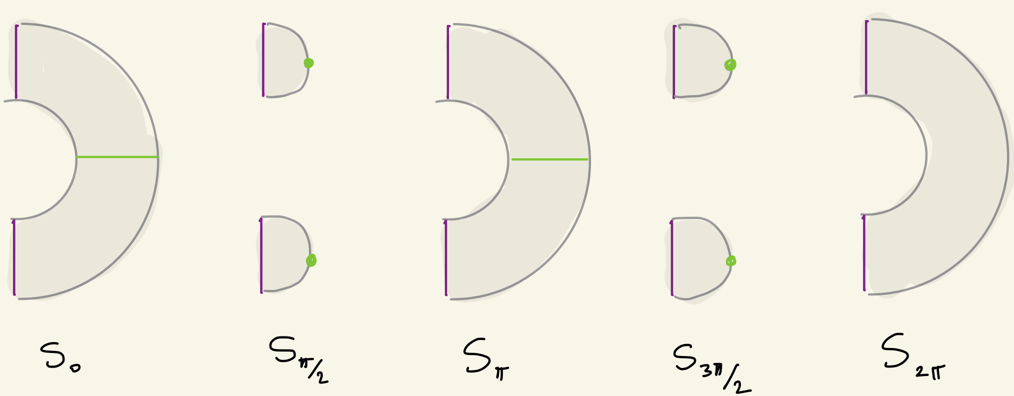
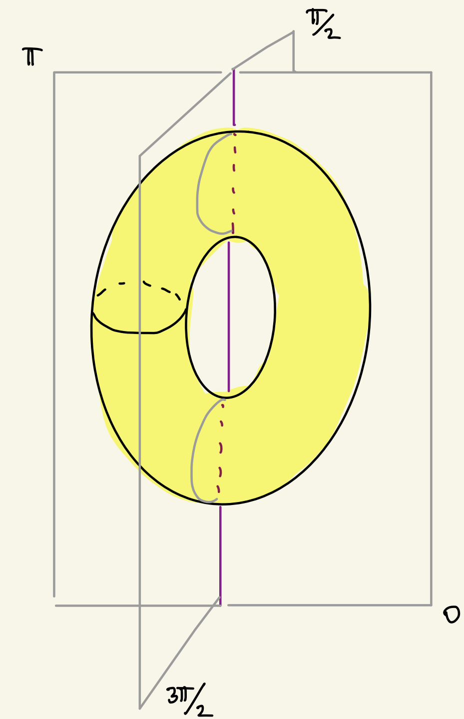
Bordered Contact Invariant

Vera Vértesi (University of Vienna)

A. Alishahi, V. Földvári, K. Hendricks, J. Licata, I. Petkova





MOTIVATION

Contact invariant in Heegaard Floer homology

M^3 / diffeo $\xleftarrow{1:\text{many}}$ Heegaard decomposition / stab $\rightsquigarrow$ $HF(M^3)$ doubly graded vspace

M^3 / diffeo $\xleftrightarrow{1:1}$ Heegaard decomposition

$M^3 = U \cup V$

handlebodies

invariant under stab

contact Heegaard decomposition

ξ ctct str / contacto $\xleftarrow{1:\text{many}}$ open book decomposition (B, π) / o.b. stab $\rightsquigarrow$ $c(\xi)$

ξ ctct str / contacto $\xleftrightarrow{1:1}$ open book decomposition (B, π)

$\in$

invariant under o.b stab

MOTIVATION

Properties:

- Ozsváth - Szabó: ξ Stein fillable $\Rightarrow c(\xi) \neq 0$
- Ozsváth - Szabó: ξ overtwisted $\Longrightarrow c(\xi) = 0$
- Ghiggini - Honda - Van Horn-Morris:

  ξ contains Giroux torsion $\Longrightarrow c(\xi) = 0$

All local reasons ($\Leftarrow$ Honda - Kazez - Matić: partial open books)

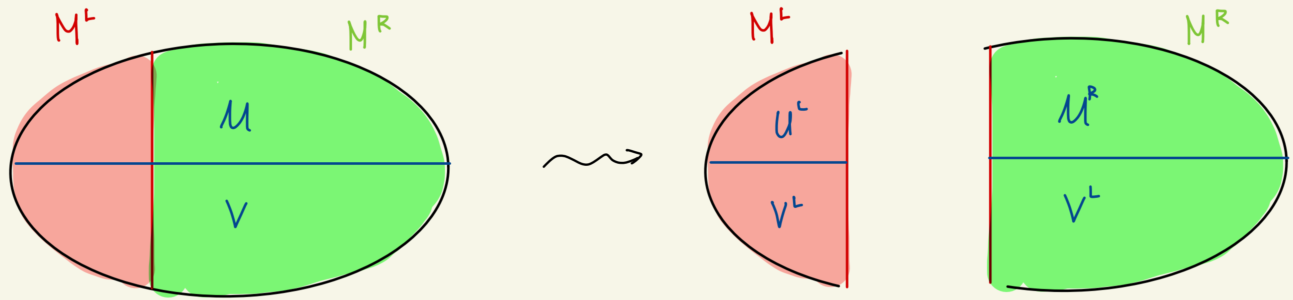
Conjecture (maybe: Ghiggini):

ξ contains $\frac{1}{2}$ Giroux torsion along a separating torus

$\Longrightarrow c(\xi) = 0$ (really global $c(T^3, \eta) \neq 0$)

MOTIVATION

bordered Heegaard Floer homology

$M =$ 

Σ ↖ separating surface

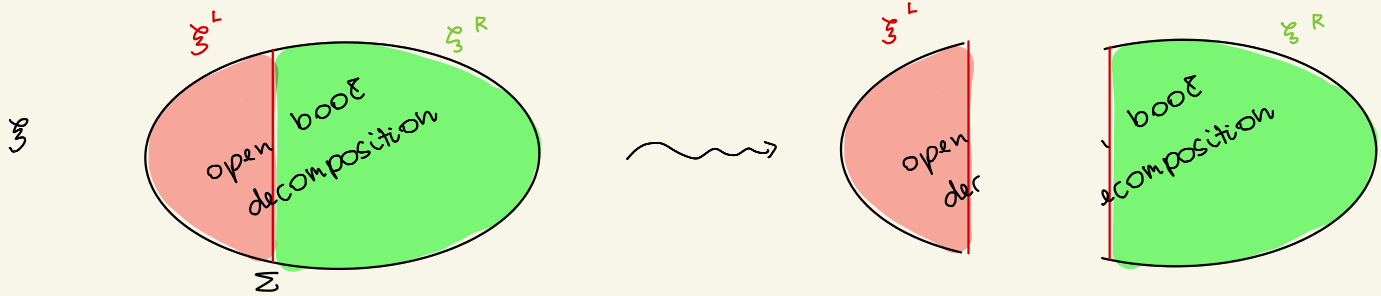
bordered 3-manifolds

(M, Σ) bordered 3-mfd $\xleftarrow{1\,:\,\text{many}}$ $\xleftrightarrow{1\,:\,1}$ bordered Heegaard decomp / stab $\xrightarrow[\text{indep.}]{}$ $CFA(M)$, $CFD(M)$

A_∞-modules over an A_∞-algebra associated to Σ

& $CFA(M^L, \Sigma) \,\tilde{\otimes}\, CFD(M^R, -\Sigma) = CF(M) \longleftarrow H_*(CF(M)) = HF(M)$

MOTIVATION



ctct 3-mfds w/ foliated bdry

& foliated open books

$(\xi, \mathcal{F}_\xi) \xleftarrow[\text{1: many}]{\text{Licata, V}}$ foliated open book / stab $\xrightarrow{\text{AFHLPV}}$ $c_A(\xi, \mathcal{F}_\xi)$, $c_D(\xi, \mathcal{F}_\xi)$

$\longleftrightarrow$

contact str
w/ char. foliation on Σ

& $c_A(\xi^L, \mathcal{F}_\xi) \otimes c_D(\xi^R, \mathcal{F}_\xi) = c(\xi)$

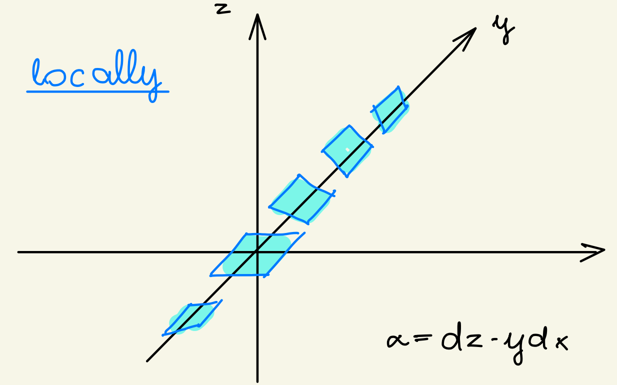
Contact Structures

$(M^3, \xi^2 = \ker \alpha)$ non-integrable 2-plane distribution

$$\Updownarrow$$

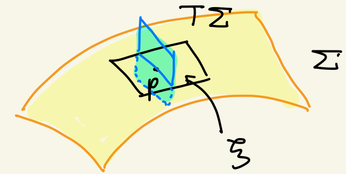
$$\alpha \wedge d\alpha > 0$$

locally



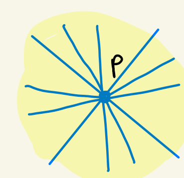
$\Sigma \hookrightarrow (M^3, \xi) \rightsquigarrow \alpha|_\Sigma$ induces a foliation:

characteristic foliation $\mathcal{F}_\xi$



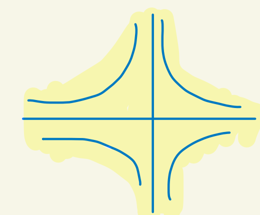
Fact (Giroux): $d(\alpha|_\Sigma) \neq 0$ at singular pts

$\Rightarrow$ the isolated singularities of $\mathcal{F}_\xi$ are either



elliptic

or



hyperbolic

but no

center

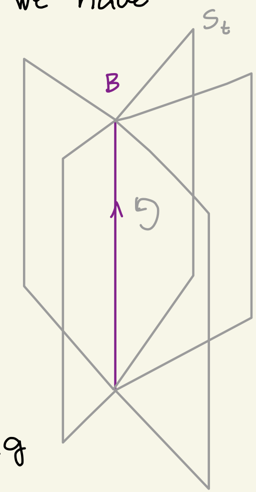
Thm (Giroux): $\mathcal{F}_\xi$ determines ξ in a nbhd of Σ.

Open Book Decompositions

M^3 $\qquad (B, \pi)$

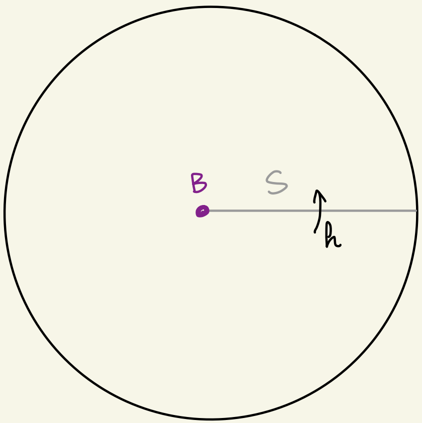
$B^1 \hookrightarrow M \qquad \pi : M \setminus B \longrightarrow S^1$ fibration, $S_t := \pi^{-1}(t)$ & near B we have

binding $\qquad$ page



$\Rightarrow$ We can think of $M \setminus B$ as a mapping cylinder of (S, h):

$$S \times I \,/\, (x,1) \sim (h(x), 0)$$



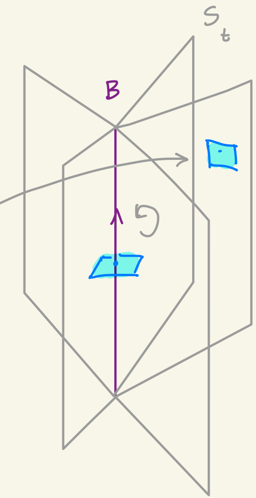
& we obtain M by further identifying

$$(x,t) \sim (x,t') \qquad \begin{array}{l} x \in \partial S \\ t, t' \in I \end{array}$$

COMPATIBILITY

(B,π) **supports** $\xi = \ker \alpha$ if
- $\alpha > 0$ on B
- $d\alpha > 0$ on S_t

ξ is 'nearby' $T S_t$

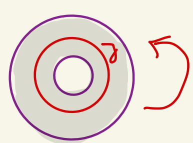


examples 1) & 2) both supports ξ_{st}

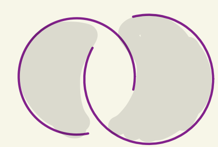
Giroux correspondence:

$$\text{ctct str } \xi \,/\, \text{isotopy} \quad \xleftrightarrow{1:1} \quad \text{open book decomposition } (B,\pi) \,/\, \text{o.b. stab}$$

local operation, connect sum w/



$h = D_\gamma$ on



$\subseteq \mathbb{R}^3$

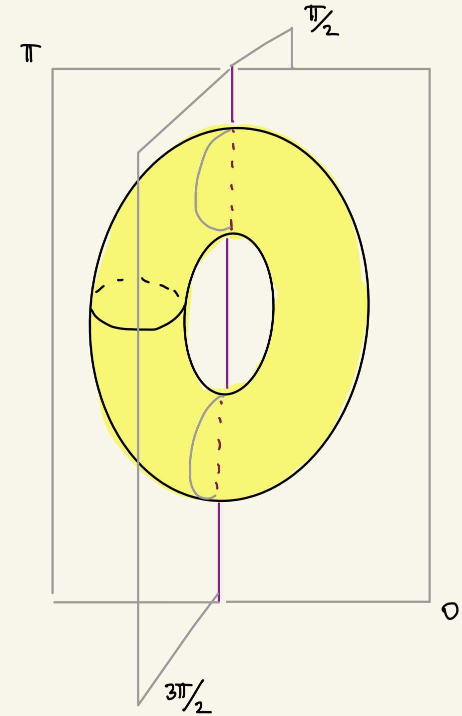
FOLIATED OPEN BOOK — EXAMPLE

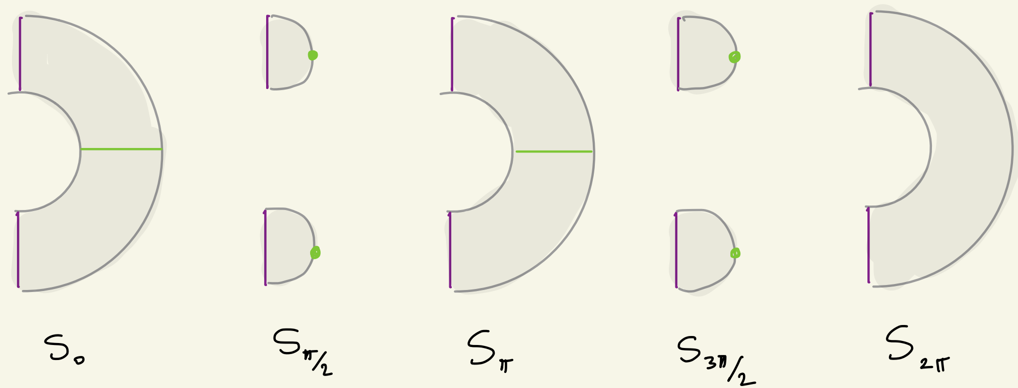
M = solid torus $\subseteq \mathbb{R}^3$

$B = \{z\text{-axis}\} \cap M$

π = angle : $M \setminus B \longrightarrow S^1$

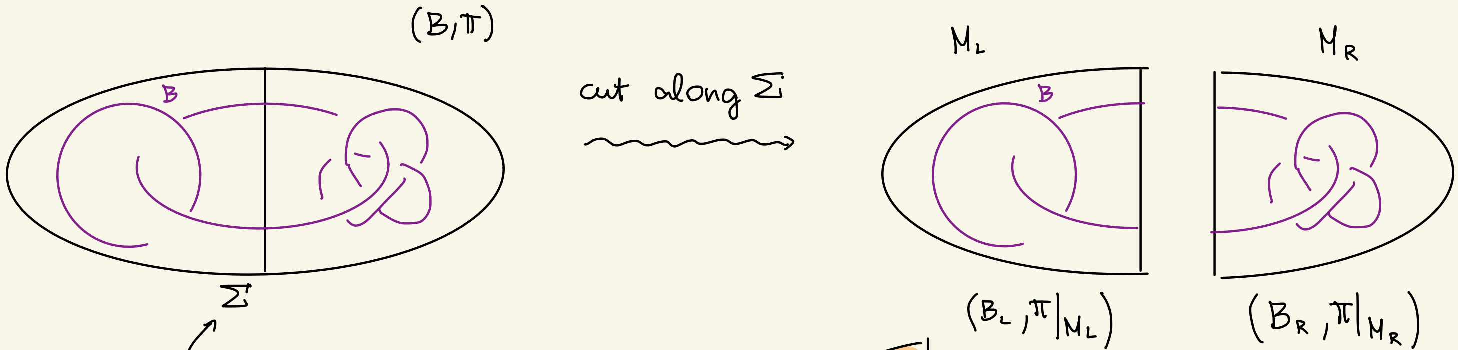
$\Rightarrow S_t = \pi^{-1}(t)$ changes in time





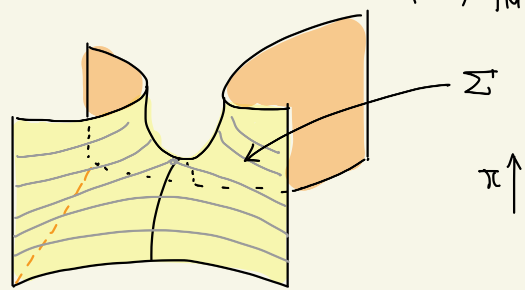
$\pi|_{\partial M}$ gives a (singular) foliation $\mathcal{F}_\pi$ for ∂M

FOLIATED OPEN BOOKS - IDEA



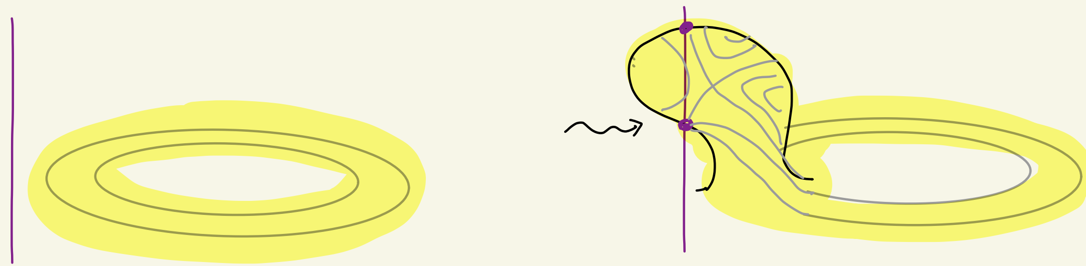
in "good position":

small isotopy
- $B \pitchfork \Sigma$
- $\pi|_\Sigma$ is Morse



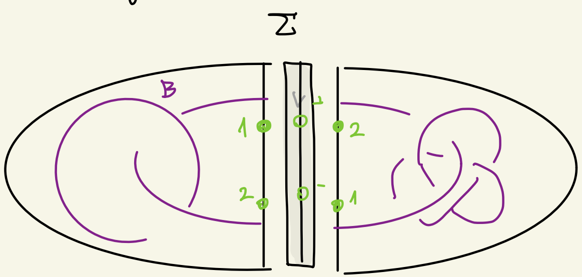
- level sets of $\pi|_\Sigma$ have no closed components

"big isotopy":



FOLIATED OPEN BOOKS

modify π near Σ by introducing a canceling pair of index 1 & 2 critical pts. for every critical pt of $\pi|_{\partial M}$



Def: (B,π) is a foliated open book for $(M,\partial M)$ if:

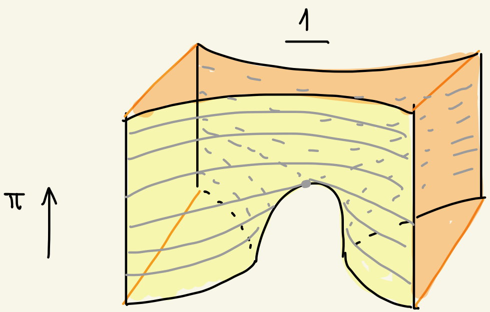
- $(B^1,\partial B) \hookrightarrow (M^3,\partial M)$
- $\pi: M\setminus B \longrightarrow S^1$ $\quad$ S^1-valued Morse function w/ $\quad \to \mathrm{Crit}(\pi) = \mathrm{Crit}(\pi|_{\partial M})$

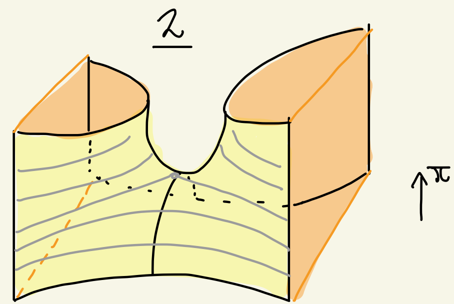
  $\to$ the level sets of $\pi|_{\partial M}$ have closed leaves

FOLIATED OPEN BOOKS

$\Rightarrow$ $\pi|_{\partial M}$ has no min / max $\Rightarrow$ indices of critical pts of $\pi|_{\partial}$ are 1

$\Rightarrow$ ———— " ———— π are

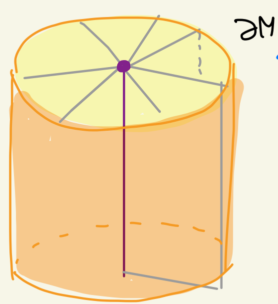




hyperbolic points

$\Rightarrow (\pi|_{\partial M})^{-1}(t)$ foliates ∂M : $\mathcal{F}_\pi$

near $B \cap \partial M$:

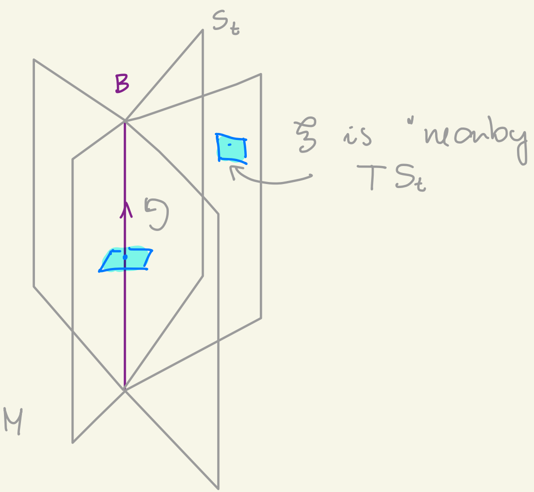


elliptic pt

COMPATIBILITY

(B,π) **supports** $\xi = \ker \alpha$ on $(M, \partial M)$ if

- $\alpha > 0$ on B
- $d\alpha > 0$ on $S_t = \pi^{-1}(t)$
- $\mathcal{F}_\xi \underset{\text{equiv.}}{\overset{\text{top}}{\sim}} \mathcal{F}_\pi$ ← this just says that ξ is "nearby" TS_t near ∂M



about $\mathcal{F}_\xi \underset{\text{equiv.}}{\overset{\text{top}}{\sim}} \mathcal{F}_\pi$:

We would like to say "the same" or $\exists$ small ~~diffeo~~ homeo isotopic to id that takes $\mathcal{F}_\xi$ to $\mathcal{F}_\pi$

but !

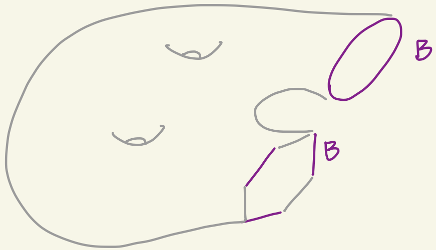
We can never have diffeo near the hyperbolic pts $\left(\operatorname{div}(\mathcal{F}_\xi) \neq 0 \leftrightarrow \operatorname{div}(\mathcal{F}_\pi) = 0\right)$

Thm (Licata-V)

$$\text{ctct str w/ fixed } \mathcal{F}_\xi \Big/ \substack{\text{isotopy} \\ \text{rel } \partial M} \quad \overset{1:1}{\longleftrightarrow} \quad \substack{\exists \text{ OB w/} \\ \text{fixed } \pi|_{\partial M}} \Big/ \text{internal stab.}$$

Abstract Foliated Open Books – Idea

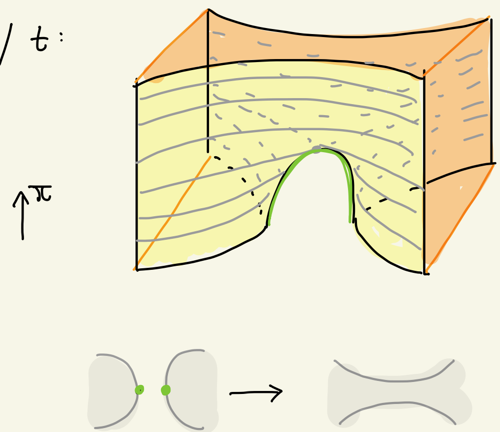
$\pi^{-1}(t) = S_t$

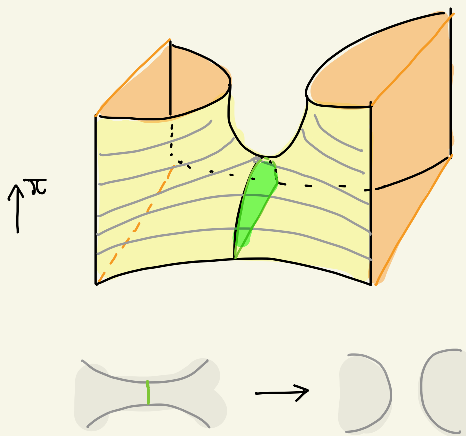


$\longleftarrow (\pi|_{\partial M})^{-1}(t) =$ leaves of $\mathcal{F}_\pi$

! no circles !

& S_t changes w/ t:

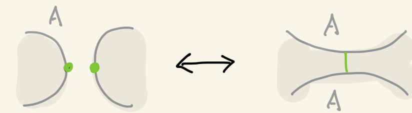




Abstract Foliated Open Books

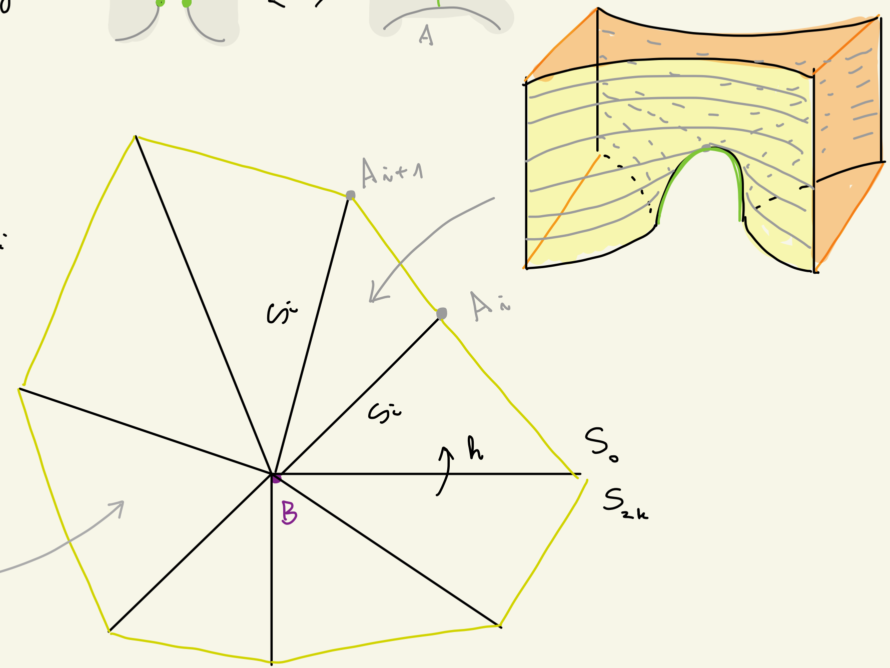
Def: an <u>abstract foliated open book</u> is $(S_0, S_1, \dots, S_{2k}, h)$ w/

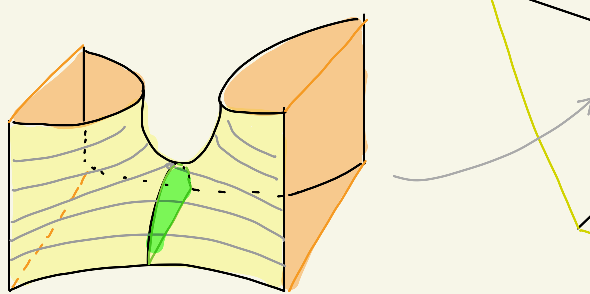
- S_i surfaces w/ polygonial bdry $= B \cup A_i$ w/ no circles in A_i
- S_{i-1} & S_i are related by



- $h : S_{2k} \xrightarrow{\cong} S_0$ fixing B

$\rightsquigarrow$ Can reconstruct $(M, \partial M, \mathcal{F})$:

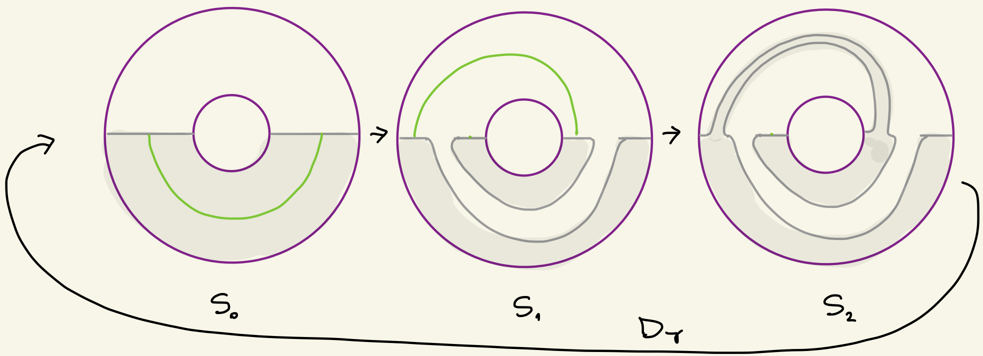




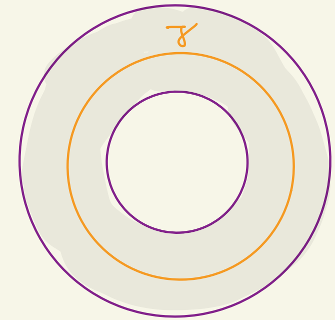
Abstract Foliated Open Books - Examples

1) take any abstract open book (S, h)

& $2k$ subsets $S_i \subseteq S$ w/ $h(S_{2k}) = S_0$

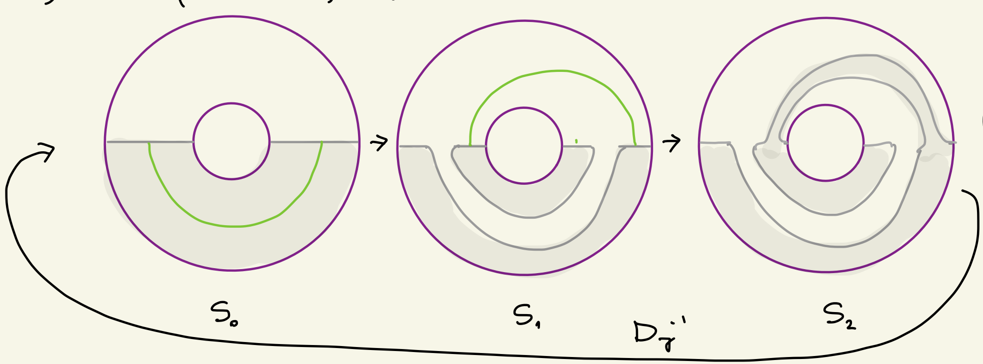


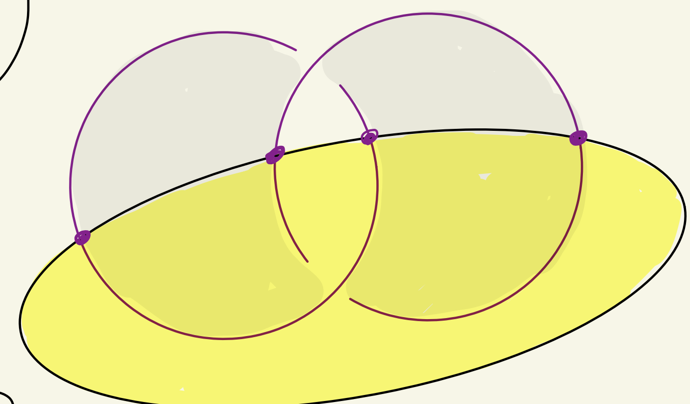
$S =$ annulus



$h = D_\gamma$
$\shortparallel$
Dehn twist along γ

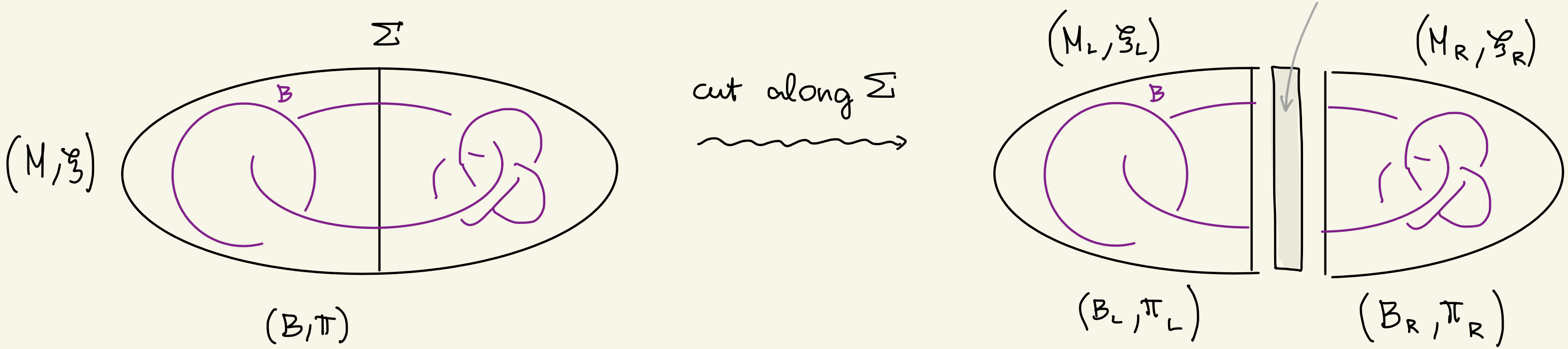
2) for $(\text{anulus}, D_\gamma^{-1})$





← neighbourhood of OT disc

PROPERTIES - CUTTING

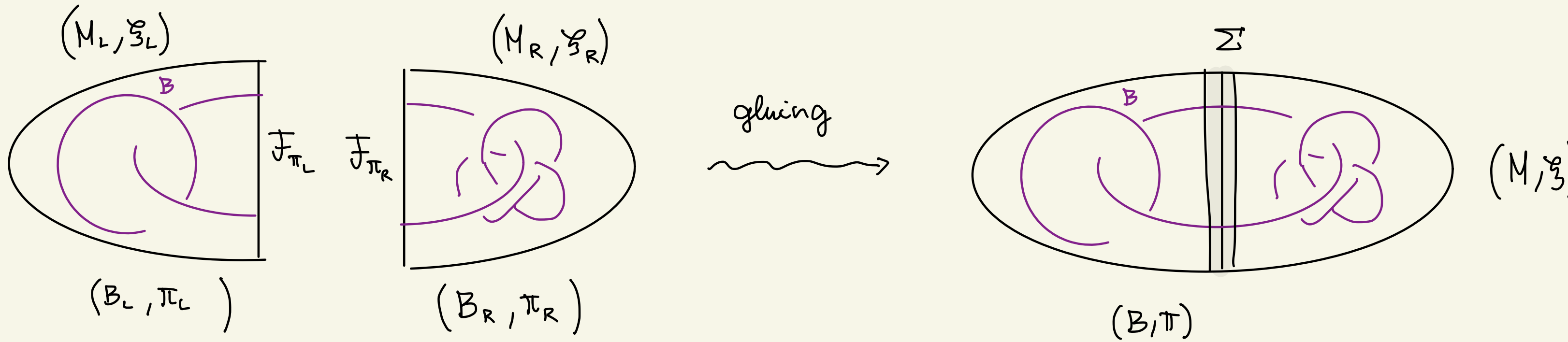


If
- $B \pitchfork \Sigma$
- $\pi|_\Sigma$ is Morse
- the level sets of $\pi|_\Sigma$ has no circles

$\Rightarrow$ $(M_L, \xi_L = \xi|_{M_L})$ is supported by $(B_L = B \cap M_L, \pi_L = \pi'|_{M_L})$ & $(M_R, \xi_R = \xi|_{M_R})$ is supported by $(B_R = B \cap M_R, \pi_R = \pi'|_{M_R})$

after modifying π near Σ & then cutting out a nbhd of Σ

PROPERTIES - GLUING



If $\partial M_L \xrightarrow[\text{rev}]{\text{or.}} \partial M_R$

$\mathcal{F}_{\pi_L} \longrightarrow -\mathcal{F}_{\pi_R}$

$\Longrightarrow$ there is a standard piece on $\Sigma \times I$.
(depends on $\mathcal{F}$)

s.t. $M = M_L \cup \Sigma \times I \cup M_R$

has an open book (B, π) w/

$(B,\pi)|_{M_L} = (B_L, \pi_L)$ supports $\xi|_{M_L} = \xi_L$

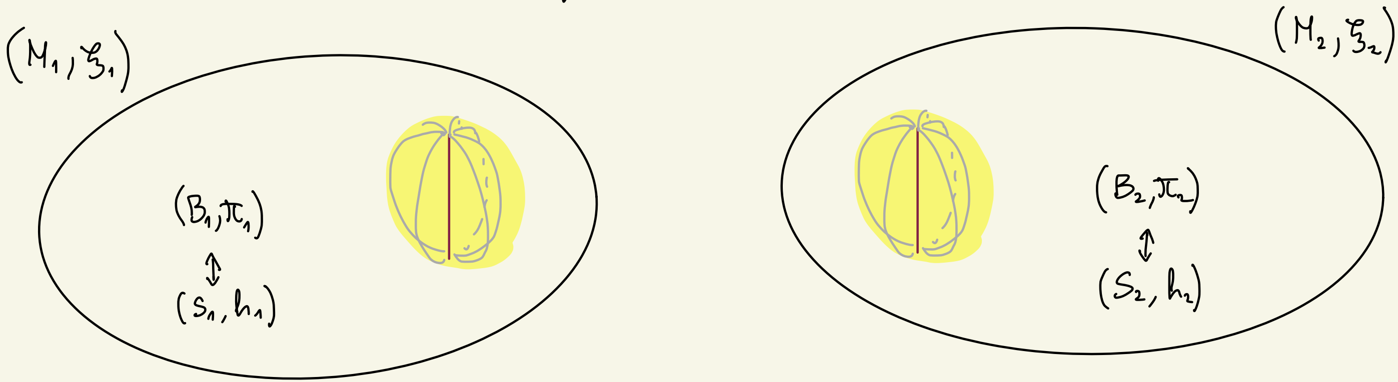
&

$(B,\pi)|_{M_R} = (B_R, \pi_R)$ supports $\xi|_{M_R} = \xi_R$

APPLICATIONS - ADDITIVITY OF THE SUPPORT NORM

(M^3,ξ) closed contact : $sn(\xi) = \min\{-\chi(S)-1 : (S,h) \text{ supports } \xi\}$

Question: How does $sn(\xi)$ behave under connected sum ?



$\rightsquigarrow$ the pages for the o.b for $(M_1 \# M_2, \xi_1 \# \xi_2)$ after gluing are $S_1 \natural S_2$

$\Longrightarrow$ $sn(\xi) \leqslant sn(\xi_1) + sn(\xi_2)$

Question: Is $sn(\xi) = sn(\xi_1) + sn(\xi_2)$?

Applications - Additivity of the Support Norm

<u>Example (Özbagci)</u>: M $\mathbb{Z}$-homology sphere, ξ overtwisted

$$(M,\xi)\#(S^3,\xi_{-1/2}) \cong (M,\xi)$$

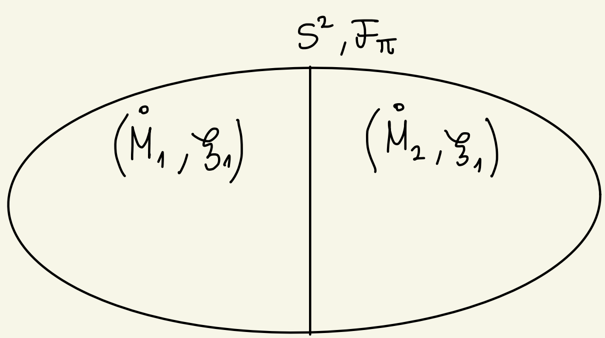
$d_3 = -\frac{1}{2}$ compute

but $\mathrm{sn}(S^3,\xi_{-1/2}) \neq 0$

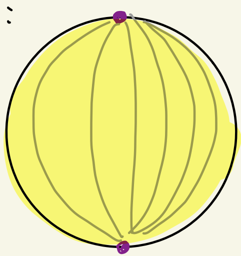
<u>Thm (V)</u>: (M_1,ξ_1) & (M_2,ξ_2) tight $\Longrightarrow$ $\mathrm{sn}(\xi) = \mathrm{sn}(\xi_1) + \mathrm{sn}(\xi_2)$

$(M,\xi) =$

(B,π)



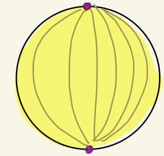
if $\mathcal{F}_\pi$ on S^2 is:

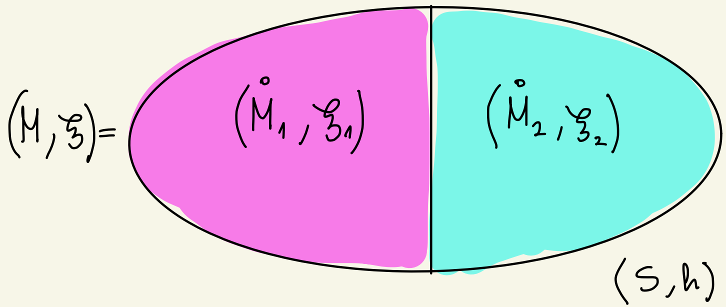


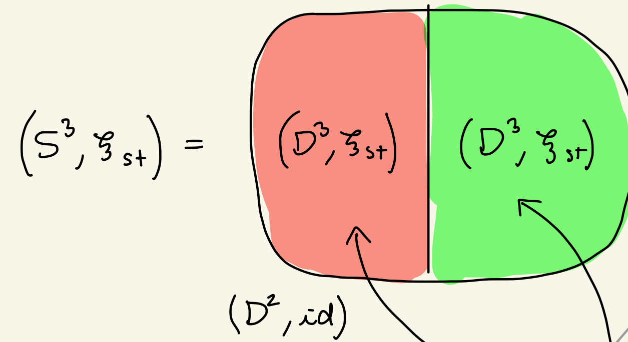
$\Rightarrow \checkmark$

APPLICATIONS - ADDITIVITY OF THE SUPPORT NORM

Why are we done?

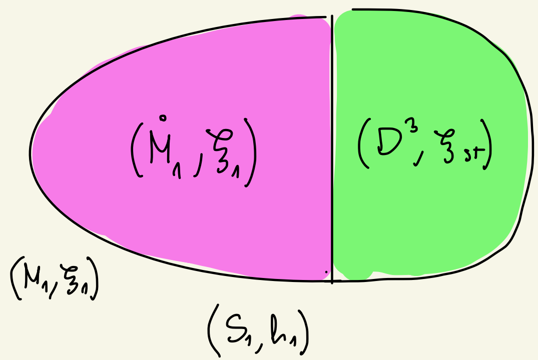


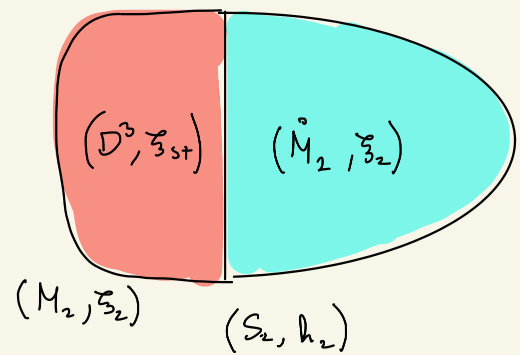


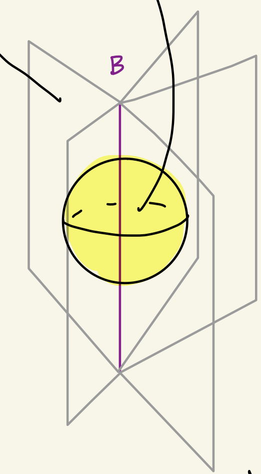


cut both $\rightsquigarrow$

& glue

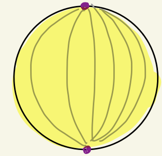




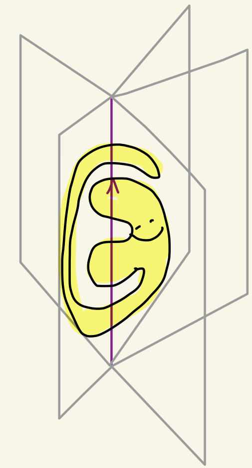


$\Rightarrow \quad S = S_1 \natural S_2 \quad \Longrightarrow \quad \chi(S) = \chi(S_1) + \chi(S_2) - 1 \quad \Longrightarrow \quad sn(\xi) \geqslant sn(\xi_1) + sn(\xi_2)$

APPLICATIONS - ADDITIVITY OF THE SUPPORT NORM

But! We cannot alway simplify to .

instead:

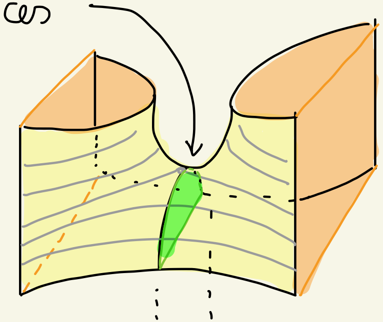


ξ_{st}

Lemma: Any foliation $\mathcal{F}$ (coming from a tight ξ) can be realised in (D^2, id)

idea of Proof: tight $\Rightarrow$ the graph on elliptic pts formed by these separatrices is a tree.

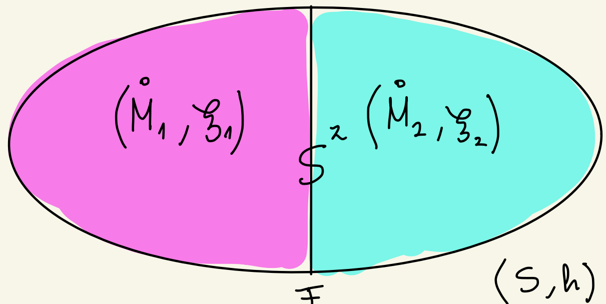
$\rightsquigarrow$ can do induction...



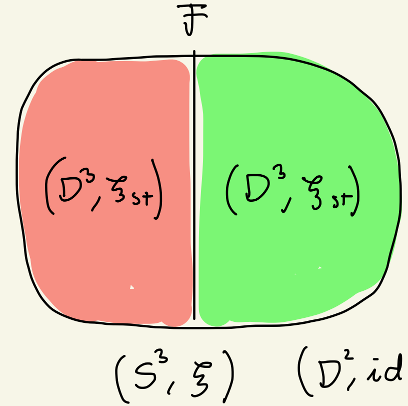
elliptic point

APPLICATIONS - ADDITIVITY OF THE SUPPORT NORM

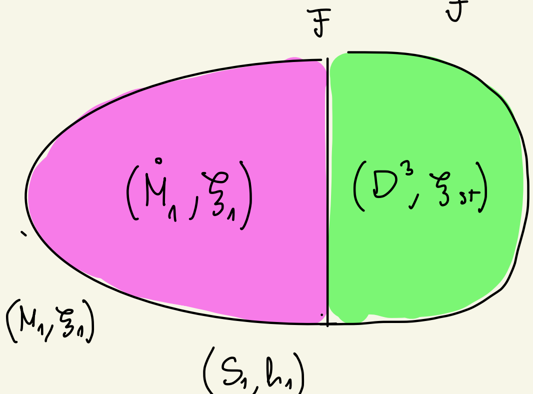
Proof: Given $(M,\xi) =$

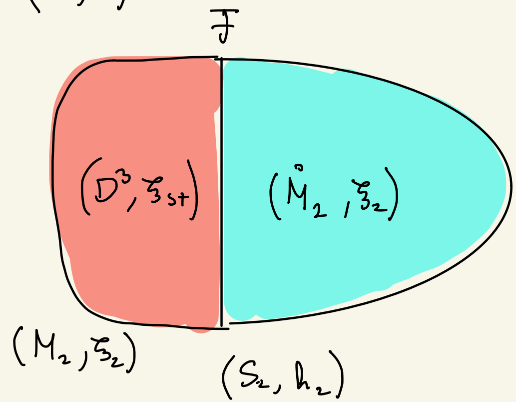


Lemma $\rightsquigarrow$



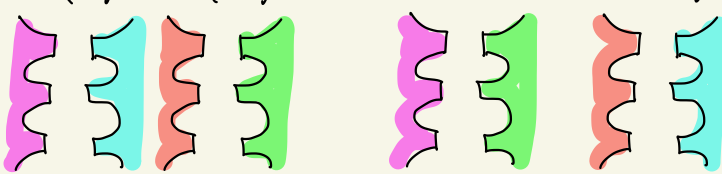
cut both $\rightsquigarrow$ & glue





$$\Rightarrow \quad \chi(S) + \chi(D^2) = \chi(S_1) + \chi(S_2) \qquad \Longrightarrow \quad sn(\xi) \geq sn(\xi_1) + sn(\xi_2)$$

pages



∎

CONTACT INVARIANTS (work in progress)

(M, ξ) closed ctct 3-manifold $\xrightarrow[\text{Szabó}]{\text{Ozsváth}}$ $c(\xi) \in HF(-M)$

(M, ξ, Γ) ctct 3-mfd w/ convex ∂ $\xrightarrow[\text{-Matić}]{\text{Honda-Kazez}}$ $EH(\xi, \Gamma) \in SFH(-(M, \Gamma))$

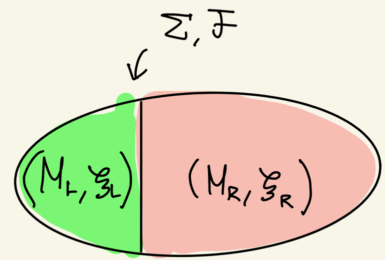
$(M, \xi, \mathcal{F}_\xi)$ ctct 3-mfd w/ char. foliation on ∂ (this is also cx)

$\xrightarrow[\text{Petkova, Licata, V}]{\text{Alishahi, Földvári, Hendricks}}$ $c_A(\xi) \in CFA(-(M, \partial M))$

$c_D(\xi) \in CFD(-(M, \partial M))$

$\mathcal{F}_\xi$ parametrizes ∂M

<u>Thm</u> (AFHLPV) $(M, \xi) = (M_L, \xi_L, \mathcal{F}) \underset{\Sigma}{\cup} (M_R, \xi_R, \overline{\mathcal{F}})$



$\Rightarrow$ $CFA(-M_L, \Sigma) \tilde{\otimes} CFD(-M_R, \Sigma) = CF(-M)$

& $c_A(\xi_L) \tilde{\otimes} c_D(\xi_R) = c(\xi)$

CONNECTION TO SUTURED INVARIANT

Thm: (AFHLPV) $CFA(-M) \xrightarrow{R.\text{Zarev}} SFH(-M)$ & $CFD(-M) \xrightarrow{R.\text{Zarev}} SFH(-M)$

$$c_A(\xi) \longmapsto EH(\xi) \qquad\qquad c_D(\xi) \longmapsto EH(\xi)$$

&

$$\begin{tikzcd}
CFA(-M_L) \ni c_A(\xi_L) \arrow[d, "\text{Zarev}"] & \tilde{\otimes} & CFD(-M_R) \ni c_D(\xi_R) \arrow[d, "\text{Zarev}"] & = & CF(-M) \ni c(\xi) \arrow[d, "H_*(\ )"] \\
SFH(-M) \ni EH(\xi_L) & \otimes & SFH(-M) \ni EH(\xi_R) \arrow[rr, "HKM"] & & HF(-M) \ni c(\xi)
\end{tikzcd}$$

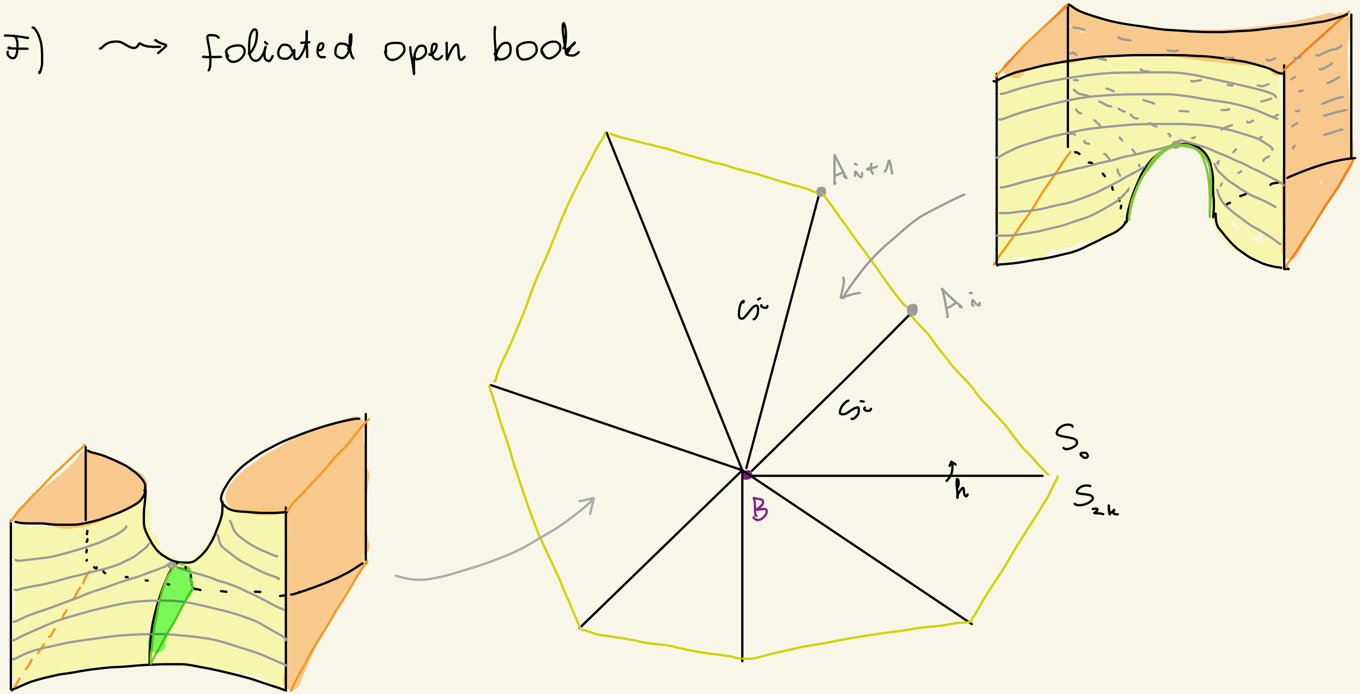
commutes

the definition of this map includes a "sufficiently complicated" so it is impossible to compute it even in concrete examples

this diagram makes this computation possible

Contact Invariant in Bordered Floer Homology - Idea

$(M, \xi, \mathcal{F}) \rightsquigarrow$ foliated open book

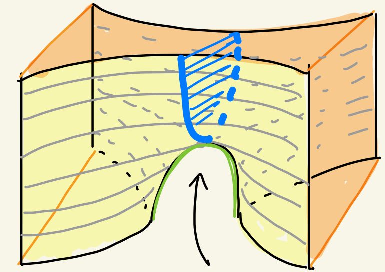


Contact Invariant in Bordered Floer Homology – Idea

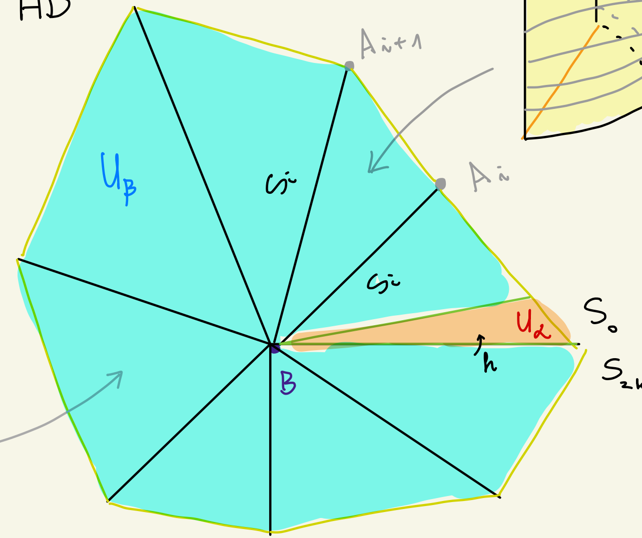
$(M, \xi, \mathcal{F}) \rightsquigarrow$ foliated open book

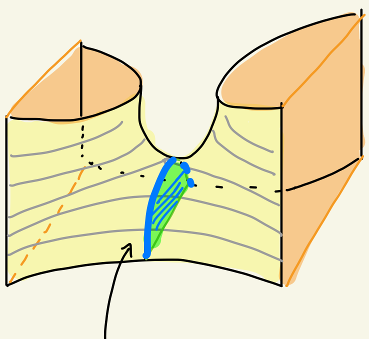
$\rightsquigarrow$ "multipointed" β-bordered HD

(= sutured bordered HD)



parametrisation





parametrisation

basepoints $= N(\partial S_0 \cap \partial M) = N(\tilde{\pi}^{-1}(0)) \subset \partial M$

! depends on the time-parametrisation of $\pi : M \setminus B \to S^1$!

FURTHER DIRECTIONS

- contact invariant in $CFDA(-M)$

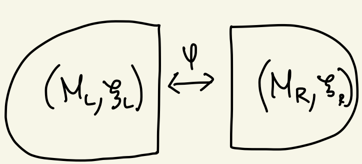
  $\rightsquigarrow$ description of $CFA(M)$ & $CFD(M)$ in terms of SFH:

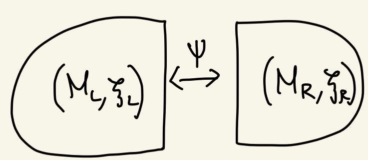
  Thm (Zarev): $CFA(M) = \bigoplus_{I} SFH(M, \Gamma_I)$

  elementary div curves

  Conjecture (Zarev): the $\mathcal{A}_\infty$-maps can be described by gluing standard $(\partial M \times I, \Gamma_I \cup -\Gamma_J)$-pieces.

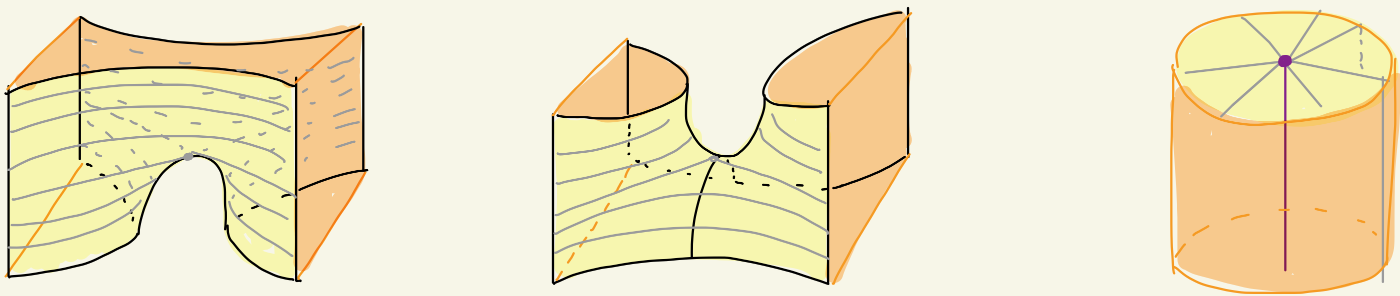
- gives a good guess of what bordered ECH should be.
- can be used to give global results as well:





can distinguish these

THANKS FOR YOUR ATTENTION!



QUESTIONS?