On depth 1 Frege systems

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Overview

1. Bounded depth Frege systems, the separation problem, and Bounded Arithmetic
2. Canonical and interpolations pairs of bounded depth Frege systems
3. Monotone interpolation by game-schemas
4. Two characterizations of the interpolation pair of depth 1 Frege system
5. Generalized monotone Boolean circuits

Bounded depth Frege systems

- $\mathcal{F}_d$ - depth d Frege system = depth d Sequent Calculus for propositional logic
- literals have depth 0, conjunctions and disjunctions of literals have depth 1, etc.
- a sequent $A_1, \ldots, A_n$ is semantically $A_1 \vee \ldots A_n$
- Resolution = $\mathcal{F}_0$
- in $\mathcal{F}_1$ sequents are *sequences of conjunctions*, semantically DNFs

Canonical and interpolations pairs

[Razborov'94] Let P be a propositional proof system. The canonical pair $\mathcal{C}_P$ of P is the pair of disjoint **NP** sets (A, B) where

$$A = \{(\phi, 1^m) : \phi \text{ is satisfiable}\}$$
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[P'03] Let Δ_P be the set of triples (ϕ, ψ, π) where ψ and ϕ are propositional formulas in disjoint variables and π is a P-refutation of $\phi \wedge \psi$. The interpolation pair $\mathcal{I}_P$ is the pair of disjoint **NP** sets (A, B) where

$$A = \{(\phi, \psi, \pi) \in \Delta_{\mathcal{P}} : \ \phi \text{ is satisfiable}\}$$
$$B = \{(\phi, \psi, \pi) \in \Delta_{\mathcal{P}} : \ \psi \text{ is satisfiable}\}.$$

- polynomial separability of the canonical pair of $P =$ automatability of P
- polynomial separability of the interpolation pair of $P =$ feasible interpolation for P

Proposition

1. *The interpolation pair of $\mathcal{F}_0$ (Resolution) is polynomially separable ($\equiv$ feasible interpolation) [Krajíček, 1994]*
2. *For every $d \geq 0$, the canonical pair of the proof systems $\mathcal{F}_d$ is equivalent to the interpolation pair of $\mathcal{F}_{d+1}$. [BPT'14]*

Problem

Is the canonical pair of $\mathcal{F}_0$ (resolution) polynomially separable, i.e., is Resolution weakly automatable?
Equivalently, is the interpolation pair of $\mathcal{F}_1$ polynomially separable?

Definition (**NP** pairs of combinatorial games)

Let G be a combinatorial 2-player game with a concept of a *positional strategy*. Suppose the concept of a *positional winning strategy is in* **NP**. Then we associate a disjoint **NP** pair (A, B) with G defined by

$$A = \{G : \text{Player 1 has a positional winning strategy}\}$$
$$B = \{G : \text{Player 2 has a positional winning strategy}\}.$$

[2] Moreover, it seems that the characterization can be extended to (unbounded depth) *Frege systems*.

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- The canonical and interpolation pairs of $\mathcal{F}_d$ can be characterized by the canonical pairs of certain games [P'19].[2]
- The canonical pair of Resolution (= interpolation pair of $\mathcal{F}_1$) is also characterized by the canonical pair of the *point-line game* [BPT'14].

[2] Moreover, it seems that the characterization can be extended to (unbounded depth) *Frege systems*.

From disjoint **NP** pairs to partial monotone Boolean functions

Suppose that the definition of a game has a parameter $\bar{z} \in \{0,1\}^n$ which may be

- initial position, or
- string that determines the winning positions.

Then we call the same concept with $\bar{z}$ as a variable a *game schema*.

Let $G(\bar{z})$ be a game schema. Then it determines a partial Boolean function:

For $\bar{a} \in \{0,1\}^n$,

$$F(\bar{a}) = 1 \text{ if Player 1 has a positional winning strategy}$$
$$F(\bar{a}) = 0 \text{ if Player 2 has a positional winning strategy}$$

otherwise undefined.

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- If 1s in $\bar{a}$ are the winning positions for Player 1, then the function is monotone.
- For the point-line game where $\bar{z}$ determines the initial position, the function is monotone.
- We can compare game schemas using projections.

Basic example: monotone Boolean circuit

Let $C(\bar{z})$ be a monotone Boolean Circuit and $\bar{a} \in \{0,1\}^n$.

1. (C, a) *as a game*
   - players $\bigvee$ and $\bigwedge$
   - $\bigvee$ wants to reach an input with 1, $\bigwedge$ wants 0
2. $C(\bar{z})$ is a *game schema*

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   - $\bigvee$ wants to reach an input with 1, $\bigwedge$ wants 0
2. $C(\bar{z})$ is a *game schema*

N.B. if a player has a winning strategy then (s)he also has a *positional* winning strategy.

Monotone interpolation by game schemas

Theorem (P'19)

Let $\Phi(\bar{x}, \bar{z})$ and $\Psi(\bar{y}, \bar{z})$ be two CNF formulas whose only common variables are $\bar{z}$ and they occur in Φ only positively and in Ψ only negatively. Let an $\mathcal{F}_d$ refutation of $\Phi(\bar{x}, \bar{z}) \wedge \Psi(\bar{y}, \bar{z})$ be given.

Then it is possible to construct in polynomial time a depth $d + 1$ game schema $S(\bar{z})$ such that for every assignment $\bar{a} : \bar{z} \to \{0, 1\}$,

- *if $\Phi(\bar{x}, \bar{a})$ is satisfiable, then Player I has a positional wining strategy in $S(\bar{a})$ and*
- *if $\Psi(\bar{y}, \bar{a})$ is satisfiable, then Player II has a positional wining strategy in $S(\bar{a})$.*

Depth 2 games and game schemas

Definition (Depth 2 game)

Two players alternate filling a $2 \times m$ matrix

$$\begin{pmatrix} u_1 & u_2 & \dots & u_{m-1} & u_m \\ v_1 & v_2 & \dots & v_{m-1} & v_m \end{pmatrix}$$

in the order $u_1 u_2 \dots u_{m-1} u_m v_m v_{m-1} \dots v_2 v_1$, $u_i, v_j \in A$.

Legal moves (and positional strategies) are

- for u_i, determined by i, u_{i-1},
- for v_i, determined by i, u_i, and v_{i+1}.

Player 1 wins if $v_1 \in W$, otherwise Player 2.

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Definition (depth 2 game schema)

same, except W is not fixed.

equivalent definition

1. In the 1st round players alternate constructing a monotone Boolean circuit C.
2. In the 2nd round they play the game determined by C and an input $a \in \{0,1\}^n$.

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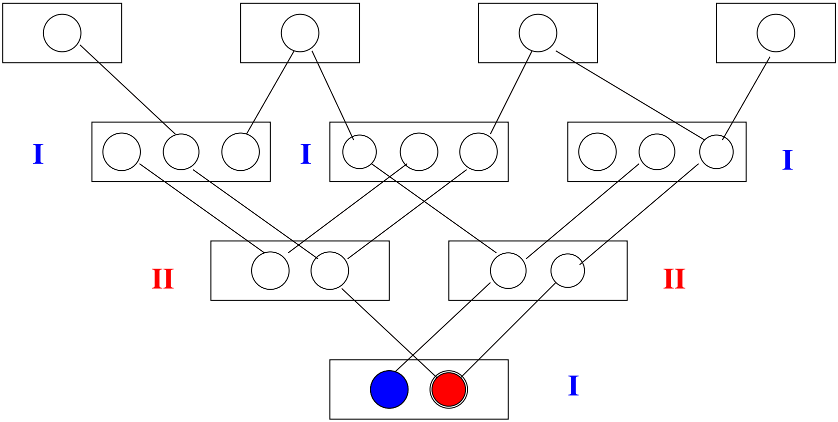
In more detail

- the circuit they construct will be a *straight-line program*
- at step i, the player can choose an instruction from some fixed set I_i (e.g., I_i can be $\{y_k := x_l \wedge y_p, \quad y_k := y_q \vee y_r\}$)
- they construct the program in the *reverse order*.

The point-line game

- DAG (G, E) (nodes and arrows)
- nodes labeled B and W (the players)
- for every node A, a set P_A (points of A)
- for every arrow $A \to B$, a partial matching between P_A and P_B (lines)
- one source
- each sink has exactly one point
- game starts with black and white pebbles on the points of the source
- players pick arrows and move pebbles along the lines
- the winner is whose pebble ends up in a sink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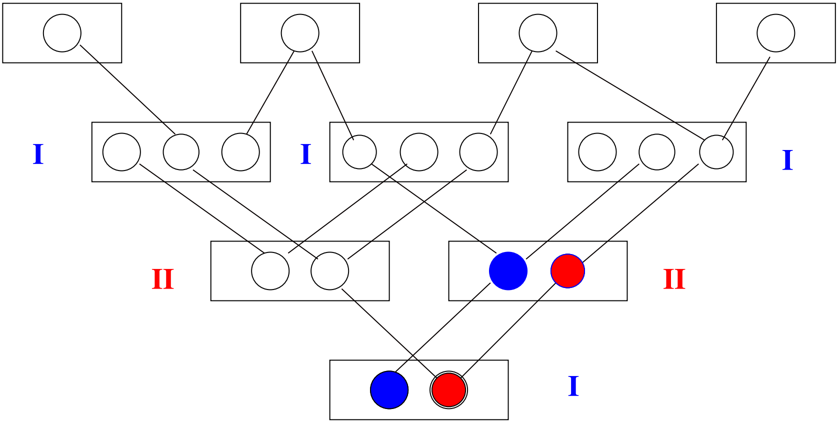
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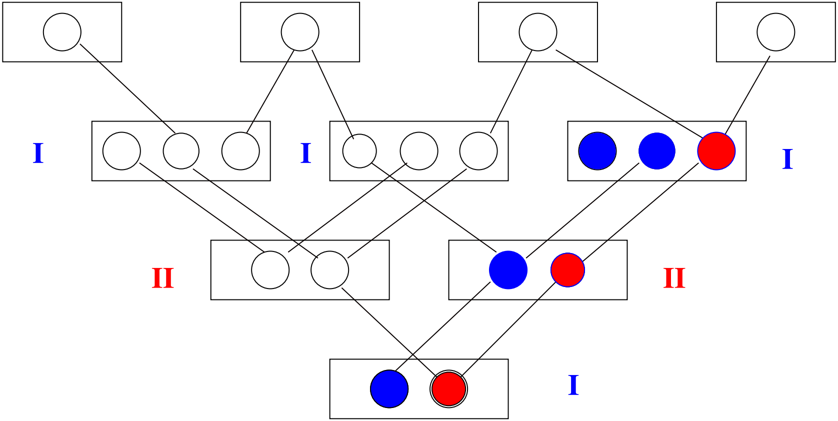
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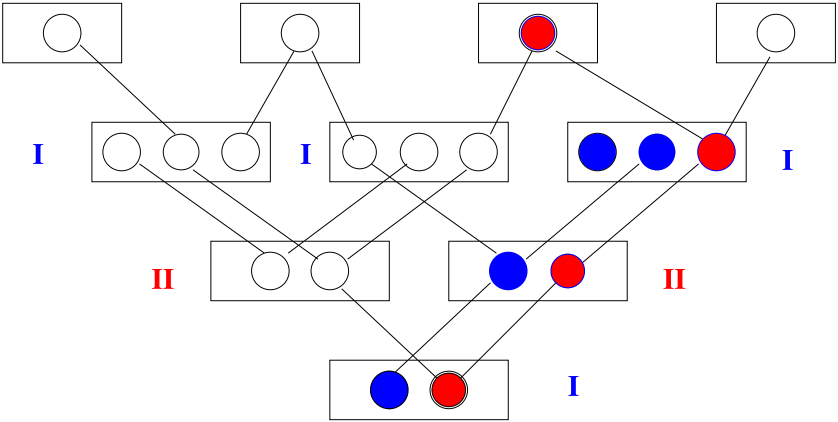
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A different way of playing the point-line game

- do not move pebbles, only construct the path
- after reaching a leaf, determine the color by following the lines back

Proposition

Point-line game schemas and depth 2 game schemas are reducible to each other using projections and at most polynomial increase of the size of the games.

Proof (only the easy direction - simulation of point-line games by depth 2 games).

We will use the definition of depth 2 games based on circuits.

Let a point-line game G be given. Think of the points as variables. When a player decides to go from node P to node Q where $p_1 \to q_1, \ldots, p_k \to q_k$ is the matching of lines, then in the depth 2 game the player will play

$$q_1 := p_1, \ldots, q_k := p_k$$

Furthermore, we may assume w.l.o.g. that there is a unique sink in the point-line game. The variable assigned to the point y in it will be the output variable of the constructed circuit.

The resulting circuit will contain instructions

$$r_2 := r_1, r_3 := r_2, \ldots, y := r_{m-1},$$

where $r_1, \ldots, r_{m-1}, y$ is the path from point r_1 in the input node to y, the point in the sink.

Generalized monotone Boolean circuits

Monotone Boolean circuits as a calculus

Axioms: $0, 1, x_1, x_2, \ldots$

Rules:

$$\frac{f \quad g}{f \wedge g} \qquad \frac{f \quad g}{f \vee g}$$

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Generalized monotone Boolean circuits as a calculus

$+$ substitution rule:

$$\frac{f(y_1, \ldots, y_r)}{f(z_1, \ldots, z_r)}$$

where $y_1, \ldots, y_r$ are distinct variables and $z_1, \ldots, z_r$ are variables or constants.

The *size* of the (generalized) circuit is the *length of the derivation* (not counting substitutions).

Proposition

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Proof idea.

By induction, construct generalized circuits for all nodes of the given point-line game schema. Nodes of Black (White) will correspond to $\vee$ (to $\wedge$). Substitutions are determined by matchings between the nodes. □

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Remark. If the graph of the point-line game is a tree, we can eliminate substitutions and get a monotone Boolean formula.

Problems

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2. Extend the calculus defining generalized monotone circuits to include information about *positional* winning strategies.
3. Use 2. to prove a lower bound on point-line game schemas representing a partial monotone function.

WPHP has quasipolynomial $\mathcal{F}_1$ proofs

Corollary

1. *Depth 2 game schemas are exponentially more powerful than monotone Boolean circuits.*
2. *Generalized monotone circuits are exponentially more powerful than monotone Boolean circuits.*

Corollary

For every n, there exists an $m = n^{O(n)}$ and a formula $\phi(\bar{x}, \bar{y})$ where $\bar{x}$ occur negatively in ϕ, $|\bar{x}| = n$, such that every monotone circuit $C(\bar{x})$ such that for every $\bar{a} \in \{0,1\}^n$,

$$C(\bar{a}) = 1 \text{ if } \phi(\bar{a}, y) \text{ has a } \mathcal{F}_1 \text{ refutation of size } \le m$$
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Problem

Can we prove the same for Resolution?

Thank you