Some results on blowup and scattering for critical wave maps

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Wave maps

Wave maps from $\mathbb{R}^{1+d}_{t,x}$ to a Riemannian manifold $(\mathcal{N}, g)$ are critical points of the Lorentz invariant Lagrangian

$$L(u_t, \nabla u) = \int_{\mathbb{R}^{d+1}} \Big(-|u_t|_g^2 + \sum_{j=1}^{d} |u_{x_j}|_g^2 \Big)\, dtdx_1 \cdot \ldots \cdot dx_d$$

if $\mathcal{N} \hookrightarrow \mathbb{R}^m$ then extrinsic formulation is

$$\Box\Psi \perp T_\Psi\mathcal{N}, \quad \Psi \in \mathbb{R}^m$$

For the sphere $\mathcal{N} = \mathbb{S}^{m-1}$ the WM equation is

$$u_{tt} - \Delta u = u(-|u_t|^2 + |\nabla u|^2)$$

The right-hand side is the second fundamental form with Lorentz signature. If second fundamental form vanishes, obtain free wave equation. Contrast with the Euclidean analogue, the harmonic maps $-\Delta u = u|\nabla u|^2$ which are critical points of the Dirichlet energy.

Symmetry reduction: $u \circ R = R^k \circ u$ for any rotation R of Euclidean space, with fixed integer $k \neq 0$.

Wave maps, conserved energy, symmetries, criticality

Energy

$$E(u) = \int_{\mathbb{R}^d} \left(|u_t|_g^2 + \sum_{j=1}^{d} |u_{x_j}|_g^2 \right) dx_1 \cdot \ldots \cdot dx_d$$

is conserved (for smooth wave maps). Dilation symmetry $u \mapsto u_\lambda := u(t/\lambda, x/\lambda)$ takes wave maps to wave maps, and

$$E(u_\lambda) = \lambda^{-2+d} E(u)$$

Critical dimension $d = 2$. Supercritical case: $d \geq 3$ **Shatah** 1980s showed finite time blowup. **Donninger-Glogic** 2017: $d \geq 8$ Self-similar blowup into negatively curved manifolds, stability. Critical case: Negatively curved manifold such as hyperbolic plane: global existence (**Tataru-Sterbenz, Tao, Krieger**-S. 2009). For positive curvature such as Sphere: finite time blowup (**Krieger-S.-Tataru** 2006, **Raphael-Rodnianski** 2008).

Equivariant wave maps

Special case $\mathcal{N} = \mathbb{S}^2 \subset \mathbb{R}^3$, 1-equivariant solutions:

$$\Psi(t, r\cos\theta, r\sin\theta) = \big(\sin(u(t,r))\cos\theta, \sin(u(t,r))\sin\theta, \cos(u(t,r))\big).$$

with $u(t,r)$ angle from north pole. WM equation is reduced to a semi-linear one:

$$\begin{cases} \partial_t^2 u(t,r) = \partial_r^2 u(t,r) + \frac{1}{r}\partial_r u(t,r) - \frac{1}{2r^2}\sin(2u(t,r)), \\ (u(t_0,r), \partial_t u(t_0,r)) = (u_0(r), \dot{u}_0(r)). \end{cases} \qquad \text{(WM)}$$

Conserved energy takes the form

$$E(\boldsymbol{v}) := \pi \int_0^\infty \left(\dot{v}^2 + (\partial_r v)^2 + \frac{1}{r^2}(\sin(v))^2\right) r\mathrm{d}r.$$

Finite energy enforces boundary conditions: $v(t,0) \in \pi\mathbb{Z}$, $v(t,\infty) \in \pi\mathbb{Z}$. So (WM) connects north pole (say) to either north or south pole. Topological degree of the map is m where $v(t,0) = 0$, $v(t,\infty) = m\pi$.

Wellposedness of equivariant WM

Solve (WM) in

$$\mathcal{H}_0 := \mathcal{H} \times L^2, \qquad \|v\|_{\mathcal{H}}^2 := \int_0^\infty \left((\partial_r v)^2 + \frac{1}{r^2} v^2\right) r\mathrm{d}r$$

in 0 degree class. Finite energy E in 0-degree is equivalent with $\mathcal{H}_0$. Local well-posedness in $\mathcal{H}_0$

$$\forall \boldsymbol{u}_0 \in \mathcal{H}_0,\ \exists! \boldsymbol{u} \in C((T_-, T_+); \mathcal{H}_0), \qquad T_- < t_0 < T_+.$$

- The energy is conserved; the flow is reversible.
- Let $\lambda > 0$. For $\boldsymbol{v} = (v, \dot{v}) \in \mathcal{H}_0$ we denote

$$\boldsymbol{v}_\lambda(r) := \left(v\left(\frac{r}{\lambda}\right), \frac{1}{\lambda}\dot{v}\left(\frac{r}{\lambda}\right)\right).$$

We have $\|\boldsymbol{v}_\lambda\|_{\mathcal{H}_0} = \|\boldsymbol{v}\|_{\mathcal{H}_0}$ and $E(\boldsymbol{v}_\lambda) = E(\boldsymbol{v})$. Moreover, if $\boldsymbol{u}(t)$ is a solution of (WM) on the time interval $[0, T_+)$, then $\boldsymbol{w}(t) := \boldsymbol{u}(\frac{t}{\lambda})_\lambda$ is a solution on $[0, \lambda T_+)$.

Stationary states – k-equivariant harmonic maps

Harmonic maps are stationary wave maps. In each equivariance and degree class (homotopy class) unique harmonic map.

- Explicit radially symmetric solutions of $\partial_r^2 u(r) + \frac{1}{r}\partial_r u(r) - \frac{k^2}{2r^2}\sin(2u(r)) = 0$:

$$Q_\lambda(r) := 2\arctan\left(\frac{r^k}{\lambda^k}\right), \qquad \boldsymbol{Q}_\lambda := (Q_\lambda, 0) \in \mathcal{H}_0.$$

- $E(\boldsymbol{Q}_\lambda) = 4k\pi$; orbital stability
- $\boldsymbol{Q}_\lambda$ are, up to sign and translation by π, all the equivariant stationary states.

In degrees 0 and 1, $k = 1$ full dynamics is known below energies $2E(Q)$, resp. $3E(Q)$

Theorem 1 – Côte, Kenig, Lawrie, S. (2015)

Let $\boldsymbol{u}_0$ be such that $E(\boldsymbol{u}_0) < 2E(Q)$ and of degree 0. Then the solution $\boldsymbol{u}(t)$ of (WM) with initial data $\boldsymbol{u}(0) = \boldsymbol{u}_0$ exists globally and scatters in both time directions. Sharp: finite time blowup if $E(\boldsymbol{u}_0) > 2E(Q)$.

Threshold in degree 1

Theorem 1 – Côte, Kenig, Lawrie, S. (2015)

Let $\boldsymbol{u}_0$ s.t. $E(\boldsymbol{u}_0) < 3E(Q)$ and degree 1, $k = 1$.

(1) **Finite time blow-up:** The solution $\vec{u}(t)$ blows up at $t_0 > 0$,

$$\vec{u}(t) = \vec{\varphi} + \left(Q\left(\cdot/\lambda(t)\right), 0\right) + \vec{\epsilon}(t) \tag{1}$$

$\vec{\epsilon}(t) \in \mathcal{H}_0$ and $\vec{\epsilon}(t) \to 0$ in $\mathcal{H}_0$ as $t \to t_0$, $\lambda(t) = o(t_0 - t)$, $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}_0$.

(2) **Global Solution:** $\exists\, \lambda : [0, \infty) \to (0, \infty)$ with $\lambda(t) = o(t)$ as $t \to \infty$, a solution $\vec{\varphi}_L(t) \in \mathcal{H}_0$ to the linearized (WM) s.t.

$$\vec{\psi}(t) = \vec{\varphi}_L(t) + \left(Q\left(\cdot/\lambda(t)\right), 0\right) + \vec{\epsilon}(t) \tag{2}$$

$\vec{\epsilon}(t) \in \mathcal{H}_0$ and $\vec{\epsilon}(t) \to 0$ in $\mathcal{H}_0$ as $t \to \infty$.

Unclear what happens for $E(\boldsymbol{u}_0) > 3E(Q)$.

Two bubble construction in degree 0

It turns out that there exist non-scattering solutions of threshold energy.

Theorem 1 – Jendrej (2016)

Let $k \geq 3$. There exists a solution $\boldsymbol{u} : (-\infty, T_0] \to \mathcal{H}_0$ of (WM),

$$\lim_{t \to -\infty} \left\| \boldsymbol{u}(t) - \left(-\boldsymbol{Q} + \boldsymbol{Q}_{\kappa|t|^{-\frac{2}{k-2}}}\right) \right\|_{\mathcal{H}_0} = 0, \qquad \kappa \text{ constant } > 0.$$

- An analogous result holds for the critical radial Yang-Mills equation (exponential concentration rate); the same would be the case for (WM) with $k = 2$.
- Concentration of one bubble: Krieger, S. and Tataru (2008) (Donninger, Krieger 2013), Raphaël and Rodnianski (2012).
- Strong interaction of bubbles: the concentration of one bubble driven by stationary one.
- Only possible choice of signs, no $Q + Q_\lambda$ bubble exists.
- Uniqueness?
- Is T_0 finite? What happens as $t \to T_0$?

There is only one possible dynamical behavior of a non-scattering solution.

Theorem 2 – Jendrej, Lawrie (Inventiones 2018)

Fix any equivariance class $k \geq 2$. Let $\boldsymbol{u}(t) : (T_-, T_+) \to \mathcal{H}_0$ be a solution of (WM) such that

$$E(\boldsymbol{u}) = 2E(\boldsymbol{Q}) = 8\pi k.$$

Then $T_- = -\infty$, $T_+ = +\infty$ and one of the following alternatives holds:

- $\boldsymbol{u}(t)$ scatters in both time directions,
- $\boldsymbol{u}(t)$ scatters in one time direction; in the other time direction, there exist $\iota \in \{-1, 1\}$ and continuous functions $\mu(t), \lambda(t) > 0$ such that

$$\|\boldsymbol{u}(t) - \iota(-\boldsymbol{Q}_{\mu(t)} + \boldsymbol{Q}_{\lambda(t)})\|_{\mathcal{H}_0} \to 0,$$
$$\mu(t) \to \mu_0 \in (0, +\infty), \qquad \lambda(t) \to 0 \text{ (at a specific rate)}.$$

Comments

- They obtain $\lambda(t) \sim |t|^{-\frac{2}{k-2}}$ for $k \geq 3$ and $\exp(-Ct) \leq \lambda(t) \leq \exp(-t/C)$ for $k = 2$
- In particular, the two-bubble solutions from Theorem 1 scatter in forward time, which provides an example of an orbit connecting different types of dynamical behavior for positive and negative times. Cf. to 9-set theorem.
- Non-existence of solutions which form a pure two-bubble in both time directions is reminiscent of the work of Martel and Merle for gKdV and seems to be a typical feature of models which are not completely integrable
- They conjecture that there exists a unique (up to rescaling and sign change) non-scattering solution of threshold energy
- Jendrej-Lawrie theorem the only complete dynamical classification in a setting allowing more than one bubble, except for completely integrable models.

One-pass theorem in two soliton dynamics

The most novel aspect of the proof is a no-return theorem, based on a nonlinear effect. In the earlier one-pass theorem with Nakanishi we used the exponential instability of the linearized flow to obtain the crucial contradiction in the virial argument. This effect is completely absent in the two-bubble dynamics of (WM). Instead, Jendrej and Lawrie rely on second order effects in the energy/virial functionals based on soliton interaction, to prove an ejection lemma for trajectories close to a two bubble configuration $-Q_\lambda + Q_\mu$ with $\lambda/\mu \ll 1$. Combined with the concentration compactness theory, and the $2E(Q)$ threshold theorem in degree 0 from above they show that a two-bubble configuration in the past will exist for all positive forward times and must in fact scatter.

KST Blowup solutions (2006, 2015 for full range of ν)

Let $Q(r) = 2\arctan r$ be the harmonic map.

Theorem *$\nu > 0$ arbitrary and $t_0 > 0$ small, $\lambda(t) = t^{-1-\nu}$ and N large. There exists u^e satisfying*

$$u^e \in C^{\nu+1/2-}(\{t_0 > t > 0,\ |x| \le t\})$$
$$\mathcal{E}_{\mathrm{loc}}(u^e)(t) \lesssim (t\lambda(t))^{-2}\,|\log t|^2 \quad \text{as} \quad t \to 0$$

and blow-up solution u to critical (WM) in $[0, t_0]$ s.t.

$$u(t,r) = Q(\lambda(t)r) + u^e(t,r) + \epsilon(t,r), \qquad 0 \le r \le t$$

where ϵ decays at $t = 0$:

$$\epsilon \in t^N H^{1+\nu-}_{\mathrm{loc}}(\mathbb{R}^2), \qquad \epsilon_t \in t^{N-1} H^{\nu-}_{\mathrm{loc}}(\mathbb{R}^2),$$
$$\mathcal{E}_{\mathrm{loc}}(\varepsilon)(t) \lesssim t^N \quad \text{as} \quad t \to 0$$

By Struwe's theorem: $\nu = 0$ impossible, and all excess energy must be evacuated out of the light cone (by smallness)

The approximate solution

Iterative construction of the approximate solution $Q(\lambda(t)r) + u^e(t,r)$. Let $R = \lambda(t)r, \ \tau = \int_t^\infty \lambda(s)\, ds$

$$u_{2k-1}(r,t) = Q(R) + \frac{c_k}{\tau^2} R \ln(1+R^2) + O\left(\frac{R^{-1}(\ln(1+R^2))^2}{\tau^2}\right)$$

$$e_{2k-1} = O\left(\frac{R(\log(2+R))^{2k-1}}{t^2\tau^{2k}}\right)$$

The latter being the size of the nonlinear error, with rapid decay in τ. Near the boundary of the light cone $r \le t, \ R \le \nu\tau$

$$u^e(t,r) = O(\tau^{-1}\log\tau)[(1-r/t)^{\frac{1}{2}+\nu}\log(1-r/t) + \text{more regular terms}]$$

Singularity at $r = t$ intrinsic to the construction. Blowup comes from shock on the light cone.

The iteration for the exact solution

Seek exact solution $u(t,r) = u_{2k-1} + R^{-\frac{1}{2}}\varepsilon$ whence

$$-\varepsilon_{tt} + \varepsilon_{rr} + \frac{1}{r}\varepsilon - \frac{\cos(2Q(r\lambda(t)))}{r^2}\varepsilon = N_{2k-1}(\varepsilon) + e_{2k-1}$$

with $N_{2k-1}(\varepsilon)$ the error resulting from $\cos(2Q(R))$. In rescaled variables

$$\Big(- (\partial_\tau + \dot\lambda \cdot \lambda^{-1} R\partial_R)^2 + \frac{1}{4}(\dot\lambda \cdot \lambda^{-1})^2 + \frac{1}{2}\partial_\tau(\dot\lambda \cdot \lambda^{-1})\Big)\varepsilon - \mathcal{L}\varepsilon$$
$$= \lambda^{-2} R^{\frac{1}{2}}(N_{2k-1}(\varepsilon) + e_{2k-1}) \qquad (\star)$$
$$\mathcal{L} = -\partial_R^2 + \frac{3}{4R^2} - 8(1+R^2)^{-2}$$

Then solve $(\star)$ via the Fourier transform associated with $\mathcal{L}$. Resonance mode $\psi(R) = R\partial_R Q(R) = \frac{2R}{1+R^2} \notin L^2(R\,dR)$ makes spectral measure singular at zero energy. $\mathcal{L}(R^{\frac{1}{2}}\psi) = 0$. But $\mathcal{L} \geq 0$ in contrast to u^5 equation where Q is exponentially unstable.

The Fourier transform relative to $\mathcal{L}$

From Stone's formula relating spectral measure/resolvent:

$$\mathcal{F} : f \longrightarrow \hat{f}(\xi) = \lim_{b\to\infty} \int_0^b \phi(r,\xi) f(r)\, dr$$

is a unitary operator from $L^2(\mathbb{R}^+)$ to $L^2(\mathbb{R}^+, \rho)$ and its inverse is given by

$$\mathcal{F}^{-1} : \hat{f} \longrightarrow f(r) = \lim_{\mu\to\infty} \int_0^\mu \phi(r,\xi) \hat{f}(\xi)\, \rho(\xi)\, d\xi$$

with spectral measure

$$\rho(\xi) = \frac{1}{\pi} \mathrm{Im}\; m(\xi + i0)\, \chi_{[\xi>0]}$$

and Weyl function

$$m(\xi) = \frac{W(\theta(.,\xi),\, \psi^+(.,\xi))}{W(\psi^+(.,\xi), \phi(.,\xi))}$$

and ϕ, θ a suitable fundamental system, ϕ determined either by boundary condition at origin (limit circle) or an L^2 condition (in limit point case).

Stability of these blowup solutions, equivariant case

Rigidity of KST blowup under equivariant perturbations which do not change regularity on the light cone:

Theorem (Krieger-Miao Duke 2020): $0 < \nu \ll 1$ *small,* u^ν *KST blowup solutions. There is* $\delta_0 > 0$ *small , s.t. for* $(\varepsilon_0, \varepsilon_1) \in H^4_{\mathbb{R}^2} \times H^3_{\mathbb{R}^2}$ *with*

$$\left\|(\varepsilon_0(r)e^{i\theta}, \varepsilon_1(r)e^{i\theta})\right\|_{H^4_{\mathbb{R}^2} \times H^3_{\mathbb{R}^2}} < \delta_0,$$

data $(u^\nu(t_0, \cdot) + \varepsilon_0, \partial_t u^\nu(t_0, \cdot) + \varepsilon_1)$ *at* t_0 *give (WM)* u *on* $(0, t_0] \times \mathbb{R}^2$ *s.t.* $u(t, r) = Q(\lambda(t)r) + \tilde{\varepsilon}(t, r)$ *with* $(\tilde{\varepsilon}(t, \cdot), \tilde{\varepsilon}_t(t, \cdot)) \in H^{1+\nu-}_{\mathbb{R}^2} \times H^{\nu-}_{\mathbb{R}^2}$ *for all* $t \in (0, t_0]$ *and with*

$$\lim_{t \to 0} \int_0^t \left(\tilde{\varepsilon}_t^2 + \tilde{\varepsilon}_r^2 + \frac{\sin^2 \tilde{\varepsilon}}{2r^2} \right) rdr = 0$$

Perturbed solutions exhibit same blow up point, regularity.

Equivariant stability argument I

Basic ansatz, switch to "super symmetric" linearized operator:

$$u(t,r) = u^{\nu}(t,r) + \varepsilon(t,r)$$

$$\mathcal{L}\varepsilon = \left(-\partial_R^2 - \frac{1}{R}\partial_R + \frac{1}{R^2}\frac{1-6R^2+R^4}{(1+R^2)^2}\right)\varepsilon \quad \text{in} \quad L^2(R\,dR)$$

$$\mathcal{D} := \partial_R + \frac{1}{R} - \frac{2}{R(R^2+1)}, \quad \mathcal{L} = \mathcal{D}^*\mathcal{D},\ \mathcal{D}\phi_0 = 0,$$

Solve for $\mathcal{D}\varepsilon$ (differentiated wave map) and not ε, thus

$$\varepsilon = \mathcal{D}^{-1}(\mathcal{D}\varepsilon) + c(\tau)\phi_0(R),$$

$$\mathcal{D}^{-1}(g) := \phi_0(R)\int_0^R (\phi_0(s))^{-1} g(s)\,ds$$

Linearized operator is $\tilde{\mathcal{L}} = \mathcal{D}\mathcal{D}^*$, which has no 0-energy issue and a more regular spectral measure.

Equivariant stability argument II

Coupled system for $\mathcal{D}\varepsilon, c(\tau)$:

$$-\left(\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)^2 + 3\frac{\lambda'}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\right)\mathcal{D}\varepsilon - \tilde{\mathcal{L}}\mathcal{D}\varepsilon$$

$$= \lambda^{-2}\mathcal{D}\left(N(\varepsilon)\right) - \frac{4R}{(R^2+1)^2}\left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\varepsilon$$

$$- \frac{\lambda'}{\lambda}\frac{4R}{(R^2+1)^2}\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\varepsilon - \frac{\lambda'}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\left(\frac{4R}{(R^2+1)^2}\varepsilon\right)$$

$$+ \left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\mathcal{D}\varepsilon$$

$$\left(\partial_\tau + \frac{\lambda'}{\lambda}\right)^2 c(\tau) + \frac{\lambda'}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda}\right)c(\tau) + h(\tau) + \lambda^{-2}n(\tau) = 0.$$

where h, n computed from nonlinearity at the origin.

Stability of these blowup solutions, nonequivariant case

A recent result by Duyckaerts-Jia-Kenig-Merle (IMRN 2018) characterises blow up solutions $u : \mathbf{R}^{2+1} \longrightarrow \mathbb{S}^2$ whose data are close in energy to the family of ground states (i.e. Q up to symmetries) and which blow up at the origin:

$$\vec{u}(t,x) = \mathcal{R}_{h(t)}^{\alpha(t),\beta(t)} \mathcal{L}_v(\vec{Q}_{\lambda(t)}) + \epsilon(t,x)$$

where $\mathcal{R}_{h(t)}^{\alpha(t),\beta(t)}$ is a general rotation in $SO(3)$ and $\mathcal{L}_v$ a Lorentz transform.

Based on Grinis soliton resolution and an exterior energy estimate. Ongoing work by Krieger-Miao-S. to **construct** such examples. In fact, we prove a nonequivariant version of the Krieger-Miao 2020 theorem: complete rigidity of the KST blowup solutions close to self-similar rate, under small sufficiently regular perturbations.

Question: How do Raphael-Rodnianski (smooth) blow up solutions behave under nonequivariant perturbations? Does the blowup vanish?

The wave map nonequivariant stability argument

Starting from equivariant solution

$$\Phi(t,r,\theta) := (\cos\theta \sin U(t,r), \sin\theta \sin U(t,r), \cos U(t,r)).$$

introduce the orthonormal frame (i.e, fix a gauge)

$$E_1 = \partial_U = (\cos\theta \cos U, \sin\theta \cos U, -\sin U)$$
$$E_2 = (\sin U)^{-1}\partial_\theta = (-\sin\theta, \cos\theta, 0)$$

Write a nonequivariant perturbation of Φ in the form

$$\Psi := \Phi + \Pi_{\Phi^\perp}\varphi + a\left(\Pi_{\Phi^\perp}\varphi\right)\Phi, \quad a\left(\Pi_{\Phi^\perp}\varphi\right) = \sqrt{1 - |\Pi_{\Phi^\perp}\varphi|^2} - 1$$
$$\Pi_{\Phi^\perp}\varphi = \varphi_1(t,r,\theta)E_1 + \varphi_2(t,r,\theta)E_2$$

Write (WM) as a system for (φ_1, φ_2).

Linear operator, angular momenta

$$\mathfrak{L}\,\varphi := \begin{pmatrix} \left(\partial_R^2 + \frac{1}{R}\partial_R + \frac{1}{R^2}\partial_\theta^2 - \frac{R^4-6R^2+1}{R^2(R^4+2R^2+1)}\right)\varphi_1 - \frac{2-2R^2}{R^2(1+R^2)}\partial_\theta\varphi_2 \\ \left(\partial_R^2 + \frac{1}{R}\partial_R + \frac{1}{R^2}\partial_\theta^2 - \frac{R^4-6R^2+1}{R^2(R^4+2R^2+1)}\right)\varphi_2 + \frac{2-2R^2}{R^2(1+R^2)}\partial_\theta\varphi_1 \end{pmatrix}.$$

Or as a Fourier expansion

$$\varphi_1(t,R,\theta) = \sum_n \widehat{\varphi}_1(n,t,R)e^{in\theta},$$
$$\varphi_2(t,R,\theta) = \sum_n \widehat{\varphi}_2(n,t,R)e^{in\theta}$$

For fixed n, the linear operator acting on Fourier modes is

$$\mathfrak{L}_n\,\widehat{\varphi}(n) :=$$
$$\begin{pmatrix} \left(\partial_R^2 + \frac{1}{R}\partial_R - \frac{n^2}{R^2} - \frac{R^4-6R^2+1}{R^2(R^4+2R^2+1)}\right)\widehat{\varphi}_1(n) - in\frac{2-2R^2}{R^2(1+R^2)}\widehat{\varphi}_2(n) \\ \left(\partial_R^2 + \frac{1}{R}\partial_R - \frac{n^2}{R^2} - \frac{R^4-6R^2+1}{R^2(R^4+2R^2+1)}\right)\widehat{\varphi}_2(n) + in\frac{2-2R^2}{R^2(1+R^2)}\widehat{\varphi}_1(n) \end{pmatrix}.$$

Symmetries generate six zero modes

Space dilation

$$\mathfrak{L}\begin{pmatrix} R(1+R^2)^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Space translations, Lorentz transforms

$$\mathfrak{L}\begin{pmatrix} (1+R^2)^{-1}\cos\theta \\ -(1+R^2)^{-1}\sin\theta \end{pmatrix} = \mathfrak{L}\begin{pmatrix} (1+R^2)^{-1}\sin\theta \\ (1+R^2)^{-1}\cos\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Rotations of $\mathbb{S}^2$

$$\mathfrak{L}\begin{pmatrix} 0 \\ R(1+R^2)^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathfrak{L}\begin{pmatrix} -\cos\theta \\ (1-R^2)(1+R^2)^{-1}\sin\theta \end{pmatrix} = \mathfrak{L}\begin{pmatrix} \sin\theta \\ (1-R^2)(1+R^2)^{-1}\cos\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Only lead to zero modes for angular momenta $0, \pm 1$. So $|n| \geq 2$ not affected.

Solving for $\widehat{\varphi_1}(n,\tau,R), \widehat{\varphi_2}(n,\tau,R)$

Insert ansatz into extrinsic (WM) system

$$\Psi_{tt} - \Delta\Psi + (|\Psi_t|^2 - |\nabla\Psi|^2)\Psi = 0$$

yields

$$-\left(\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)^2 + \frac{\lambda'}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\right)\varphi + \mathfrak{L}\varphi = \lambda^{-2}\mathfrak{N}(\varphi),$$

$$\mathfrak{N}(\varphi) =: \begin{pmatrix} \mathfrak{N}(\varphi_1) \\ \mathfrak{N}(\varphi_2) + \frac{(1+\nu)^2}{\nu^2\tau^2}\frac{4R^2}{1+2R^2+R^4}\varphi_2 \end{pmatrix}.$$

Linear operator acting on $\widehat{\varphi_1} \pm i\widehat{\varphi_2}$ is diagonal, of the form

$$\begin{bmatrix} H_n^+ & 0 \\ 0 & H_n^- \end{bmatrix}, \qquad H_n^\pm = \partial_R^2 + \frac{1}{R}\partial_R - f_n(R) + g_n(R)$$

$$f_n(R) := \frac{n^2}{R^2} + \frac{R^4-6R^2+1}{R^2(R^4+2R^2+1)}, \quad g_n(R) := n\frac{2-2R^2}{R^2(R^2+1)}$$

Summing over the frequencies

One needs to conduct the spectral analysis of $H_n^{\pm}$ on $L^2(R\,dR)$ with precise understanding of the constants in terms of n, and for all energies. We solve the infinite system indexed by the angular momentum by applying the distorted Fourier transform associated with each $H_n^{\pm}$. In the end all estimates are summed over the angular momentum. We cannot allow any exponential losses in n.

Theorem: *large* k, *if* $\|\varphi_i\|_{H^k(\mathbb{R}^2)} + \|\partial_t \varphi_i\|_{H^{k-1}(\mathbb{R}^2)} < \delta_0$, $i = 1, 2$, *then with the KST* Φ *the initial ansatz (with* ∂_t *)*

$$\Psi := \Phi + \Pi_{\Phi^\perp}\varphi + a\,(\Pi_{\Phi^\perp}\varphi)\,\Phi$$

$$\Pi_{\Phi^\perp}\varphi = \varphi_1(t_0, r, \theta)E_1(\Phi) + \varphi_2(t_0, r, \theta)E_2(\Phi)$$

give a (nonequivariant) solution blowing up at $(0, 0)$

$$\vec{u}(t, x) = \mathcal{R}_{h(t)}^{\alpha(t),\beta(t)} \mathcal{L}_{v(t)}(\vec{Q}_{\lambda(t)}) + \epsilon(t, x)$$

where $\mathcal{R}_{h(t)}^{\alpha(t),\beta(t)}$ *is a general rotation in* $SO(3)$ *and* $\mathcal{L}_{v(t)}$ *a Lorentz transform. All six parameters converge as* $t \to 0+$.

Some general remarks on the argument

- Nullform structure of (WM) is essential, otherwise the light cone produces unmanageable singularities
- Nonlinear analysis completely different from the standard non-equivariant methods (Tataru, Tao space)
- Gauge is implicit in the choice of frame
- All estimates are performed in the Fourier variables, which are different for each angular momentum n
- We do not work in the energy norm, bootstrap in a much stronger norm expressed on the Fourier side. It is different for each angular momentum.
- Nonlinear analysis is specific to the particular rigid KST blowup. It would be impossible to carry this out if the blowup location in space-time changed. Such as it most likely does for the Raphael-Rodnianski solutions.
- Not clear how to avoid fibration in angular momenta: symmetries only affect $n = 0, \pm 1$ but all modes coupled.

Dispersive estimates on manifolds

- Consider wave or Schrödinger evolution on (hyper)surface of revolution with conic ends. How does local geometry affect the long-time behavior of the flow?
- Examples: one sheeted hyperboloid with a unique periodic geodesic, exponentially unstable. Now glue in a sphere in place of the neck, with lots of stable geodesics. How is the estimate affected?
- Answer: local geometry only affects the constant in front of the standard Euclidean power law in terms of growth in the angular momentum (polynomial for the hyperboloid, exponential for the sphere glued in). The polynomial bounds can be summed, leading to a finite number of angular derivatives on the data.
- This relies on the same WKB (or Liouville-Green) techniques, and was established in 2005-2011 (Costin, Donninger, S., Soffer, Tanveer) in a series of papers. In particular, we obtained the decay law for linear waves on a Schwarzschild black hole background.

Dispersive estimates on manifolds II

Theorem [Soffer-Staubach-S 06]: *Let* $\mathcal{M}$ *be a surface which is asymptotically conical at both ends as defined above. For each* $\ell \geq 0$ *and all* $0 \leq \sigma \leq \sqrt{2}\ell$, *there exist constants* $C(\ell, \mathcal{M}, \sigma)$ *and* $C_1(\ell, \mathcal{M}, \sigma)$ *such that for all* $t > 0$

$$\|w_\sigma \, e^{it\Delta_\mathcal{M}} f\|_{L^\infty(\mathcal{M})} \leq \frac{C(\ell, \mathcal{M}, \sigma)}{t^{1+\sigma}} \Big\| \frac{f}{w_\sigma} \Big\|_{L^1(\mathcal{M})}$$

$$\|w_\sigma \, e^{it\sqrt{-\Delta_\mathcal{M}}} f\|_{L^\infty(\mathcal{M})} \leq \frac{C_1(\ell, \mathcal{M}, \sigma)}{t^{\frac{1}{2}+\sigma}} \Big(\Big\| \frac{\partial_x f}{w_\sigma} \Big\|_{L^1(\mathcal{M})} + \Big\| \frac{f}{w_\sigma} \Big\|_{L^1(\mathcal{M})} \Big)$$

provided $f = f(x, \theta) = e^{i\ell\theta} \tilde{f}(x)$ *where* $\tilde{f}$ *does not depend on* θ. *Here* $w_\sigma(x) := \langle x \rangle^{-\sigma}$ *are weights on* $\mathcal{M}$.

Dispersive estimates on manifolds III

Theorem [Donninger-Soffer-S 09]: *Let $\mathcal{M}$ be asymptotically conical at both ends as above and suppose that $\mathcal{M}$ has a unique periodic geodesic and is uniformly convex near it. Then for all $t > 0$, and any $\varepsilon > 0$,*

$$\|w_{1+\varepsilon} e^{it\Delta_{\mathcal{M}}} w_{1+\varepsilon} f\|_{L^2(\mathcal{M})} \leq \frac{C(\mathcal{M}, \varepsilon)}{\langle t \rangle} \|(1 - \partial_\theta^2) f\|_{L^2(\mathcal{M})}$$

$$\|w_1 e^{it\Delta_{\mathcal{M}}} w_1 f\|_{L^\infty(\mathcal{M})} \leq \frac{C(\mathcal{M}, \varepsilon)}{t} \|(1 - \partial_\theta^2)^{2+\varepsilon} f\|_{L^1(\mathcal{M})}$$

provided $f = f(x, \theta)$ is Schwartz on $\mathcal{M}$. Analogous statement for the wave equation, including a Schwarzschild background.

Semiclassical spectral problem

We face singular perturbation problem, of a universal character

$$-\frac{1}{(n+1)^2}\partial_R^2 f + V(R)f = \frac{E^2}{(n+1)^2}f,$$
$$V(R) := \frac{1}{R^2} - \frac{1}{4(n+1)^2R^2} - \frac{4n}{(n+1)^2}\frac{1}{R^2(R^2+1)} - \frac{8}{(n+1)^2}\frac{1}{(R^2+1)^2}.$$

Technique needs to use $V > 0$ and $V' < 0$, such as WKB.
Switching to semiclassical notation, we introduce $\hbar := \frac{1}{n+1}$ and write $V(R) = V_n(R) = V(R;\hbar)$ as

$$V(R) = V(R;\hbar) = \frac{1}{R^2}\left(1 - \frac{\hbar^2}{4} - \frac{4\hbar}{R^2+1} + \frac{4\hbar^2(1-R^2)}{(R^2+1)^2}\right)$$
$$:= \frac{1}{R^2}\left(1 + \frac{15\hbar^2}{4} - 4\hbar\right) + \hbar\varepsilon(R^2;\hbar)$$

with $\varepsilon(R^2;\hbar) := 4(R^2+1)^{-1} - 4\hbar(R^2+3)(R^2+1)^{-2}$.

Turning point analysis

Scale E out with $x := \hbar ER$. Define $\tilde{f}(x) := f(R)$, then

$$-\hbar^2 \tilde{f}''(x) + Q(x)\tilde{f}(x) = 0, \quad Q(x) := \hbar^{-2}E^{-2}V\left(\frac{x}{\hbar E}\right) - 1.$$

More precisely, with $\alpha := \hbar E$,

$$Q(x,\alpha;\hbar) = x^{-2}\left(1 + \frac{15\hbar^2}{4} - 4\hbar\right) + \alpha^{-2}\varepsilon\left(\frac{x^2}{\alpha^2};\hbar\right) - 1.$$

Langer modification: add a multiple of x^{-2} to the potential:

$$Q_0(x;\alpha,\hbar) := Q(x;\alpha,\hbar) + \frac{\hbar^2}{4x^2} = x^{-2}(1-2\hbar)^2 + \alpha^{-2}\varepsilon\left(\frac{x^2}{\alpha^2};\hbar\right) - 1.$$

Turning point $Q_0(x_t;\alpha,\hbar) = 0$ is unique and near 1.
Most delicate case is $0 < \alpha < 1$, $0 < x < 1$.

Liouville Green transform from 1837

Given $-\hbar^2 y''(x) + Q(x)y(x) = 0$, $x \in I \subset \mathbb{R}$ interval. Approximate form of fundamental system? Set $y(x) = (\xi'(x))^{-\frac{1}{2}} w(\xi(x))$. Then

$$-\hbar^2 \ddot{w}(\xi) + Q(x)(\xi'(x))^{-2} w(\xi) = \hbar^2 \Big[\frac{3}{4}\Big(\frac{\xi''(x)}{\xi'(x)^2}\Big)^2 - \frac{1}{2}\frac{\xi'''(x)}{\xi'(x)^3}\Big] w(\xi)$$

If $\xi'(x) = \sqrt{Q(x)}$, then $y(x) \simeq Q(x)^{-\frac{1}{4}} e^{\pm\hbar^{-1}\int^x \sqrt{Q(u)}\,du}$. Need $Q > 0$ on I. If $Q(x_0) = 0$, $Q' > 0$, set $Q(x)(\xi'(x))^{-2} = \xi$. Then $\xi(x) = \mathrm{sign}(x - x_0)\Big(\frac{3}{2}\int_{x_0}^x \sqrt{|Q(u)|}\,du\Big)^{\frac{2}{3}}$, smooth since x_0 is a simple zero. Then $y(x) \simeq (\xi/Q)^{\frac{1}{4}} \mathrm{Ai}(\hbar^{-\frac{2}{3}}\xi)$, $(\xi/Q)^{\frac{1}{4}} \mathrm{Bi}(\hbar^{-\frac{2}{3}}\xi)$.
In our case we apply this transformation globally since Q is monotone with a unique zero (turning point). It is essential to obtain an exact fundamental system of the form leading order $\times(1 + \hbar a(\xi, \alpha, \hbar))$ with uniform bounds on a and its derivatives.

Fourier basis

Fourier transform associated with $-\mathcal{H}_n^+$, $n \gg 1$:

$$\hat{f}(\xi) = \int_0^\infty \phi_n(R,\xi) f(R)\, dR, \qquad f(R) = \int_0^\infty \phi_n(R,\xi)\hat{f}(\xi)\rho_n(d\xi)$$

with

$$\phi_n(R;\xi) = \hbar^{\frac{1}{3}}\alpha^{-\frac{1}{2}}(-\tau/Q_0)^{\frac{1}{4}}(\tau)\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tau)(1+\hbar a_0(-\tau;\alpha,\hbar))$$

One has $\phi_n(R;\xi) \sim \xi^{\frac{n-1}{2}} R^{n-\frac{1}{2}}$ as $R \to 0+$. To the right of the turning point $\phi_n(R;\xi) = -c_1\hbar^{-\frac{1}{6}}\xi^{-\frac{1}{4}}(-\tau/Q_0)^{\frac{1}{4}}(\tau)$

$$\mathrm{Re}\left(\left(1+\hbar\Xi(\xi;\hbar)\right)\left(\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tau)+i\mathrm{Bi}(-\hbar^{-\frac{2}{3}}\tau)\right)(1+\hbar a_1(\tau;\alpha,\hbar))\right)$$

$$\sim -c_2\xi^{-\frac{1}{4}}\,\mathrm{Re}\left(\left(1+\hbar\Xi(\xi;\hbar)\right)e^{i\frac{\pi}{4}}e^{i\xi^{\frac{1}{2}}R}\right) \quad \text{as} \quad R\to\infty,$$

where $c_1, c_2 > 0$ are absolute constants and $|\partial_\xi^k \Xi(\xi;\hbar)| \le C_k\,\xi^{-k}$ for all $k \ge 0$ uniformly in $\hbar$. The spectral measure ρ_n is purely absolutely continuous with density satisfying

$$\frac{1}{2} \le \frac{d\rho_n(\xi)}{d\xi} \le 2, \qquad \left|\frac{d^\ell}{d\xi^\ell}\frac{d\rho_n(\xi)}{d\xi}\right| \le C_\ell\,\xi^{-\ell}, \quad \forall\, \xi > 0,\ \ell \ge 0$$