Almost global well-posedness for quasilinear strongly coupled wave-Klein-Gordon systems in two space dimensions

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This is joint work with A. Stingo

Introduction

We consider real solutions for the wave-Klein-Gordon system

$$\begin{cases} (\partial_t^2 - \Delta_x)u(t,x) = \mathbf{N_1}(v, \partial v) + \mathbf{N_2}(u, \partial v) \\ (\partial_t^2 - \Delta_x + 1)v(t,x) = \mathbf{N_1}(v, \partial u) + \mathbf{N_2}(u, \partial u) \end{cases} \qquad (t,x) \in [0, +\infty) \times \mathbb{R}^2,$$

with initial conditions

$$\begin{cases} (u,v)(0,x) = (u_0(x), v_0(x)) \\ (\partial_t u, \partial_t v)(0,x) = (u_1(x), v_1(x)). \end{cases}$$

Here the nonlinearities $\mathbf{N_1}(\cdot,\cdot)$ and $\mathbf{N_2}(\cdot,\cdot)$ are combinations of the classical quadratic null forms

$$\begin{cases} Q_{ij}(\phi,\psi) = \partial_i\phi\partial_j\psi - \partial_j\phi\partial_i\psi, \\ Q_{0i}(\phi,\psi) = \partial_t\phi\partial_i\psi - \partial_t\psi\partial_i\phi, \\ Q_0(\phi,\psi) = \partial_t\phi\partial_t\psi - \nabla_x\psi\cdot\nabla_x\phi. \end{cases}$$

- Physical models related to general relativity have shown the importance of studying such systems.
- Very few results are known at present in low (2) space dimensions.

Vector fields associated with the WKG system

Translation in the coords direct.:	$\partial_t,\ \partial_1,\ \partial_2$
Rotations in x:	$\Omega_{ij} = x_j\partial_i - x_i\partial_j$
Hyperbolic rotations:	$\Omega_{0i} = t\partial_i + x_i\partial_t$
Scaling:	$\mathscr{S} = t\partial_t + r\partial_r$

Here $1 \leq i \neq j \leq 2$, $r = |x|$ and $\partial_r = \frac{x}{r}\cdot\nabla_x$

- We denote $Z := \{\Omega_{ij}, \Omega_{0i}\}$ *Lorentz vector fields.*
- We denote $\mathcal{Z} := \{\partial_0, \partial_1, \partial_2, \Omega_{ij}, \Omega_{0i}\}$ the full set of vector fields associated to the symmetries of the linear problem.

Notation

For a multiindex $\gamma = (\alpha, \beta)$ we denote $\mathcal{Z}^\gamma = \partial^\alpha Z^\beta$ and define the size

$$|\gamma| = |\alpha| + h|\beta|$$

Here $h \in \mathbb{N} \rightsquigarrow$ balance between Lorentz v.f. and reg. derivatives

Energy functionals and Functional Spaces

The linear system WKG has an associated conserved energy

$$E(t;u,v) = \int_{\mathbb{R}} u_t^2 + u_x^2 + v_t^2 + v_x^2 + v^2 \, dx$$

The system linear WKG system is a well-posed linear evolution in the space $\mathcal{H}^0$ with norm

$$\|(u[t],v[t])\|^2_{\mathcal{H}^0} := \|u\|^2_{\dot{H}^1} + \|u_t\|^2_{L^2} + \|v\|^2_{H^1} + \|v_t\|^2_{L^2}$$

where we use the following notation for the Cauchy data in WKG system at time t:

$$(u[t],v[t]) := (u(t),u_t(t),v(t),v_t(t))$$

Higher order functional spaces

The higher order energy spaces for the linear WKG system are the spaces $\mathcal{H}^n$ endowed with the norm

$$\|(u_0, u_1, v_0, v_1)\|^2_{\mathcal{H}^n} := \sum_{|\alpha| \leq n} \|\partial_x^\alpha (u_0, u_1, v_0, v_1)\|^2_{\mathcal{H}^0},$$

where $n \geq 1$. We use the energy spaces for the nonlinear system!

Higher order counterparts of the energy functionals:

a) the energy $E^n(t, u, v)$ measures the regularity in the function space $\mathcal{H}^n$ of the solutions that carry n derivatives,

$$E^n(t, u, v) := \sum_{|\alpha| \leq n} E\left(t; \partial^\alpha u, \partial^\alpha v\right)$$

b) the energy $E^{[n]}(t, u, v)$ which in addition to regular derivatives, keeps track of Z vector fields applied to the solution,

$$E^{[n]}(t, u, v) := \sum_{|\gamma| \leq n} E\left(t; \mathcal{Z}^\gamma u, \mathcal{Z}^\gamma v\right)$$

Scaling, criticality and local well-posedness

Scaling

We define the notion of criticality by means of the scaling symmetry matched at the highest order:

$$\begin{cases} u(t,x) \to \lambda^{-1} u(\lambda t, \lambda x) \\ v(t,x) \to \lambda^{-1} v(\lambda t, \lambda x). \end{cases}$$

This, leads to the critical Sobolev space $\mathcal{H}^{s_c}$ with $s_c = d/2 + 1$.

Hyperbolic quasilinear system

Thus, it is not too difficult to show that in two dimensions WKG is locally well-posed in $\mathcal{H}^n$ for $n \geq 4$ (or $\mathcal{H}^{3+\epsilon}$ if we do not restrict ourselves to integers). Lower regularity than that would require different set of tools.

Control norms

To describe the lifespan of the solutions we define the control norms

- The following is a scale invariant quantity:

$$A := \sum_{|\alpha|=1} \|D_x^\alpha u\|_{L^\infty} + \sum_{|\alpha|=1} \|D_x^\alpha v\|_{L^\infty} + \|u_t\|_{L^\infty} + \|v_t\|_{L^\infty}$$

  This needs to remain small throughout in order to guarantee the hyperbolicity of the system.

- The following norm (and in particular its smallness) assures the propagation of higher regularity.

$$B := \sum_{|\alpha|\leq 2} \|D_x^\alpha u\|_{L^\infty} + \sum_{|\alpha|\leq 2} \|D_x^\alpha v\|_{L^\infty} + \|u_{tt}\|_{L^\infty} + \|v_{tt}\|_{L^\infty}$$

Main question:

Study the long time well-posedness problem for the nonlinear WKG system for small and localized initial data.

Theorem

Let $h \geq 7$. Assume that the initial data $(u[0], v[0])$ for WKG equation satisfies

$$\|(u[0], v[0])\|_{\mathcal{H}^{2h}} + \|x\partial_x(u[0], v[0])\|_{\mathcal{H}^h} + \|x^2\partial_x^2(u[0], v[0])\|_{\mathcal{H}^0} \leq \epsilon \ll 1.$$

Then the WKG equation is almost globally well-posed in $\mathcal{H}^{2h}$, with L^2 bounds as follows:

$$E^{[2h]}(t, u, v) \lesssim \epsilon^2,$$

and pointwise bounds

$$|\partial^j v| \lesssim \epsilon \langle t + r\rangle^{-1}, \qquad j = \overline{0, 3},$$

$$|\partial^j u| \lesssim \epsilon \langle t + r\rangle^{-\frac{1}{2}} \langle t - r\rangle^{-\frac{1}{2}}, \qquad j = \overline{1, 3},$$

$$|\partial^j Zu| \lesssim \epsilon, \qquad j = \overline{0, 2}.$$

- Forthcoming global result, under the same assumptions.
- We used only minimal x^2 type decay, but we did not attempt to fully optimize the choice of h

What is known about the well-posedness for WKG

$3D$ WKG results:

Gorgiev '90 , Katayama '12.

Related models:

KG systems - Delort '04, '09, '12,'15, '16, Einstein's field equations, Dirac-Klein-Gordon system, etc: LeFloch , Ma '14, '16, Wang '16, massive Dirac-Klein-Gordon system: Bejenaru-Herr(s),Candy-Herr(l).

Global existence of solutions to WKG systems in 3D:

Quasilinear quadratic nonlinearities satisfying suitable conditions, initial data are small, smooth and compactly support → method by Tataru '01 and then used by LeFloch under name: *hyperboloidal foliation method*; Ionescu-Pausader'17

2D WKG results:

Global existence of small amplitude solutions in lower space dimensions → Ma: '17, 19. (semilinear, compactly supported data); Stingo '18 (only Q_0 null forms)

The scaling vector field S

- The main difficulty on this type of system, compared with the pure wave or Klein-Gordon systems, is the lack of symmetry. The conformal Killing vector filed $S = x_\alpha \partial_\alpha$ of the linear wave operator is no longer conformal Killing with respect to the linear Klein-Gordon operator.

- This prevents any possibility of naive combination of the methods for wave equations with those for Klein-Gordon equations.

Quadratic resonant interactions

Wave equation: dispersion relation

$$\omega_W(\xi) = \pm|\xi|$$

Klein-Gordon equation: dispersion relation

$$\omega(\xi)_{KG} = \pm\sqrt{|\xi|^2 - 1}$$

- Two wave resonant interactions for the wave eq alone occur only in between parallel waves (null condition helps).
- Two wave resonant interactions for the KG equation alone or mixed wave - KG never occur.
- However, in the last two cases there is a near resonance for almost parallel waves in the high frequency limit, which becomes stronger in a quasilinear setting.

Quadratic resonances and normal forms

Suppose that N_1 and N_2 are of Ω_{ij} type. Then $u \times v$ interactions do not cancel at second order along paralel directions: they lead to an unbounded bilinear symbol in the normal form transformation

$$c(\eta, \xi) = \frac{2\langle \xi, \eta \rangle \, \xi \wedge \eta}{|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2 + |\xi|^2}$$

$\rightarrow$ If ξ is at frequency ≈ 1 and η is very large then the symbol of $C(u, v)$ can become unbounded as the angle in between ξ and η (let's call it θ) becomes very small:

$$c(\xi, \eta) = \frac{\eta^2 \, \theta}{\eta^2 \, \theta^2 + 1} \approx \eta \text{ or } \approx \frac{1}{\theta}$$

This says that the normal form introduces a derivative every time is used. Hence a normal form approach cannot be used!

Sketch of the proof

Standard approach has two main steps

- (i) vector field fixed time energy estimates,
- (ii) fixed time pointwise bounds derived from energy estimates (Klainerman-Sobolev inequalities).

Novelty: a twist of the standard approach

- Energy estimates are space-time L^2 *local energy bounds*, localized to dyadic regions $C^{\pm}_{TS}$, where T stands for dyadic time, S for the dyadic distance to the cone, and $\pm$ for the interior/exterior cone.
- Similarly, pointwise bounds are akin to Sobolev embeddings or interpolation inequalities in the same type of regions.

$$C^{+}_{TS} := \{(t,x) \,:\, S \leq t - r \leq 2S,\, T \leq t \leq 2T\}\,, \text{ where } 1 \leq S \lesssim T$$
$$C^{-}_{TS} := \{(t,x) \,:\, S \leq r - t \leq 2S,\, T \leq t \leq 2T\}\,, \text{ where } 1 \leq S \lesssim T$$

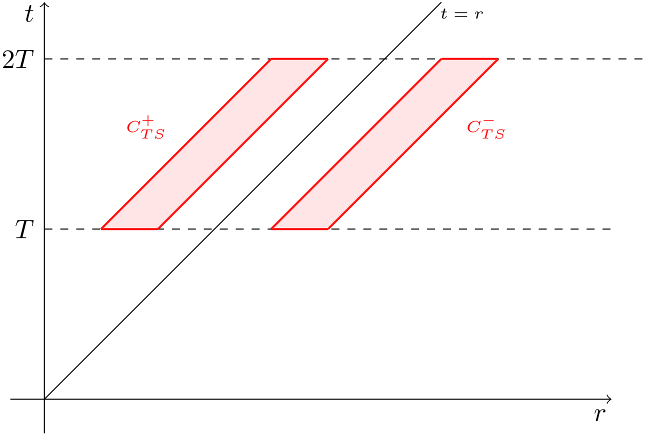

Figure: 1D vertical section of space-time regions $C^{\pm}_{TS}$

$\rightarrow$ Metcalfe - Tataru - Tohaneanu

Exterior region: $C_T^{out} := \{T \leq t \leq 2T,\ r \gg T\}$, treated directly

Prerequisites for the proof

These have to do with the local in time theory for our evolution:

- Local well-posedness in $\mathcal{H}^4$ (also in $\mathcal{H}^n$ for $n \geq 4$).
- Continuation of $\mathcal{H}^4$ solutions for as long as $\partial^2(u, v)$ remain bdd+ propagation of higher regularity, i.e. bounds in $\mathcal{H}^n$ $\forall n$.
- Uniform finite speed of propagation as long as $|\nabla v|$ stays pointwise small.

Our proof is set up as a bootstrap argument, where the bootstrap assumption is on pointwise decay bounds for the solution:

$$|Zu| \leq C\epsilon \langle t-r \rangle^{\frac{\delta}{2}},$$

$$\begin{gathered} |\partial u| \leq C\epsilon \langle t+r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{1}{2}+\frac{\delta}{2}}, \\ |Z\partial^j u| \leq C\epsilon, \qquad j = \overline{1,2}, \\ |\partial^{j+1} u| \leq C\epsilon \langle t+r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{1}{2}-\delta}, \qquad j = \overline{1,2}, \\ |\partial^j v| \leq C\epsilon \langle t+r \rangle^{-1}, \quad j = \overline{1,3}. \end{gathered}$$

Part 1 of the proof: Energy Estimates

Energy estimates

Consider a solution (u, v) to WKG in a time interval $[0, T_0]$, which is a-priori assumed to satisfy the pointwise bootstrap assumptions.
Then (u, v) satisfies the energy estimates in $[0, T_0]$:

$$E^{[2h]}(u, v)(t) \lesssim \langle t \rangle^{\tilde{C}\epsilon} E^{[2h]}(u, v)(0), \qquad t \in [0, T_0].$$

- $\tilde{C}$ is a large constant -depends on C in our bootstrap assumption, $\tilde{C} \approx C$. However, the implicit constant in energy estimates cannot depend on C.
- The time T_0 is arbitrary!

Part 2 of the proof: Uniform Bounds

Pointwise bounds

Assume (u,v) a sol to WKG in a time interval $[0,T_0]$, such that the energy bounds hold

$$E^{[2h]}(u,v)(t) \lesssim \epsilon \langle t \rangle^{\tilde{C}\epsilon}, \qquad t \in [0,T_0].$$

Then we show (u,v) satisfies the pointwise bounds

$$\begin{aligned}
\|Zu\|_{L^\infty} &\leq \epsilon \langle t \rangle^{\tilde{C}\epsilon}, \\
|\partial u| &\leq \epsilon \langle t \rangle^{\tilde{C}\epsilon} \langle t+r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{1}{2}}, \\
\|Z\partial^j u\|_{L^\infty} &\leq \epsilon \langle t \rangle^{\tilde{C}\epsilon}, \qquad j = \overline{1,2}, \\
|\partial^j u| &\leq \epsilon \langle t \rangle^{\tilde{C}\epsilon} \langle t+r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{1}{2}-2\delta}, \qquad j = \overline{2,3}, \\
|\partial^j v| &\leq \epsilon \langle t \rangle^{\tilde{C}\epsilon} \langle t+r \rangle^{-1}, \quad j = \overline{0,3}.
\end{aligned}$$

Lifespan T_0 is again arbitrary

Conclusion of the proof

In both steps, the time T_0 is arbitrary. However, in order to close the bootstrap argument one needs to recover the bootstrapped assumptions/ bounds from what we need to show. This requires

$$T_0^{\epsilon \tilde{C}} \ll C \rightarrow T_0 \ll e^{\frac{c}{\epsilon}},$$

i.e. our almost global result.

- Previous work in higher D is done in higher regularity setting (large number of v.f) both in the energy estimates and in the pointwise bounds, and the argument works as above.
- Both steps require only fixed time bounds, and the pointwise bounds are akin to an improved form of the Sobolev embeddings.

This approach fails in $2+1$ dimensions because there is less dispersive decay, and the problem is strongly quasilinear! Thus, analysis must be adapted to the light cone geometry!

Energy estimates

- (a) for the linearized equation
- (b) for the solution and its higher derivatives
- (c) for the vector fields applied to the solution

- The main work goes into the energy estimates for the linearized system.
- Equations for higher derivatives $\partial^\alpha(u, v)$ and vector fields $\mathcal{Z}^\beta(u, v)$ are interpreted as the linearized equations with source terms.
- Source terms are estimated perturbatively using the null structure and interpolation inequalities.

Linearized WKG

(U,V) = linearized variables:

$$\begin{cases} (\partial_t^2 - \Delta_x) U(t,x) = \mathbf{N_1}(v,\partial V) + \mathbf{N_1}(V,\partial v) + \mathbf{N_2}(u,\partial V) + \mathbf{N_2}(U,\partial v) + F \\ (\partial_t^2 - \Delta_x + 1) V(t,x) = \mathbf{N_1}(v,\partial U) + \mathbf{N_1}(V,\partial u) + \mathbf{N_2}(u,\partial U) + \mathbf{N_2}(U,\partial u) + G \end{cases}$$

- Fixed time energy estimate for the homogeneous linearized equations

$$E(U,V)(t) \lesssim t^{C\epsilon} E(U,V)(0), \quad t \in [0,T]$$

- Replace linear energy $E(U,V)$ with $E^{quasi}(U,V)$: better adapted to lin. pb.

$$E^{quasi}(U,V) := E\,(U,V) + \int_{\mathbb{R}^2} B_1(\partial v; \partial U, \partial V) + B_2(\partial u; \partial U, \partial V)\, dx$$

and

$$E^{quasi}(U,V)(t) \lesssim t^{C\epsilon} E^{quasi}(U,V)(0), \quad t \in [0,T]$$

- Time dyadic version

$$\sup_{t\in[T,2T]} E^{quasi}(U,V)(t) \lesssim (1+\epsilon C) E^{quasi}(U,V)(T).$$

Space-time norms

Additional space-time bound

$$\sup_{1\leq S\lesssim T}\int_{C_{TS}}\frac{1}{S}\left\{\left(V_j+\frac{x_j}{r}V_t\right)^2+\left(U_j+\frac{x_j}{r}U_t\right)^2+V^2\right\}dxdt\lesssim\sup_{t\in[T,2T]}E^{quasi}(U,V)(t).$$

- Helps to bound the time derivative of the energy

$$\frac{d}{dt}E^{quasi}(U,V)=\int N(\partial^2u,\partial U,\partial V)+N(\partial^2v,\partial U,\partial V)\,dx$$

- Proved using Alinhac's ghost weight method with weights adapted to each C_{TS}

Final** space-time norm:

$$\|(U,V)\|^2_{X_T}:=\sup_{t\in[T,2T]}E^{quasi}(U,V)(t)+\sup_{1\leq S}S^{-1}\left(\|\mathcal{T}(U,V)\|^2_{L^2_{C_{TS}}}+\|V\|^2_{L^2_{C_{TS}}}\right)$$

**Uniform energy bounds on hyperboloids are also included in X_T^+, but omitted for simplicity.

Energy bounds for the inhomogeneous problem

(i) Uniform in time bound

$$\sup_{t\in[T,2T]} E^{quasi}(U,V)(t) \lesssim (1+\epsilon C) E^{quasi}(U,V)(T) + \|(F,G)\|_{Y^T}$$

(ii) Space-time bound

$$\|(U,V)\|_{X^T} \lesssim E^{quasi}(U,V)(T) + \|(F,G)\|_{Y^T}.$$

where the norm Y^T for the source term is given by

$$\|(F,G)\|_{Y^T} = \sup_{1\leq S\leq T} T^{\frac{1}{2}} \|(F,G)\|_{L^2(C_{TS})}.$$

Klainerman-Sobolev inequalities

Theorem

Let $h \geq 7$. Assume that the functions (u, v) in C_T^{in} satisfy the bounds

$$\|\mathcal{Z}^\gamma(u, v)\|_{X^T} \leq 1, \qquad |\gamma| \leq 2h,$$

as well as

$$\|\mathcal{Z}^\gamma(\Box u, (\Box + 1)v)\|_{Y^T} \leq 1, \qquad |\gamma| \leq h.$$

Then they also satisfy the pointwise bounds

$$|\partial u| \lesssim \langle t \rangle^{-\frac{1}{2}} \langle t - r \rangle^{-\frac{1}{2}},$$

$$|Zu| \lesssim 1,$$

$$|\partial^j u| \lesssim \langle t \rangle^{-\frac{1}{2}} \langle t - r \rangle^{-\frac{1}{2} - \delta}, \qquad j = \overline{2, 3}$$

$$|Z\partial^j u| \lesssim \langle t - r \rangle^{-\delta} \qquad j = \overline{1, 2},$$

$$|\partial^j v| \lesssim \langle t \rangle^{-1}, \quad j = \overline{0, 3}.$$

Main elements of the proof

- Separate proofs in each of the dyadic regions $C^{\pm}_{TS}$.
- Separate arguments for the wave and KG equations
- Use hyperbolic coordinates to represent $C^{\pm}_{TS}$ as a unit size region
- Differentiate between interior and exterior regions relative to the cone
- Vector fields give bounds for derivatives along hyperboloids
- Use the equations to capture information about the scaling derivative
- Use Gagliando-Niremberg-Sobolev inequalities or frequency localized Bernstein's inequalities on $C^{\pm}_{TS}$.
- (optional, more efficient) Use L^2 bounds on hyperboloids in the case of C^{+}_{TS} (inside the cone)

Pointwise bounds for KG inside the cone

Spherical hyperbolic coordinates in $\mathbb{H}^2 \times \mathbb{R}$:

$$\begin{cases} t = e^{\sigma}\cosh(\phi) \\ x_1 = e^{\sigma}\sinh(\phi)\sin(\theta) \\ x_2 = e^{\sigma}\sinh(\phi)\cos(\theta) \end{cases}$$

The KG equation in the new coordinates:

$$-e^{2\sigma}(\Box + 1) = -e^{2\sigma} - \left(\partial_\sigma + \frac{1}{2}\right)^2 + \frac{1}{4} + \partial_\phi^2 + \frac{1}{\sinh^2\phi}\partial_\theta^2 + \frac{\cosh\phi}{\sinh\phi}\partial_\phi,$$

L^2 bounds for the KG sol and vf(KG sol) on hyperboloids H intersected with unit size regions C_{ST}^+:

$$\|\mathcal{Z}^\alpha v\|_{L_h^2(H)} + \|\mathcal{Z}^\alpha \mathcal{T} v\|_{L_h^2(H)} \lesssim T^{-1}, \qquad |\alpha| \leq 2h,$$
$$\|\mathcal{Z}^\alpha \nabla v\|_{L_h^2(H)} \lesssim S^{-\frac{1}{2}} T^{-\frac{1}{2}}, \qquad |\alpha| \leq 2h.$$

Here Z includes $\partial_\phi, \partial_\theta$, i.e. a unit frame on H. Now use Bernstein/Sobolev and interpolation inequalities on $H \cap C_{TS}^+$.

Pointwise bounds for Wave equation outside the cone

Spherical hyperbolic coordinates in $\mathbb{H}^2_{out} \times \mathbb{R}$:

$$\begin{cases} t = e^{\sigma} \sinh(\phi) \\ x_1 = e^{\sigma} \cosh(\phi) \sin(\theta) \\ x_2 = e^{\sigma} \cosh(\phi) \cos(\theta), \end{cases}$$

Wave equation in the new coordinates:

$$-\Box = e^{-2\sigma} \left(\partial_\sigma^2 - \partial_\phi^2 + \frac{1}{\cosh^2(\phi)} \partial_\theta^2 - \partial_\sigma + \frac{\sinh(\phi)}{\cosh(\phi)} \partial_\phi \right).$$

L^2 bounds for the Wave soln and vf(Wave soln) in C^-_{TS} regions:

$$\| \mathcal{Z}^\alpha Z u \|_{L^2_h} \lesssim 1, \qquad |\alpha| \leq 2h$$

$$\| \mathcal{Z}^\alpha (\partial_\sigma - \partial_\phi) u \|_{L^2_h} \lesssim S^{\frac{1}{2}} T^{-\frac{1}{2}}, \qquad |\alpha| \leq 2h$$

$$\| \mathcal{Z}^\alpha (\partial_\sigma - \partial_\phi)(\partial_\sigma + \partial_\phi + 1) u \|_{L^2_h} \lesssim S^{\frac{1}{2}} T^{-\frac{1}{2}}, \qquad |\alpha| \leq h.$$

Here Z includes $\partial_\phi, \partial_\theta$. Now use Bernstein/Sobolev and interpolation inequalities on C^-_{TS}, first two bounds for Zu and last two for $(\partial_\sigma - \partial_\phi)u$.

Thank you.