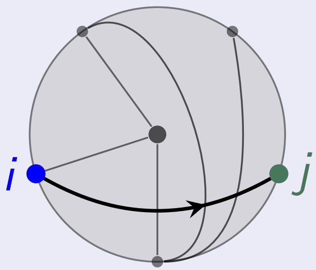
Proposition (G., Musiker, Vogel)

The arc from vertex blue to vertex green with k self-intersections

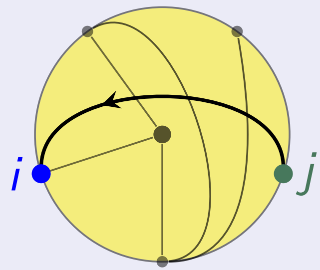
$$=$$

the arc from vertex blue to vertex green with $k-1$ self-intersections

$+$



$+$



Proof: Progression formulas and induction.

Progression formulas

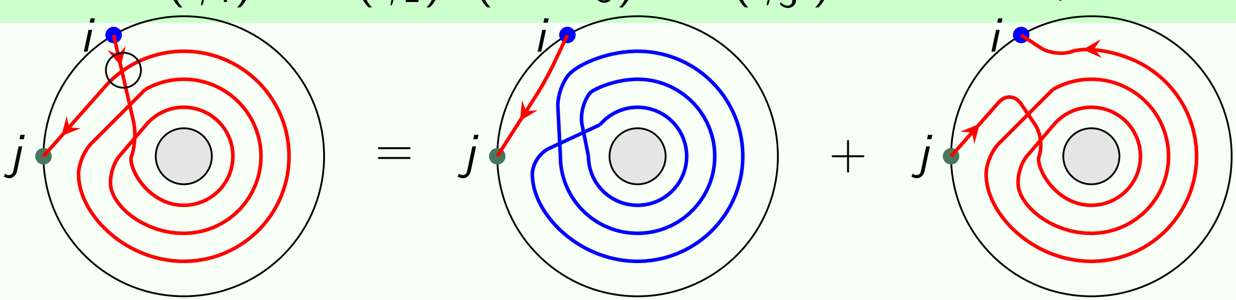
Theorem (G., Musiker, and Vogel)

Let γ_1 be an arc starting and finishing at vertices i and j. For $k = 1, 2, \ldots$ and $1 \leq m \leq k-1$, we have

$$x(\gamma_k) = x(\gamma_m)x(Brac_{k-m}) + x(\gamma^C_{k-2m+1}), \text{ where:}$$

- *for $r \geq 0$, γ^C_{-r} is the curve γ_{r+1} with a kink, so that $x(\gamma^C_{-r}) = -x(\gamma_{r+1})$, and*
- *a* **bracelet** *$Brac_k$ is obtained by following a (non-contractible, non-self-crossing, kink-free) loop k times, creating $(k-1)$ self-crossings.*

$$x(\gamma_4) = x(\gamma_1)x(Brac_3) + x(\gamma^C_3) \text{ for } k = 4,\ m = 1$$



Recent papers on infinite friezes

- Preprint July 2020
  "Infinite friezes and triangulations of annuli"
  – Baur, Çanakçı, Jacobsen, Kulkarni, Todorov

- U of Minnesota REU project 2020
  "Infinite frieze pattern and dissections on annuli"
  – Chen (mentor: Banaian)

A great survey on friezes:

"Coxeter's frieze patterns at the crossroads of algebra, geometry and combinatorics"
- Sophie Morier-Genoud arXiv: 1503.05049

Thank you!

Infinite friezes of cluster algebras from surfaces

Emily Gunawan, Gregg Musiker and Hannah Vogel

Cluster algebras from surfaces

A **cluster algebra** (Fomin – Zelevinsky 2000) is a subring of $\mathbb{Q}(x_1, \dots, x_n)$ with a distinguished set of generators, called **cluster variables**. Cluster variables are produced by an iterative process which can be read off from a directed graph or a skew-symmetric matrix.

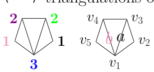
A **cluster algebra from a surface** (Fomin – Shapiro – Thurston 2006) is modeled by a Riemann surface S & marked points.

- arcs (with no self-crossing) $\longleftrightarrow$ cluster variables.
- Cluster algebras of type A, $\tilde{A}$, D, and $\tilde{D}$ arise from polygons, annuli, once-punctured polygons, and twice-punctured polygons, respectively.

Finite frieze patterns

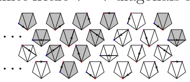
A (Conway – Coxeter) **frieze pattern** is an array such that the top row is a row of 1s and every diamond $\begin{smallmatrix} & b & \\ a & & d \\ & c & \end{smallmatrix}$ satisfies $ad - bc = 1$.

Theorem (Conway – Coxeter 1970s): Finite frieze patterns $\longleftrightarrow$ triangulations of polygons.

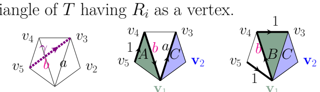


1 1 1 1 1 1 1 1 …
Row 2 … **3** **1** **2** **2** **1** 3 1
2 2 1 3 1 2 2 …
… 1 1 1 1 1 1 1 1

Theorem (Broline – Crowe – Isaacs 1970s): Entries of a finite frieze $\longleftrightarrow$ diagonals of a polygon.



Definition: Let T be a triangulation and γ an arc. Let $R_1, R_2, \dots, R_r$ be the boundary vertices to the right of γ. A **BCI tuple** for γ is a pairwise distinct r-tuple $(t_1, \dots, t_r)$ such that the i-th entry t_i is a triangle of T having R_i as a vertex.



T and γ $\quad (t_1, t_2) = (A, C)$ $\quad (t_1, t_2) = (B, C)$

$1\, b^{-1}\, a \quad + \quad 1\, b^{-1}\, 1$

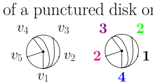
$x_\gamma = \frac{a}{b} + \frac{1}{b}$

Theorem (Caldero – Chapoton 2006): The cluster variables of a cluster algebra from a polygon (type A) form a finite frieze pattern. (Example: type A_2)

… 1 1 1 1 1 1 …
$\frac{1+a+b}{ab}$ a $\frac{1+b}{a}$ $\frac{1+a}{b}$ b $\frac{1+a+b}{ab}$
… $\frac{1+a}{b}$ b $\frac{1+a+b}{ab}$ a $\frac{1+b}{a}$ $\frac{1+a}{b}$ …
1 1 1 1 1 1 1

Infinite frieze patterns

Theorem (Baur – Parsons – Tschabold 2015-2016): Every *infinite* frieze arises from a triangulation of a punctured disk or an annulus/ infinite strip.



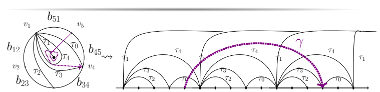
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
4 1 2 3 2 4 1 2 3 2 4 1 2 3 2 4 1 2
3 1 5 5 7 3 1 5 5 7 3 1 5 5 7 3 1 5
2 2 8 17 5 2 2 8 17 5 2 2 8 17 5 2 2
Level 1 3 3 27 12 3 3 3 27 12 3 3 3 27 12 3 3 3

4 10 19 7 4 4 10 19 7 4 4 10 19 7 4 4 1
13 7 11 9 5 13 7 11 9 5 13 7 11 9 5 13
9 4 14 11 16 9 4 14 11 16 9 4 14 11 16
Level 2 5 5 17 35 11 5 5 17 35 11 5 5 17 35 11
6 6 54 24 6 6 6 54 24 6 6 6 54 24

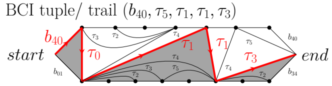
7 19 37 13 7 7 19 37 13 7 7 19 37 13
Level 3 22 13 20 15 8 22 13 20 15 8 22 13 20 1
15 7 23 17 25 15 7 23 17 25 15 7 23
8 8 26 53 17 8 8 26 53 17 8 8 2
9 9 81 36 9 9 9 81 36 9 9 9

10 28 55 19 10 10 28 55 19 10 10 2
Level 4 31 19 29 21 11 31 19 29 21 11 31
21 10 32 23 34 21 10 32 23 34
11 11 35 71 23 11 11 35 71 23
12 12 108 48 12 12 12 108 48 1

Generalized arcs to cluster algebra elements



A BCI tuple/ trail $(b_{40}, \tau_5, \tau_1, \tau_1, \tau_3)$



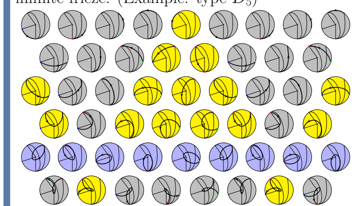
with weight $b_{40}\, x_0^{-1}\, x_1\, x_1^{-1}\, x_3 = \dfrac{1\, x_1\, x_3}{x_0\, x_1}$

The 11 BCI tuples correspond to the 11 terms of

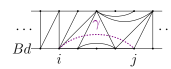
$$x_\gamma = \frac{\mathbf{x_0 x_1 x_4} + 2x_1x_3x_4 + 2x_0^2 + 4x_0x_3 + 2x_3^2}{x_0x_1x_4}$$

Infinite frieze

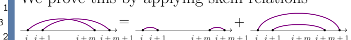
Theorem 1: The cluster algebra elements corresponding to generalized arcs between the same boundary of a punctured disk or annulus form an infinite frieze. (Example: type D_5)



Proof:
Given a triangulation T of the infinite strip, the arc $\gamma(i,j)$ from i to j on a boundary component Bd corresponds to the (i,j)-th entry in the infinite frieze arising from T.



We prove this by applying skein relations



Complementary arcs

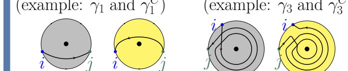
Definition: For $1 \le i, j \le n$ and $k \in \mathbb{N}$, let $\gamma_k(i,j)$ denote the arc that lifts to the infinite strip cover as follows:

$$\gamma_k(i,j) = \begin{cases} \gamma\,(i, j + (k-1)n) & \text{if } i < j \\ \gamma\,(i, j + kn) & \text{if } i \ge j \end{cases}.$$

That is, $\gamma_k(i,j)$ is the generalized arc that starts at i and finishes at j (possibly $i = j$) with $(k-1)$ self-intersections such that the boundary Bd is to the right of the curve as we trace it. Let the arc **complementary** to $\gamma_k = \gamma_k(i,j)$ be defined as

$$\gamma_k(i,j)^C = \begin{cases} \gamma\,(j, i + kn) & \text{if } i < j \\ \gamma\,(j, i + (k-1)n) & \text{if } i \ge j \end{cases}$$

(example: γ_1 and γ_1^C) (example: γ_3 and γ_3^C)



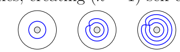
Progression formulas

Theorem 2: Let $\gamma_1 = \gamma$ be an arc from i to j (possibly $i = j$) or a boundary edge from i to $i+1$. For $k \in \mathbb{N}$ and $1 \le m \le k-1$, we have

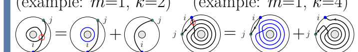
$$x(\gamma_k) = x(\gamma_m)x(Brac_{k-m}) + x(\gamma^C_{k-2m+1}),$$

where:

- for $r \ge 0$, γ^C_{-r} is the curve γ_{r+1} with a contractible kink, so that $x(\gamma^C_{-r}) = -x(\gamma_{r+1})$, and
- a **bracelet** $Brac_k$ is obtained by following a (non-contractible, non-self-crossing, kink-free) loop k times, creating $(k-1)$ self-crossings.



(example: m=1, k=2) (example: m=1, k=4)



$x(\gamma_2) = x(\gamma_1)x(Brac_1) + x(\gamma_1^C)$, $x(\gamma_4) = x(\gamma_1)x(Brac_3) + x(\gamma_3^C)$

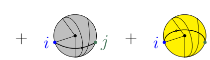
Arithmetic progression

Proposition 1: Let $\mathcal{F}$ be an infinite frieze from a punctured disk. Then

(the arc from i to j with k self-intersections)

=

(the arc from i to j with $k-1$ self-intersections)



Proof: Progression formulas and induction.

Complementary arc differences

Proposition 2: Let $\gamma_1 = \gamma$ be an arc from i to j (possibly $i = j$) or a boundary edge from i to $i+1$. Define $c_k := x\,(\gamma_k) - x\,(\gamma_k^C)$. Let $c_0 := c_1$. Then, for $k \ge 2$, we have

1. $c_k = (Brac_{k-1} - Brac_{k-2})c_1 + c_{k-2}$,
2. $c_k = c_1(1 + \Sigma_{i=0}^{k-1}(-1)^{i+\alpha}s_i)$, where $\alpha = 1$ if k is even and $\alpha = 0$ otherwise.

Note that, if $i = j$, then $c_1 = x(\gamma_1)$.

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