A stochastic Prékopa-Leindler inequality for log-concave functions

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BIRS - Geometric Tomography

Brunn-Minkowski

For compact $K, L \subset \mathbb{R}^n$ and $\lambda \in (0,1)$:

$$|K+L|^{1/n} \geq |K|^{1/n} + |L|^{1/n}.$$

Equivalently, as a *rearrangement inequality*:

$$|K+L| \geq |K^* + L^*|,$$

where $A^* = r_A B_2^n$ such that $|A^*| = |A|$.

Stochastic Brunn-Minkowski

Stochastic Model:

- $K \subset \mathbb{R}^n$ convex body.
- $\{X_i\}_{i=1}^N$ i.i.d. uniformly in K $\left(X_i \sim \frac{1}{|K|}\mathbb{1}_K\right)$
- Random polytope: $[K]_N = \mathrm{conv}\{X_1, \ldots, X_N\}$

Theorem (Paouris & P., 2017)

Let $K, L \subset \mathbb{R}^n$ be convex bodies and $N, M > n$. Then for all $\alpha > 0$

$$\mathbb{P}\left(|[K]_N + [L]_M| > \alpha\right) \geq \mathbb{P}\left(|[K^*]_N + [L^*]_M| > \alpha\right).$$

When $L = \{0\}$, get [Busemann,'53], [Groemer '74] for random polytopes:

$$\mathbb{E}|[K]_N| \geq \mathbb{E}|[K^*]_N|.$$

Where does convexity enter the picture?

Use linear images of **convex sets** $C \subseteq \mathbb{R}^N$:

$$\int_{(\mathbb{R}^n)^N} |[X_1, \ldots, X_N]C| \prod_{i=1}^{N} f_i(x_i) dx_1 \ldots dx_N,$$

- $f_1, \ldots, f_N$ are any densities on $\mathbb{R}^n$.
- Can intertwine operations: convex hull, Minkowski sums, $p-sums$, Orilicz sums, via choice of C: [Paouris, P. '12]
- Rearrangement inequalities: [Rogers, 58], [Brascamp-Lieb-Luttinger, '74], [Christ, 84]

Prékopa-Leindler

Let $f, g, h : \mathbb{R}^n \to [0, \infty)$ be integrable, $\lambda \in (0, 1)$. If

$$h(\lambda x + (1-\lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}, \quad \forall x, y \in \mathbb{R}^n$$

then

$$\int h \geq \left(\int f\right)^\lambda \left(\int g\right)^{1-\lambda}.$$

As a *rearrangement inequality*, [Brascamp-Lieb, '76]:

$$\int_{\mathbb{R}^n} (f \star_\lambda g)(v)\, dv \geq \int_{\mathbb{R}^n} (f^* \star_\lambda g^*)(v)\, dv,$$

$$(f \star_\lambda g)(v) := \sup\{f(x)^\lambda g(y)^{1-\lambda} : v = \lambda x + (1-\lambda)y\}$$

Recent variants: [Melbourne '19]

A stochastic Prékopa-Leindler inequality

Stochastic Model:

- $f : \mathbb{R}^n \to [0,\infty)$ integrable, log-concave.
- $\{(X_i, Z_i)\}_{i=1}^N \subset \mathbb{R}^n \times [0,\infty)$ i.i.d uniform in

$$G_f := \{(x,z) \in \mathbb{R}^n \times [0,\infty) : z \leq f(x)\}$$

- $[f]_N$ - least log-concave majorant above $\{(X_i, Z_i)\}$:

$$[f]_N(x) = e^{\sup\{z : (x,z) \in H_f)\}},$$

where

$$H_{f,N} = \mathrm{conv}\{(X_1, \log Z_1), \ldots, (X_N, \log Z_N)\}.$$

Equivalently,

$$[f]_N(x) = \sup\left\{\prod_i Z_i^{c_i} : x = \sum_i c_i X_i, c_i \geq 0, \sum_i c_i = 1\right\}.$$

A stochastic Prékopa-Leindler inequality

Theorem (P., Rebollo Bueno)

Let $f, g : \mathbb{R}^n \to [0, \infty)$ be integrable log-concave functions, $\lambda \in (0, 1)$, and $N, M > n + 1$. Then for all $\alpha > 0$

$$\mathbb{P}\left(\int_{\mathbb{R}^n} ([f]_N \star_\lambda [g]_M)(v)dv > \alpha\right) \geq \mathbb{P}\left(\int_{\mathbb{R}^n} ([f^*]_N \star_\lambda [g^*]_M)(v)dv > \alpha\right)$$

For one function, we get a stochastic functional Groemer-type inequality:

$$\mathbb{P}\left(\int_{\mathbb{R}^n} [f]_N(x)\, dx > \alpha\right) \geq \mathbb{P}\left(\int_{\mathbb{R}^n} [f^*]_N(x)\, dx > \alpha\right).$$

Ingredients in the proof

Reduction to bodies of revolution:

[Artstein-Klartag-Milman, '04]

[Klartag, '07]

[Artstein-Klartag-Schütt-Werner, '12]

Approximation of log**-concave functions:**

- $f : \mathbb{R}^n \to [0, \infty)$ is s-concave if $f^{1/s}$ is concave (non-standard).
- Let $f : \mathbb{R}^n \to [0, \infty)$ be log-concave, then

$$f_s(x) := \left(1 + \frac{\log f(x)}{s}\right)_+^s$$

is s-concave. This way $f_s \le f$, $\forall\ s > 0$, and $f_s \overset{s\to\infty}{\longrightarrow} f$ locally uniformly on $\mathbb{R}^n$.

Ingredients in the proof

By Brunn's principle: For $s \in \mathbb{N}$, $f : \mathbb{R}^n \to [0, \infty)$ is s-concave on $\mathbb{R}^n$ if and only if it is a marginal of the uniform measure on a convex body in $\mathbb{R}^{n+s}$.

- Let $f : \mathbb{R}^n \to [0, \infty)$ and set

$$\mathcal{K}_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^s : x \in \overline{\mathrm{supp} f}, |y| \leq f^{1/s}(x)\}.$$

- $f(x) = \left|B_2^s\right|^{-1} \int_{\mathbb{R}^s} \mathbb{1}_{\mathcal{K}_f}(x, y)\, dy,$
- $\int_{\mathbb{R}^n} f(x)\, dx \simeq |\mathcal{K}_f|.$

Ingredients in the proof

Let $f, g : \mathbb{R}^n \to [0, \infty)$ s-concave.

- Homothety:

$$(\lambda \cdot_s f)(x) := \lambda^s f\left(\frac{x}{\lambda}\right) \Longrightarrow \mathcal{K}_{\lambda \cdot_s f} = \lambda \mathcal{K}_f.$$

- s-Minkowski Sum:

$$(f \oplus_s g)(v) := \sup_{v=x+y} \{(f(x)^{1/s} + g(y)^{1/s})^s\} \Longrightarrow \mathcal{K}_{f \oplus_s g} = \mathcal{K}_f + \mathcal{K}_g.$$

Crucial property:

$$f \star_{\lambda,s} g := (\lambda \cdot_s f) \oplus_s ((1-\lambda) \cdot_s g) \Longrightarrow \mathcal{K}_{f \star_{\lambda,s} g} = \lambda \mathcal{K}_f + (1-\lambda)\mathcal{K}_g.$$

and $f \star_{\lambda,s} g \overset{s \to \infty}{\longrightarrow} f \star_\lambda g$.

Rearrangement inequality

$F : (\mathbb{R}^n)^N \to [0, \infty)$ is **Steiner convex** if for each $\theta \in \mathbb{S}^{n-1}$ and $Y = \{y_1, \dots, y_N\} \subset (\theta^\perp)^N$

$$F_{Y,\theta}(t_1, \dots, t_N) := F(y_1 + t_1\theta, \dots, y_N + t_N\theta)$$

is even and quasi-convex.

Theorem (Rogers '58 & Brascamp-Lieb-Luttinger '74 & Christ '84)

Let $F : (\mathbb{R}^n)^N \to [0, \infty)$ *be Steiner convex and* $f_i : \mathbb{R}^n \to [0, \infty)$, $i = 1, \dots, N$. *Then*

$$\int_{(\mathbb{R}^n)^N} F(x_1, \dots, x_N) \prod_{i=1}^N f_i(x_i)\, d\mathbf{x} \geq \int_{(\mathbb{R}^n)^N} F(x_1, \dots, x_N) \prod_{i=1}^N f_i^*(x_i)\, d\mathbf{x}$$

Some examples of Steiner convex functionals

- [Busemann, 53]: $(x_1, \ldots, x_n) \longmapsto |\det(x_1, \ldots, x_n)|$.
- [Groemer, '76] $(x_1, \ldots, x_N) \longmapsto |\mathrm{conv}\{x_1, \ldots, x_N\}|$.
- [Pfiefer, '82]: For $r > 0$
$$(x_1, \ldots, x_N) \longmapsto |\mathrm{conv}\{B_r(x_1), \ldots, B_r(x_N)\}|\,.$$
- [Paouris, P. '12]: For $C \subset \mathbb{R}^N$ convex set
$$(x_1, \ldots, x_N) \longmapsto |[x_1 \cdots x_N]C|$$
where $[x_1 \cdots x_N]C = \left\{ \sum_{i=1}^{N} c_i x_i : (c_i) \in C \right\}$.

Key lemma

Notation:

$$B^s_\rho(x) = \{(x, \widehat{z}) \in \mathbb{R}^n \times \mathbb{R}^s : |\widehat{z}| \leq \rho\}.$$

Lemma

Let $\{(x_i, z_i)\}_{i=1}^N \subset \mathbb{R}^n \times [0, \infty)$, $w_i = (x_i, z_i)$. Denote by $T_{\{w_i\}}$, the least s-concave function above $\{w_i\}$. Then

$$\mathcal{K}_{T_{\{w_i\}}} = \mathrm{conv}\{B^s_{\rho_1}(x_1), \ldots, B^s_{\rho_N}(x_N)\}, \qquad \textit{where } \rho_i = z_i^{1/s}.$$

and

$$(x_1, \ldots, x_N) \longmapsto \left|\mathrm{conv}\{B^s_{\rho_1}(x_1), \ldots, B^s_{\rho_N}(x_N)\}\right|$$

is Steiner convex.

A last reduction

For compactness and degeneracy issues, we use instead

$$\mathcal{K}_{[f_\epsilon]_{N,s} \star_{\lambda,s} [g_\epsilon]_{M,s}} = \lambda \mathcal{K}_{[f_\epsilon]_{N,s}} + (1-\lambda)\mathcal{K}_{[g_\epsilon]_{M,s}},$$

where

- $f_\epsilon = f \cdot \mathbb{1}_{\{f \geq \epsilon\}}$.
- $[f_\epsilon]_{N,s} = \left(1 + \frac{\log [f_\epsilon]_N}{s}\right)^s$, $s > -\log \epsilon$.

Allows a uniform treatment for *one random sample* $\{(X_i, Z_i)\}$.

Applying the R-BLL-C rearrangement inequality

For the functional Groemer inequality:

$$\mathbb{P}\left(\int_{\mathbb{R}^n}[f]_N(x)dx>\alpha\right)=\frac{1}{\|f\|_1^N}\int_{(\mathbb{R}^n\times[0,\infty))^N}\mathbb{1}_{\{F>\alpha\}}(\overline{w})\prod_{i=1}^N\mathbb{1}_{[0,f(x_i)]}(z_i)d\overline{w}$$

$$=\frac{1}{\|f\|_1^N}\int_{[0,\infty)^N}\left(\int_{(\mathbb{R}^n)^N}\mathbb{1}_{\{F>\alpha\}}\prod_{i=1}^N\mathbb{1}_{[0,f(x_i)]}(z_i)d\overline{x}\right)d\overline{z}$$

$$\geq\frac{1}{\|f^*\|_1^N}\int_{[0,\infty)^N}\left(\int_{(\mathbb{R}^n)^N}\mathbb{1}_{\{F>\alpha\}}\prod_{i=1}^N\mathbb{1}_{[0,f^*(x_i)]}(z_i)d\overline{x}\right)d\overline{z}$$

$$=\mathbb{P}\left(\int_{\mathbb{R}^n}[f^*]_N(x)dx>\alpha\right).$$

For the stochastic Prékopa-Leindler inequality:

$$\mathcal{K}_{[f_\epsilon]_{N,s}\star_{\lambda,s}[g_\epsilon]_{M,s}} = \lambda\mathcal{K}_{[f_\epsilon]_{N,s}} + (1-\lambda)\mathcal{K}_{[g_\varepsilon]_{M,s}}$$

$$\begin{aligned}\mathcal{K}_{[f_\epsilon]_{N,s}} &= \operatorname{conv}\{B^s_{R_1}(X_1),\ldots,B^s_{R_N}(X_N)\}\\ &= \oplus_{C_N}(\{B^s_{R_i}(X_i)\}_{i=1}^N),\end{aligned}$$

$$\begin{aligned}\mathcal{K}_{[g_\epsilon]_{M,s}} &= \operatorname{conv}\{B^s_{R_{N+1}}(X_{N+1}),\ldots,B^s_{R_{N+M}}(X_{N+M})\}\\ &= \oplus_{C_M}(\{B^s_{R_i}(X_i)\}_{i=N+1}^M),\end{aligned}$$

where $C_k = \operatorname{conv}\{e_1,\ldots,e_k\}$, $k = N, M$, so

$$\begin{aligned}\mathcal{K}_{[f_\epsilon]_{N,s}\star_{\lambda,s}[g_\epsilon]_{M,s}} &= \oplus_{C_N}(\{B^s_{R_i}(X_i)\}_{i=1}^N) +_\lambda \oplus_{C_M}(\{B^s_{R_i}(X_i)\}_{i=N+1}^{N+M})\\ &= \oplus_{C_N +_\lambda \widehat{C}_M}(\{B^s_{R_i}(X_i)\}_{i=1}^{N+M}),\end{aligned}$$

where $\widehat{C}_M = \operatorname{conv}\{e_{N+1},\ldots,e_{N+M}\}$,

More generally, use $\mathcal{M}$-addition of [Gardner, Hug, Weil, '13]

Thank you!