# Strichartz estimates for the compressible Euler equation with vorticity and low regularity solutions

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**Dynamics in Geometric Dispersive Equations and the Effects of Trapping, Scattering and Weak Turbulence**

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In their standard form, the compressible Euler equations are given by

$$\begin{aligned} \mathbf{B}\varrho + \varrho\,\mathsf{div} v &= 0, \\ \varrho \mathbf{B} v + \nabla p &= 0, \\ \mathbf{B} s &= 0, \end{aligned} \tag{EE-stand}$$

where $v = v(t,x) = (v^1, v^2, v^3)$ is the fluid's velocity, $\varrho = \varrho(t,x)$ is the fluid's density, and $s = s(t,x)$ is the fluid's entropy, $(t,x) \in [0,T) \times \mathbb{R}^3$;

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For irrotational $(\mathrm{curl}\, v = 0)$ and isentropic $(s = constant)$ fluids, the Euler system can be written as a system of quasilinear wave equations of the form

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Q: Without assuming $\operatorname{curl} v = 0$ and $s = constant$, what is minimum $N_*$ to close estimates in $H^{N_*}$ (rule out shocks). $\Rightarrow$ time of classical existence depends only on low-regularity norm of the data.

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Consider a smooth solution to the compressible Euler equations, whose initial data obey the following assumptions for some real numbers $2 < N := 2 + \varepsilon \leq 5/2$, $0 < \alpha < 1$, $0 < D_{\varepsilon,\alpha} < \infty$, $0 < c_1 < c_2$, $0 < c_3$:

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Then the solution's time of classical existence $T$ depends only on $D_{\varepsilon;\alpha}$ and $\mathfrak{K}$, i.e., $T = T(D_{\varepsilon;\alpha}, \mathfrak{K}) > 0$. Moreover, the Sobolev and Hölder regularity of the data is propagated by the solution for $t \in [0, T]$ (norms that we can control are uniformly bounded by functions of $(D_{\varepsilon;\alpha}, \mathfrak{K})$ for $t \in [0, T]$).

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# Theorem (D–, Luo, Mazzone, Speck, 2019)

Consider a smooth solution to the compressible Euler equations, whose initial data obey the following assumptions for some real numbers $2 < N := 2 + \varepsilon \leq 5/2$, $0 < \alpha < 1$, $0 < D_{\varepsilon,\alpha} < \infty$, $0 < c_1 < c_2$, $0 < c_3$:

1. $\|(\varrho - \bar{\varrho}, v, \mathrm{curl}\, v)\|_{H^{2+\varepsilon}(\Sigma_0)} + \|s\|_{H^{3+\varepsilon}(\Sigma_0)} \leq D_{\varepsilon,\alpha}$.
2. The variables $\mathcal{C} \sim (\mathrm{curl}\mathrm{curl} v)/\varrho$ and $\mathcal{D} \sim \partial^2 s$ verify the Hölder-norm bound $\|(\mathcal{C}, \mathcal{D})\|_{C^{0,\alpha}(\Sigma_0)} \leq D_{\varepsilon;\alpha}$.
3. Along $\Sigma_0$, the data functions are contained in the interior of a compact subset $\mathfrak{K}$ of state-space in which $\varrho \geq c_3$ and the speed of sound is bounded from below by $c_1$ and above by $c_2$.

Then the solution's time of classical existence $T$ depends only on $D_{\varepsilon;\alpha}$ and $\mathfrak{K}$, i.e., $T = T(D_{\varepsilon;\alpha}, \mathfrak{K}) > 0$. Moreover, the Sobolev and Hölder regularity of the data is propagated by the solution for $t \in [0, T]$ (norms that we can control are uniformly bounded by functions of $(D_{\varepsilon;\alpha}, \mathfrak{K})$ for $t \in [0, T]$).

Results of independent interest: sharp estimates for the characteristic (acoustic) geometry; Strichartz estimates for waves coupled to vorticity; Schauder estimates for transport-div-curl part.

# Remarks on the assumptions and the result

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Despite the presence of a wave-part, the Euler system cannot be viewed as "wave equations perturbed by smoother transported terms:" the presence of the tiniest amount of vorticity is a "game changer."

# Remarks on the assumptions and the result (cont.)

We have $(\varrho - \bar{\varrho}, v) \in H^{2+\varepsilon}(\Sigma_0)$ but also the "extra" regularity assumptions curl$v \in H^{2+\varepsilon}(\Sigma_0)$, $s \in H^{3+\varepsilon}(\Sigma_0)$ and $\mathcal{C} \sim$ curlcurl$v/\varrho$, $\mathcal{D} \sim \partial^2 s \in C^{0,\alpha}(\Sigma_0)$.

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More recently, Wang considered the isentropic ($s = constant$) case with vorticity ($\mathrm{curl}\, v \neq 0$) and further lowered the regularity to $(\varrho - \bar{\varrho}, v) \in H^{2+\varepsilon}$, $\mathrm{curl}\, v \in H^{2+\varepsilon'}$, $0 < \varepsilon' < \varepsilon$, and no Hölder assumption on the data.

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$$\mathbf{g} := -dt \otimes dt + c^{-2} \sum_{a=1}^{3} (dx^a - v^a dt) \otimes (dx^a - v^a dt),$$

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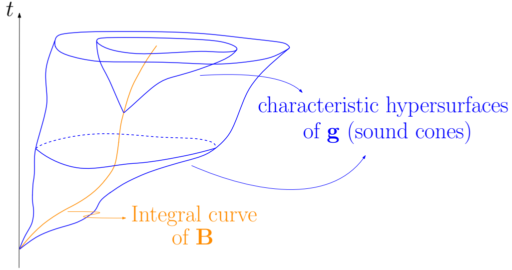


**Figure:** The characteristics of Euler's equations.

Quasilinear: our regularity assumptions are tied to the characteristics of the Euler system: transport-part and wave-part.

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$$\mathcal{C}^i := \exp(-\rho)(\text{curl}\Omega)^i + \exp(-3\rho)\frac{c^{-2}}{\overline{\varrho}}\frac{\partial p}{\partial s}S^a\partial_a v^i - \exp(-3\rho)\frac{c^{-2}}{\overline{\varrho}}\frac{\partial p}{\partial s}(\partial_a v^a)S^i,$$

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# New formulation of Euler's equations (Speck, Speck-Luk)

With $\Psi \in \{\rho, v^1, v^2, v^3, s\}$, solutions to (EE-stand) also satisfy:

$$\square_{\mathbf{g}(\vec{\Psi})}\Psi = \mathscr{L}(\vec{\Psi})[\vec{\mathcal{C}}, \mathcal{D}] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]$$

wave equations

$$\mathbf{B}\Omega^i = \mathscr{L}(\vec{\Psi}, \vec{\Omega}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}]$$

$$\mathbf{B}S^i = \mathscr{L}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}].$$

transport equations

$$\text{div}\Omega = \mathscr{L}(\vec{\Omega})[\boldsymbol{\partial}\vec{\Psi}],$$

$$\mathbf{B}\mathcal{C}^i = \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \partial\vec{\Omega}] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \partial\vec{S}] + \mathscr{Q}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi}, \vec{\Omega}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}],$$

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$$\mathbf{B}\mathcal{D} = \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \partial\vec{S}] + \mathscr{Q}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi}, \vec{S})[\partial\vec{\Omega}],$$

$$(\text{curl}\, S)^i = 0,$$

transport-div-curl equations for the entropy gradient

where $\square_{\mathbf{g}(\vec{\Psi})}$= wave operator w.r.t. $\mathbf{g}$, $\boldsymbol{\partial} = (\partial_t, \partial_i)$, $\mathscr{L}(A)[B]$ is linear in $B$ with coefficients depending on $A$, and $\mathscr{Q}(A)[B, C]$ is quadratic in $B$ and $C$ with coefficients depending on $A$.

# The wave and transport parts and main steps

From the above formulation of the Euler equations, we identify the wave variables (whose dynamics is tied to the sound cones) as $\Psi \in \{\rho, v^1, v^2, v^3, s\}$, and the transport variables (whose dynamics is tied to the flow lines of $\mathbf{B}$) as $\{\Omega, S, \mathcal{C}, \mathcal{D}\}$.

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3. Transport variables appear as source terms in the acoustic geometry estimates. Need to handle the interaction of the acoustic geometry with the transport-part (different speeds).

# Energy estimates ($s = const. \Rightarrow \mathcal{C} = e^{-\rho}\text{curl}\Omega \sim \text{curl}\Omega$)

$$\square_{\mathbf{g}(\vec{\Psi})}\Psi = \mathscr{L}(\vec{\Psi})[\text{curl}\Omega] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}] \tag{1a}$$

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$$\|\boldsymbol{\partial}\Psi\|_{H^{1+\varepsilon}(\Sigma_t)} + \|\partial\Omega\|_{H^{1+\varepsilon}(\Sigma_t)} \lesssim \exp\left(\int_0^t \left(\|\boldsymbol{\partial}\Psi\|_{L^\infty(\Sigma_\tau)} + \|\partial\Omega\|_{L^\infty(\Sigma_\tau)}\right) d\tau\right)$$

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$$\partial^{1+\varepsilon}\text{div}\Omega \sim \mathscr{L}(\vec{\Omega})[\boldsymbol{\partial}\partial^{1+\varepsilon}\vec{\Psi}] \quad (1d)$$

Control $\|\Psi\|_{H^{2+\varepsilon}(\Sigma_t)}$: take $\partial^{1+\varepsilon}$ of (1a). Need control of $\partial^{1+\varepsilon}\text{curl}\Omega$. Cannot use (1b) which gives $\mathbf{B}\partial^{1+\varepsilon}\text{curl}\Omega \sim \partial^{3+\varepsilon}\Psi$. But available from (1c) if $\|\partial^{2+\varepsilon}\Omega\|_{L^2(\Sigma_t)}$ is controlled; latter follows from (1c)-(1d) and $\|\partial\Omega\|_{L^2(\Sigma_t)} \lesssim \|\text{div}\Omega\|_{L^2(\Sigma_t)} + \|\text{curl}\Omega\|_{L^2(\Sigma_t)}$. Conclusion:

$$\|\boldsymbol{\partial}\Psi\|_{H^{1+\varepsilon}(\Sigma_t)} + \|\partial\Omega\|_{H^{1+\varepsilon}(\Sigma_t)} \lesssim \exp\left(\int_0^t \left(\|\boldsymbol{\partial}\Psi\|_{L^\infty(\Sigma_\tau)} + \|\partial\Omega\|_{L^\infty(\Sigma_\tau)}\right) d\tau\right)$$

# Energy estimates ($s = const. \Rightarrow \mathcal{C} = e^{-\rho}\mathrm{curl}\Omega \sim \mathrm{curl}\Omega$)

$$\square_{\mathbf{g}(\vec{\Psi})}\partial^{1+\varepsilon}\Psi \sim \mathscr{L}(\vec{\Psi})[\partial^{1+\varepsilon}\mathrm{curl}\Omega] \tag{1a}$$

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$$\partial^{1+\varepsilon}\mathrm{div}\Omega \sim \mathscr{L}(\vec{\Omega})[\boldsymbol{\partial}\partial^{1+\varepsilon}\vec{\Psi}] \tag{1d}$$

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Hypothesis: $\mathrm{curl}\, v_0, \Omega \in H^{2+\varepsilon}$; (1c) $\neq$ curl (1b): better structure (introduction of $\mathcal{C}$ and $\mathcal{D}$).

# Key ingredient (control of mixed spacetime norm)

From

$$\|\boldsymbol{\partial}\Psi\|_{H^{1+\varepsilon}(\Sigma_t)} + \|\partial\Omega\|_{H^{1+\varepsilon}(\Sigma_t)} \lesssim \exp\left(\int_0^t \left(\|\boldsymbol{\partial}\Psi\|_{L^\infty(\Sigma_\tau)} + \|\partial\Omega\|_{L^\infty(\Sigma_\tau)}\right) d\tau\right)$$

we can close the estimate (and thus prove the Theorem) if we control

$$\|\boldsymbol{\partial}\Psi\|_{L^1_t L^\infty_x} := \int_0^t \|\boldsymbol{\partial}\Psi\|_{L^\infty(\Sigma_\tau)}\, d\tau, \quad \|\partial\Omega\|_{L^1_t L^\infty_x} := \int_0^t \|\partial\Omega\|_{L^\infty(\Sigma_\tau)}\, d\tau,$$

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# Bootstrap assumptions

The proof is carried out by assuming the following bootstrap assumptions:

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t([0,T_*])L^\infty_x} + \sum_{\nu\geq 2} \nu^{2\delta_0}\|P_\nu\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t([0,T_*])L^\infty_x} \leq 1,$$

$$\|\partial(\vec{\Omega},\vec{S})\|^2_{L^2_t([0,T_*])L^\infty_x} + \sum_{\nu\geq 2} \nu^{2\delta_0}\|P_\nu\partial(\vec{\Omega},\vec{S})\|^2_{L^2_t([0,T_*])L^\infty_x} \leq 1,$$

and showing that they can be improved to

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2([0,T_*])L^\infty_x} + \sum_{\nu\geq 2} \nu^{2\delta_1}\|P_\nu\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2([0,T_*])L^\infty_x} \lesssim T_*^{2\delta},$$

$$\|\partial(\vec{\Omega},\vec{S})\|^2_{L^2([0,T_*])C^{0,\delta_1}_x} + \sum_{\nu\geq 1} \nu^{\delta_1}\|P_\nu\partial(\vec{\Omega},\vec{S})\|^2_{L^2([0,T_*])L^\infty_x} \lesssim T_*^{2\delta},$$

where $P_\nu =$ LP projection, $0 < \delta_0 < 8\delta_1$ depend on the parameters of the problem, $T_* > 0$ and $\delta > 0$ are sufficiently small.

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$$\|P_\lambda \partial\varphi\|_{L_t^q L_x^\infty} \lesssim \lambda^{\frac{3}{2}-\frac{1}{q}} \|\partial\varphi\|_{L^2(\Sigma_0)}, \tag{4}$$

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Estimate (4) follows from the fixed-frequency Strichartz estimate

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Estimate (5) follows from a dispersive estimate that we state next.

# The dispersive estimate

By duality, the estimate $\|P\boldsymbol{\partial}\varphi\|_{L^q_t L^\infty_x} \lesssim \|\boldsymbol{\partial}\varphi\|_{L^2(\Sigma_0)}$ follows from:

$$\|P\mathbf{B}\varphi\|_{L^\infty(\Sigma_t)} \lesssim \left\{ \frac{1}{(1+|t-1|)^{\frac{2}{q}}} + d(t) \right\} \left\{ \|\boldsymbol{\partial}\varphi\|_{L^2(\Sigma_1)} + \|\varphi\|_{L^2(\Sigma_1)} \right\},$$

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Establishing the existence and integrability properties of $d(t)$ lies at the core of our result.

Unit frequency: can replace $\|P\mathbf{B}\varphi\|_{L^2(\Sigma_t)}$ on the LHS (energy estimates).

# Decay properties and the acoustic geometry

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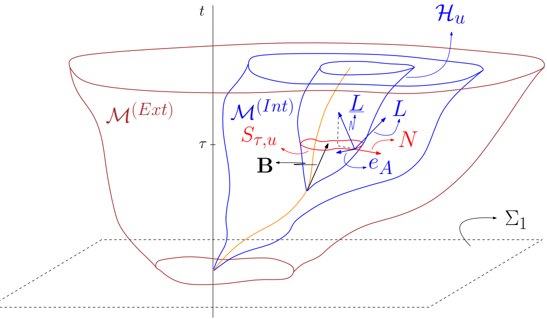


Decay: weighted energy.
Interior: multiplier $f(\widetilde{r})N$, $\widetilde{r} := t - u$ (Morawetz adapted to the acoustic geometry, integrated energy-decay).
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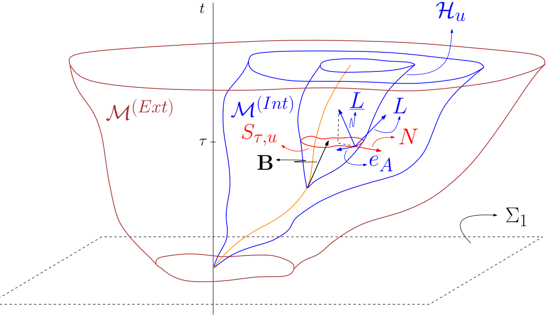


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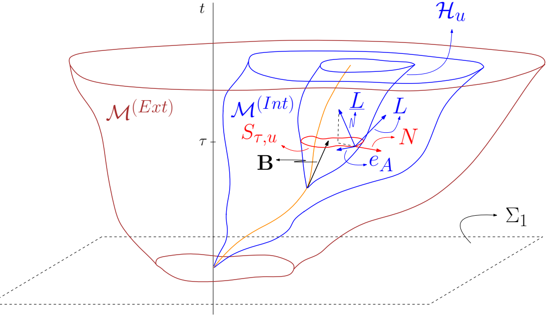


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Decay estimate can be obtained only in conjunction with appropriate estimates for the connection coefficients.

# Control of the acoustic geometry and the transport-part

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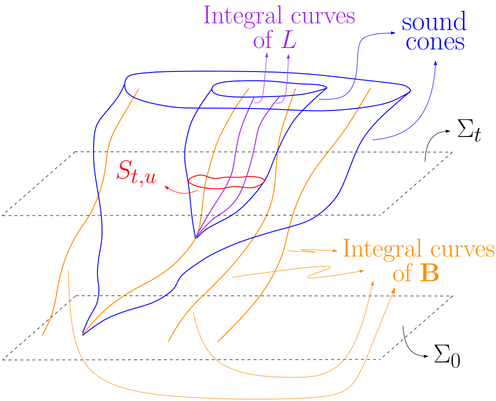

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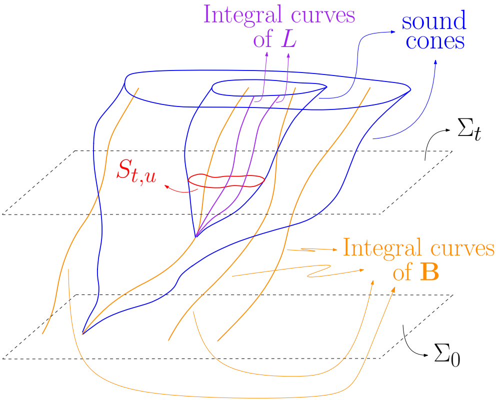


$L^2$ estimate for $\mathcal{C}$ and $\mathcal{D}$ along $\mathcal{H}_u$: $\mathbf{g}(\mathbf{B}, \mathbf{B}) = -1.$

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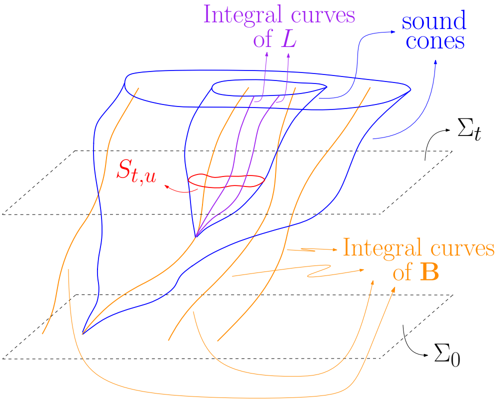


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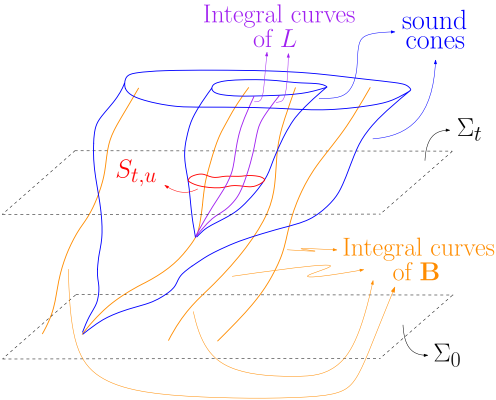


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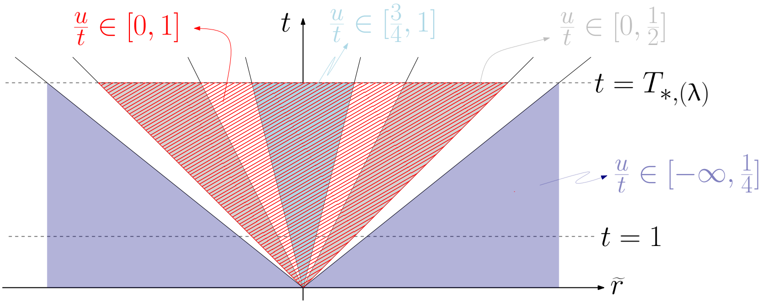
– Thank you for your attention! –

# Appendix: Conformal (weighted) energy

$$W(t,u) = \begin{cases} 1 & \text{if } \frac{u}{t} \in [0, 1/2], \\ 0 & \text{if } \frac{u}{t} \in (-\infty, -1/4] \cup [3/4, 1], \end{cases}$$

$$\underline{W}(t,u) = \begin{cases} 1 & \text{if } \frac{u}{t} \in [0, 1], \\ 0 & \text{if } \frac{u}{t} \in (-\infty, -1/4], \end{cases}$$

$$W(t,u) = \underline{W}(t,u) \text{if } t \in [1, T_{*;(\lambda)}] \text{ and } \frac{u}{t} \in (-\infty, 1/2].$$



# Appendix: Conformal (weighted) energy

$$\mathscr{C}[\varphi](t) := \int_{\widetilde{\Sigma}_t^{(Int)}} (\underline{W} - W) t^2 \left\{ |\mathbf{D}\varphi|^2 + |\tilde{r}^{-1}\varphi|^2 \right\} d\varpi_g + \int_{\widetilde{\Sigma}_t^{(Int)}} W \left\{ |\tilde{r}\mathbf{D}_L\varphi|^2 + |\tilde{r}\nabla\!\!\!/\varphi|_{g\!\!\!/}^2 + |\varphi|^2 \right\} d\varpi_g.$$

$$\mathscr{C}[\varphi](t) \leq C_\varepsilon (1+t)^{2\varepsilon} \left\{ \|\boldsymbol{\partial}\varphi\|_{L^2(\Sigma_1)}^2 + \|\varphi\|_{L^2(\Sigma_1)}^2 \right\}.$$

# Appendix: Null-structure equations

$$
\begin{aligned}
L\upsilon &= \upsilon \mathrm{tr}_{\not g}\chi, \\
Lb &= b\left\{-k_{NN} + \mathbf{g}(\mathbf{D_B B}, L)\right\}, \\
L\mathrm{tr}_{\not g}\chi + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2 &= -|\hat{\chi}|^2_{\not g} - k_{NN}\mathrm{tr}_{\not g}\chi - \mathbf{Ric}_{LL}, \\
\not{D}_L\hat{\chi}_{AB} + (\mathrm{tr}_{\not g}\chi)\hat{\chi}_{AB} &= -k_{NN}\hat{\chi}_{AB} - \left\{\mathbf{Riem}_{LALB} - \frac{1}{2}\mathbf{Ric}_{LL}\delta_{AB}\right\}, \\
\not{D}_L\zeta_A + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)\zeta_A &= -\left\{k_{BN} + \zeta_B\right\}\hat{\chi}_{AB} - \frac{1}{2}\mathrm{tr}_{\not g}\chi k_{AN} - \frac{1}{2}\mathbf{Riem}_{ALL\underline{L}}, \\
L\mathrm{tr}_{\not g}\underline{\chi} + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)\mathrm{tr}_{\not g}\underline{\chi} &= 2\not{\mathrm{div}}\,\underline{\zeta} + k_{NN}\mathrm{tr}_{\not g}\underline{\chi} - \hat{\chi}_{AB}\hat{\underline{\chi}}_{AB} + 2|\underline{\zeta}|^2_{\not g} \\
&\quad + \mathbf{Riem}_{A\underline{L}LA},
\end{aligned}
$$

# Appendix: Null-structure equations

$$
\begin{aligned}
\not{D}_{\underline{L}}\hat{\chi}_{AB} + \frac{1}{2}(\mathrm{tr}_{\not{g}}\underline{\chi})\hat{\chi}_{AB} = & -\frac{1}{2}(\mathrm{tr}_{\not{g}}\chi)\hat{\underline{\chi}}_{AB} + 2\not{\nabla}_A\zeta_B - \mathrm{di}\!\not{v}\,\zeta\delta_{AB} + k_{NN}\hat{\chi}_{AB} \\
& + \left\{2\zeta_A\zeta_B - |\zeta|^2_{\not{g}}\delta_{AB}\right\} \\
& - \left\{\hat{\underline{\chi}}_{AC}\hat{\chi}_{CB} - \frac{1}{2}\hat{\underline{\chi}}_{CD}\hat{\chi}_{CD}\delta_{AB}\right\} + \mathbf{Riem}_{AL\underline{L}B} \\
& - \frac{1}{2}\mathbf{Riem}_{CL\underline{L}C}\delta_{AB}, \\
\mathrm{di}\!\not{v}\,\hat{\chi}_A + \hat{\chi}_{AB}k_{BN} = & \frac{1}{2}\left\{\not{\nabla}_A\mathrm{tr}_{\not{g}}\chi + k_{AN}\mathrm{tr}_{\not{g}}\chi\right\} + \mathbf{Riem}_{BLBA}, \\
\mathrm{di}\!\not{v}\,\zeta = & \frac{1}{2}\left\{\mu - k_{NN}\mathrm{tr}_{\not{g}}\chi - 2|\zeta|^2_{\not{g}} - |\hat{\chi}|^2_{\not{g}} - 2k_{AB}\hat{\chi}_{AB}\right\} \\
& - \frac{1}{2}\mathbf{Riem}_{A\underline{L}LA}, \\
\mathrm{cur}\!\not{l}\,\zeta = & \frac{1}{2}\epsilon^{AB}\hat{\underline{\chi}}_{AC}\hat{\chi}_{BC} - \frac{1}{2}\epsilon^{AB}\mathbf{Riem}_{AL\underline{L}B}.
\end{aligned}
$$