Infinite-dimensional inverse problems with finite measurements

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Reconstruction Methods in Inverse Problems
Banff
June 24, 2019

Joint with Giovanni S. Alberti

Summary

- Motivations
- Calderon's problem with a finite number of measurements: global uniqueness and Lipschitz stability
- A general Lipschitz stability and reconstruction result.

G. S. Alberti, M. Santacesaria
Calderón's inverse problem with a finite number of measurements,
preprint arXiv:1803.04224.

G. S. Alberti, M. Santacesaria
Infinite-dimensional inverse problems with finite measurements,
preprint arXiv (today at 18:00, Banff time),
ResearchGate DOI: 10.13140/RG.2.2.17756.03205.

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Inverse problem

Given $y = F(x)$, determine x.

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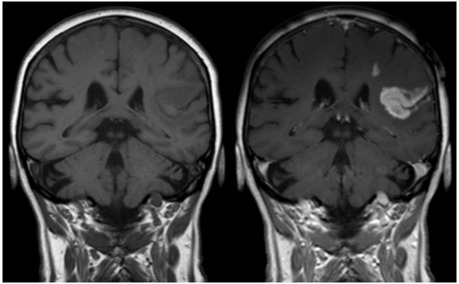
☺ $g(t) \approx t$, *well-posed problem.*

☹ $g(t) \approx |\log(t^{-1})|^{-1}$, *ill-posed problem.*

- Impose (reasonable) conditions on a ill-posed problem to make it well-posed.
- Study uniqueness from a discrete approximation of the data.

EIT for brain stroke imaging

Joint project: Univ. Helsinki, Aalto, Tampere, Kuopio and Helsinki Hospital.



- ischemic stroke: lower conductivity.
  Left: MRI image of ischemia (Hellerhoff 2010).
- haemorrhagic stroke: higher conductivity.
- Same symptoms!

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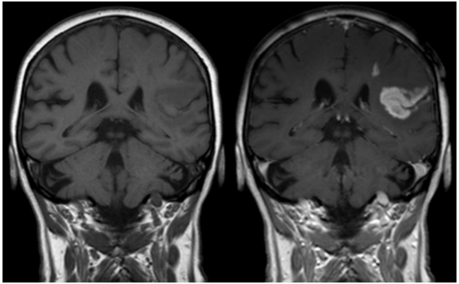

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Challenges: resistive skull layer, unknown background.

Some existing work:

- Holder 1992,
- Shi et al, 2009,
- Malone et al., 2014.

Calderón's problem for EIT

- $D \subset \mathbb{R}^d$, $d \geqslant 2$: bounded Lipschitz domain
- $\sigma \in L^\infty(D)$, $\sigma(x) \geqslant \sigma_0 > 0$: unknown conductivity
- Conductivity equation:

$$\begin{cases} -\operatorname{div}(\sigma \nabla u) = 0 & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases} \tag{1}$$

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Calderón's problem

Given Λ_σ, determine σ in D.

Some known results

Basic questions:

- Uniqueness: injectivity of $\sigma \mapsto \Lambda_\sigma$
- stability estimates: continuity of $\Lambda_\sigma \mapsto \sigma$
- reconstruction algorithm

Theoretical contributions by: Calderón, Sylvester–Uhlmann, Nachman, Novikov, Alessandrini, Astala–Päivärinta, Haberman, Caro–Rogers and many others.

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Usual reduction to the Gel'fand-Calderón inverse problem for the Schrödinger equation

$$(-\Delta + q)u = 0 \quad \text{in } D, \qquad \Lambda_q(u|_{\partial D}) = \left.\frac{\partial u}{\partial \nu}\right|_{\partial D},$$

which will be considered for the next few slides.

A finite number of measurements

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- Most results need an infinite number of measurement.
- The only exception is the reconstruction of a polygon from one measurement [Friedman-Isakov 1989].

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- 0 is not a Dirichlet eigenvalue for $-\Delta + q$ in D;
- $\|q\|_{L^\infty(D)} \leqslant R$ for some $R > 0$.

Nonlinear prolem - global uniqueness

Theorem 1 (G.S. Alberti, M.S. (2018))

Take $d \geqslant 3$ and let $D \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $\mathcal{W} \subseteq L^\infty(D)$ be a finite dimensional subspace. There exists $N \in \mathbb{N}$ such that for any $R > 0$ and $q_1 \in \mathcal{W}_R$, the following is true.

There exist $\{f_l\}_{l=1}^N \subseteq H^{1/2}(\partial D)$ such that for any $q_2 \in \mathcal{W}_R$, if

$$\Lambda_{q_1} f_l = \Lambda_{q_2} f_l, \qquad l = 1, \ldots, N,$$

then

$$q_1 = q_2.$$

Similar result for Calderón's problem as well.

Ideas of the proof

- Alessandrini's identity to go from the boundary to the interior.

$$\langle g, (\Lambda_q - \Lambda_0) f \rangle_{H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)} = \int_D q\, u_g^0 u_f^q \, dx$$

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$$\zeta_j^k \cdot \zeta_j^k = 0, \qquad \zeta_1^k + \zeta_2^k = -2\pi i k, \qquad \|r^k\|_{L^2(\mathbb{T}^d)} \leqslant c/t_k$$

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- Order the frequencies: $\rho\colon l \in \mathbb{N} \mapsto k_l \in \mathbb{Z}^d$ (bijection)
- Define the nonlinear measurement operator $U\colon L^\infty([0,1]^d) \to \ell^\infty$ by

$$(U(q))_l = \int_D q(x) e^{-2\pi i k_l \cdot x}(1 + r^{k_l}(x))\, dx$$

- $U = F + B$, where, F Fourier transform, B is a contraction (t_k large)

On the number of measurements N

- The number of measurements N depends only on $\mathcal{W}$ through
$$\|(I - P_N)FP_{\mathcal{W}}\|_{\mathcal{H}\to\ell^2} \leqslant 1/4.$$
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- It allows for an explicit calculation of N:
  - bandlimited potentials

$$N = \dim \mathcal{W}$$

  - piecewise constant potentials

$$N = O((\dim \mathcal{W})^4)$$

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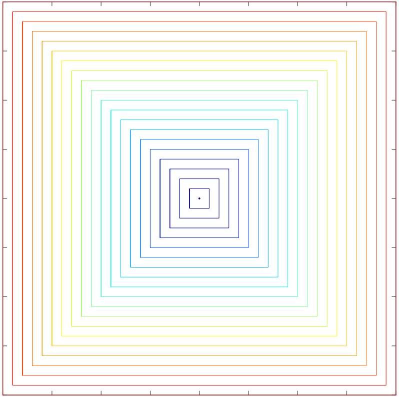
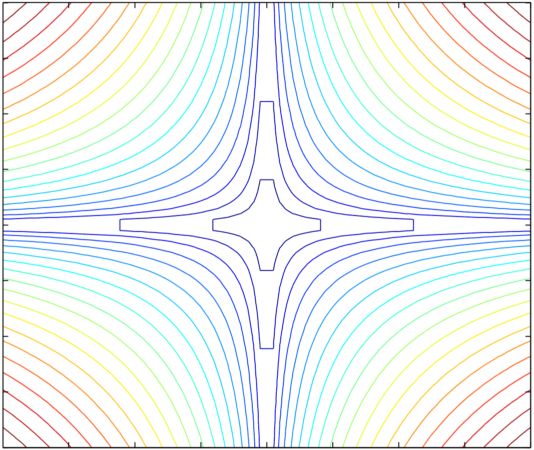
- The ordering of $\mathbb{Z}^d$ is crucial

Possible orderings of $\mathbb{Z}^d$



(a) Linear ordering



(b) Hyperbolic ordering (Jones, Adcock, Hansen, 2017)

Lipschitz stability

Theorem 2 (G.S. Alberti, M.S. (2018))

Under the same assumptions, there exist $\{f_l\}_{l=1}^N \subseteq H^{1/2}(\partial D)$ *such that for every* $q_2 \in \mathcal{W}_R$*, we have*

$$\|q_2 - q_1\|_{L^2(D)} \leqslant e^{CN^{\frac{1}{2}+\alpha}} \left\| \left(\Lambda_{q_2} f_l - \Lambda_{q_1} f_l\right)_{l=1}^N \right\|_{H^{-1/2}(\partial D)^N}$$

for some $C > 0$ *depending only on* D*,* R *and* α*.*

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- Several authors studied stability estimates with piece-wise constant unknowns with the full DN map (Alessandrini, Beretta, Francini, Gaburro, de Hoop, Scherzer, Sincich, Vessella...).
- The exponential $e^{CN^{\frac{1}{2}+\alpha}}$ is consistent with previous work (Mandache) and is related to the severe ill-posedness of this IP.
- We have also obtained a nonlinear reconstruction algorithm based on Banach fixed point theorem.

Intermezzo – open questions

- Two-dimensional case.
- Is it possible to choose $\{f_l\}_l$ independently of q? Yes [Harrach 2019]
- More realistic models (e.g. complete electrode model), numerical implementation.
- Extensions to other infinite dimensional IP, e.g. inverse scattering, elasticity. [Rüland-Sincich 2018] fractional Calderón problem.
- General Lipschitz stability result for a class of ill-posed inverse problems.

[Harrach 2019] result

- Take a finite dimensional subset of piecewise analytic conductivities: the data comes from the complete electrode model (CEM) and the input currents are independent on the conductivities.

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$$\|\sigma_1 - \sigma_2\|_{L^2(D)} \leqslant C \|P_{G_N}(\Lambda_{\sigma_1} - \Lambda_{\sigma_2})P_{G_N}\|_{L^2(\partial D) \to L^2(\partial D)},$$

  where $P_{G_N}\Lambda_{\sigma_j}P_{G_N}$ is a finite dimensional Galerkin projection, Λ_{σ_j} is the Neumann-to-Dirichlet map.

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Can this be extended to more general inverse problems?

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Examples:

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$$Q_N(y) = P_N^2 y P_N^1.$$

Lipschitz stability with finite measurements: main result

Theorem 3 (G.S. Alberti, M.S. (2019))

Let $K \subseteq A$ *be convex. Suppose there exists* $C > 0$ *such that*

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(i) *If $K \subseteq W \cap A$ is compact, where W is a finite dimensional subset of X and*

$$\lim_{N\to+\infty} (I - Q_N)F'(\xi)\tau = 0, \qquad \xi \in A, \tau \in W,$$

then

$$\lim_{N\to+\infty} s_N = 0, \qquad s_N = \sup_{\xi\in K} \|(I - Q_N)F'(\xi)\|_{W\to Y}.$$

(ii) *If $s_N \leqslant \frac{1}{2C}$, then*

$$\|x_1 - x_2\|_X \leqslant 2C\|Q_N(F(x_1)) - Q_N(F(x_2))\|_Y, \qquad x_1, x_2 \in K.$$

The *smoothing* condition: $\lim_{N\to+\infty}(I - Q_N)F'(\xi)\tau = 0$

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  Assuming that $F'(\xi)\tau : Y^1 \to Y^2$ is *compact* for every $\xi \in A, \tau \in W$ then the condition is satisfied.

On the number of measurements N

N depends on the Lipschitz constant C for the full data and on the subspace W:

$$\sup_{\xi \in K} \|(I - Q_N)F'(\xi)\|_{W \to Y} \leqslant \frac{1}{2C}$$

which can be explicitly computed in several cases.

Example I: electrical impedance tomography

Let $\mathcal{N}_\sigma$ be the Neumann-to-Dirichlet map and assume

$$\|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \leqslant C\|\mathcal{N}_{\sigma_1} - \mathcal{N}_{\sigma_2}\|_{L^2_\diamond(\partial\Omega)\to L^2_\diamond(\partial\Omega)}, \qquad \sigma_1, \sigma_2 \in K,$$

where K is a compact subset of a finite dimensional subspace of L^∞ conductivities ($L^2_\diamond(\partial\Omega) = \{f \in L^2(\partial\Omega) : \int_{\partial\Omega} f\, ds = 0\}$). Then there exists $N \in \mathbb{N}$ such that

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$\Omega \subseteq \mathbb{R}^2$ unit disk. Let P_N be the projection on the trigonometric current patterns

$$\sin(n\theta), \cos(n\theta), \text{ for } n \leqslant N, \theta \in \partial\Omega.$$

Then we have $N = O(C^2)$ (recall that for EIT $C = O(\exp(\dim W))$).

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$$\|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \leqslant C\|\mathcal{N}_{\sigma_1} - \mathcal{N}_{\sigma_2}\|_{L^2_\diamond(\partial\Omega)\to L^2_\diamond(\partial\Omega)}, \qquad \sigma_1, \sigma_2 \in K,$$

where K is a compact subset of a finite dimensional subspace of L^∞ conductivities ($L^2_\diamond(\partial\Omega) = \{f \in L^2(\partial\Omega) : \int_{\partial\Omega} f\, ds = 0\}$). Then there exists $N \in \mathbb{N}$ such that

$$\|\sigma_1 - \sigma_2\|_\infty \leqslant 2C\,\|P_N\mathcal{N}_{\sigma_1}P_N - P_N\mathcal{N}_{\sigma_2}P_N\|_{L^2_\diamond(\partial\Omega)\to L^2_\diamond(\partial\Omega)}, \qquad \sigma_1, \sigma_2 \in K.$$

$\Omega \subseteq \mathbb{R}^2$ unit disk. Let P_N be the projection on the trigonometric current patterns

$$\sin(n\theta), \cos(n\theta), \text{ for } n \leqslant N, \theta \in \partial\Omega.$$

Then we have $N = O(C^2)$ (recall that for EIT $C = O(\exp(\dim W))$).

Note that this is significantly worse than reconstructing from traces of CGO solutions, where $N = O(\dim W)$ in many cases.

Example II: inverse scattering

$$\begin{cases} \Delta u + k^2 n(x) u = 0 & \text{in } \mathbb{R}^3, \\ u = e^{ikx\cdot d} + u^s & \text{in } \mathbb{R}^3, \\ \text{radiation condition for } u^s & \end{cases}$$

- $k > 0$ is the (fixed) wavenumber, $d \in S^2$,
- $n \in L^\infty(\mathbb{R}^3; \mathbb{C})$ is the refractive index with $\operatorname{Im}(n) \geqslant 0$ in $\mathbb{R}^3$ and $\operatorname{supp}(1-n) \subseteq B$ for some open ball B.

Problem. Given the far field $u_n^\infty(\hat{x}, d) \in L^2(S^2 \times S^2)$ at fixed $k > 0$, find n in B.

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Assuming Lipschitz stability we can prove the same my measuring u^∞ on a *finite number of points* $(\hat{x}, d) \in S^2 \times S^2$.

Reconstruction

Thanks to [de Hoop, Qiu, Scherzer 2012] we can show *global* convergence of Landweber iteration in our setting.

Key idea: build a sufficiently fine lattice in the set of unknowns and find a good initial guess for local convergence using the Lipschitz stability.

Conclusions and open questions

- This can be applied to many inverse problems where the unknown belongs to a finite dimensional space
  - EIT for piecewise analytic conductivities,
  - polygonal inclusions,
  - piecewise constant on polygonal partition,
  - Inverse boundary value problems for other PDEs,
  - Inverse scattering.
- Some inverse problems where the unknown belong to a compact subspace,
  - *Increasing stability*-type estimates for the Schrödinger equation.
- Connections with regularization by discretization.

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Thank you!