A multiscale approach for inverse problems

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Multiscale decomposition of images

Tadmor, Nezzar & Vese (2004)

The Rudin-Osher-Fatemi model for denoising

$\Omega \subset \mathbb{R}^2$ fixed bounded domain; $\partial\Omega$ Lipschitz

$f \in L^2(\Omega)$ noisy image

ROF model

$\lambda_0 > 0$ fixed parameter. Solve

$$\min \left\{ \lambda_0 \|f - u\|^2_{L^2(\Omega)} + |u|_{BV(\Omega)} :\ u \in L^2(\Omega) \right\}$$

u_0 is the (unique) minimiser; $v_0 = f - u_0$ is the remainder

Remark:

$$|u|_{BV(\Omega)} = TV(u) = |Du|(\Omega); \quad \|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |u|_{BV(\Omega)}$$

The role of parameter λ_0

ROF model

$\lambda_0 > 0$ fixed parameter. Solve

$$\min \left\{\lambda_0 \|f - u\|^2_{L^2(\Omega)} + |u|_{BV(\Omega)} : u \in L^2(\Omega)\right\}$$

u_0 is the (unique) minimiser; $v_0 = f - u_0$ is the remainder

- λ_0 small: total variation of u_0 more penalised $\rightsquigarrow$ u_0 has smaller total variation (blocky reconstruction); most noise but also more detailed features are removed
- λ_0 big: fidelity term $\|f - u_0\|$ more penalised $\rightsquigarrow$ u_0 is closer to f; more detailed features are preserved, less noise is removed

The T-N-V multiscale procedure: starting point

- Start with a (relatively) small $\lambda_0 > 0$. Solve

$$\min\left\{\lambda_0\|f-u\|^2 + |u| : \ u \in L^2(\Omega)\right\}$$

u_0 is the (unique) minimiser; $v_0 = f - u_0$ is the remainder

$f = u_0 + v_0$. Let $\sigma_0 = u_0$, hence $f = \sigma_0 + v_0$

Remark: here and in what follows

$$\|\cdot\| = \|\cdot\|_{L^2(\Omega)} \qquad \text{and} \qquad |\cdot| = |\cdot|_{BV(\Omega)}$$

The T-N-V multiscale procedure: second step

- We have

$$f = u_0 + v_0. \quad \text{Let } \sigma_0 = u_0, \text{ hence } f = \sigma_0 + v_0$$

- Raise the parameter λ. Take $\lambda_0 < \lambda_1$ and replace f by the remainder v_0. Solve

$$\min \left\{ \lambda_1 \|v_0 - u\|^2 + |u| : \; u \in L^2(\Omega) \right\}$$

that is

$$\min \left\{ \lambda_1 \|f - (u_0 + u)\|^2 + |u| : \; u \in L^2(\Omega) \right\}$$

u_1 is the (unique) minimiser; $v_1 = v_0 - u_1$ is the remainder

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u_1 is the (unique) minimiser; $v_1 = v_0 - u_1$ is the remainder

$$f = u_0 + u_1 + v_1. \quad \text{Let } \sigma_1 = u_0 + u_1, \text{ hence } f = \sigma_1 + v_1$$

The T-N-V multiscale procedure: iteration

- Take $0 < \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$. By induction, for any $n \geqslant 1$ define

$$\sigma_{n-1} = \sum_{i=0}^{n-1} u_i \quad \text{and} \quad v_{n-1} = f - \sigma_{n-1}, \text{ hence } f = \sigma_{n-1} + v_{n-1},$$

and solve

$$\min \left\{ \lambda_n \| v_{n-1} - u \|^2 + |u| : \; u \in L^2(\Omega) \right\}$$

that is

$$\min \left\{ \lambda_n \| f - (\sigma_{n-1} + u) \|^2 + |u| : \; u \in L^2(\Omega) \right\}$$

u_n is the (unique) minimiser; $v_n = v_{n-1} - u_n$ is the remainder and

$$\sigma_n = \sum_{i=0}^{n} u_i \quad \text{and} \quad v_n = f - \sigma_n, \text{ hence } f = \sigma_n + v_n$$

The T-N-V multiscale decomposition

Take $0 < \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$. For any $n \geqslant 0$ we have

$$f = u_0 + u_1 + \ldots + u_n + v_n = \sigma_n + v_n.$$

Theorem — Tadmor, Nezzar & Vese (2004)

$\lambda_0 > 0$ fixed parameter. Let

$$\lambda_n = 2^n \lambda_0 \quad \text{for any } n \geqslant 0.$$

If $f \in BV(\Omega)$ then $\lim_n v_n = 0$ in $L^2(\Omega)$
that is, f has the following multiscale decomposition

$$f = \lim_n \sigma_n = \sum_{i=0}^{\infty} u_i \quad \text{in the } L^2(\Omega) \text{ sense}$$

Remark: it holds also for f in some intermediate space between $L^2(\Omega)$ and $BV(\Omega)$

Extension to nonlinear inverse problems

The Calderón inverse problem

Electrical Impedance Tomography: the conducting body

$\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) fixed bounded domain; $\partial\Omega$ Lipschitz

$0 < c_0 < c_1$ fixed constants

Classes of conductivity tensors

The anisotropic case:

$\mathcal{M}_{sym}(c_0, c_1)$ class of symmetric conductivity tensors σ, that is, $\sigma \in L^\infty(\Omega, \mathbb{M}^{N\times N}_{sym}(\mathbb{R}))$ satisfying the uniform ellipticity condition

$$0 < c_0 I_N \leqslant \sigma(x) \leqslant c_1 I_N \quad \text{for a.e. } x \in \Omega$$

The isotropic case:

$\mathcal{M}_{scal}(c_0, c_1)$ class of scalar conductivities σ, that is, $\sigma \in L^\infty(\Omega)$ satisfying the uniform ellipticity condition

$$0 < c_0 \leqslant \sigma(x) \leqslant c_1 \quad \text{for a.e. } x \in \Omega$$

The Neumann-to-Dirichlet map

- Conductivity in Ω: $\sigma \in \mathcal{M}_{sym}(c_0, c_1)$
- Prescribed current density on the boundary $\partial\Omega$:

$$g \in L^2_*(\partial\Omega) = \left\{ \psi \in L^2(\partial\Omega) : \int_{\partial\Omega} \psi = 0 \right\}$$

- Electrostatic potential in Ω: U solution to the Neumann problem

$$\begin{cases} \operatorname{div}(\sigma \nabla U) = 0 & \text{in } \Omega \\ \sigma \nabla U \cdot \nu = g & \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} U = 0 & \end{cases}$$

Neumann-to-Dirichlet map

$\Lambda(\sigma) : L^2_*(\partial\Omega) \to L^2_*(\partial\Omega)$ where

$$\Lambda(\sigma)[g] = U|_{\partial\Omega} \in L^2_*(\partial\Omega) \quad \text{for any } g \in L^2_*(\partial\Omega)$$

Electrical Impedance Tomography: the inverse problem

Inverse conductivity problem — Calderón (1980)

Determine the conductivity tensor σ from electrostatic measurements on the boundary, that is, by measuring the Neumann-to-Dirichlet map $\Lambda(\sigma)$

The forward operator:

$$\begin{array}{rcl} \Lambda : \mathcal{M} & \to & \mathcal{L}(L^2_*(\partial\Omega), L^2_*(\partial\Omega)) \\ \sigma & \mapsto & \Lambda(\sigma) \end{array}$$

where $\mathcal{M} = \mathcal{M}_{sym}$ or $\mathcal{M} = \mathcal{M}_{scal}$

Uniqueness issue

Does the Neumann-to-Dirichlet map $\Lambda(\sigma)$ uniquely determine the conductivity tensor σ? Is the forward operator Λ injective?

Uniqueness for scalar conductivities

$N = 3$; $\sigma \in \mathcal{M}_{scal}$

Kohn & Vogelius (1984) — Sylvester & Uhlmann (1987) — Isakov (1988)
Haberman & Tataru (2013) — Caro & Rogers (2016)
Haberman (2015) $\sigma \in W^{1,3}$

$N = 2$; $\sigma \in \mathcal{M}_{scal}$, Ω simply connected

Nachman (1995)
Astala & Päivärinta (2006) $\sigma \in L^\infty$

$N = 2$; $\sigma \in \mathcal{M}_{sym}$, Ω simply connected

Astala, Päivärinta & Lassas (2005) $\sigma \in L^\infty$

If $\Lambda(\sigma) = \Lambda(\sigma_1)$ then $\exists \varphi$ quasiconformal mapping with $\varphi = \mathrm{Id}$ on $\partial\Omega$ such that $\sigma_1 = \varphi_*(\sigma)$.

Setup of the inverse problem: reconstruction

- Unknown: $\widetilde{\sigma}_0 \in \mathcal{M}$, $\mathcal{M} = \mathcal{M}_{sym}(c_0, c_1)$ or $\mathcal{M} = \mathcal{M}_{scal}(c_0, c_1)$
- Exact data: $\Lambda_0 = \Lambda(\widetilde{\sigma}_0)$

Reconstruction

Numerically reconstruct $\widetilde{\sigma}_0$ from (an approximation of) $\Lambda(\widetilde{\sigma}_0)$

- Available (measured) data: $\widetilde{\Lambda} \in \mathcal{L}(L^2_*(\partial\Omega), L^2_*(\partial\Omega))$ with

$$\|\widetilde{\Lambda} - \Lambda_0\| \leqslant \varepsilon, \quad \varepsilon > 0 \text{ is the noise level}$$

where $\|\cdot\| = \|\cdot\|_{L^2\text{-}L^2} = \|\cdot\|_{\mathcal{L}(L^2_*(\partial\Omega), L^2_*(\partial\Omega))}$

Main issues

- Nonlinearity
- Ill-posedness

Variational approach: regularised minimisation problem

Regularised variational problem

$\mu_0 > 0$ fixed parameter. Solve

$$\min \left\{ \|\widetilde{\Lambda} - \Lambda(\sigma)\|^2 + \mu_0 R(\sigma) : \ \sigma \in \mathcal{M} \right\}$$

R regularisation operator; μ_0 regularisation coefficient

Choice of the regularisation operator: total variation penalisation

$$R(\sigma) = |\sigma|_{BV(\Omega)} = |\sigma|$$

Hence, for $\lambda_0 = 1/\mu_0$, solve

$$\min \left\{ \lambda_0 \|\widetilde{\Lambda} - \Lambda(\sigma)\|^2 + |\sigma| : \ \sigma \in \mathcal{M} \right\}$$

$\sigma_0 = u_0$ is a minimiser

Why the L^2-L^2-norm instead of the natural one?

Continuity with respect to G-convergence – R. (2015)

Let $\sigma_n, \sigma \in \mathcal{M}_{sym}(c_0, c_1)$ such that σ_n G-converges to σ. Then

$$\|\Lambda(\sigma_n) - \Lambda(\sigma)\|_{L^2\text{-}L^2} \to 0.$$

Hölder continuity with respect to the L^1 norm

For any $\sigma_1, \sigma_2 \in \mathcal{M}_{sym}(c_0, c_1)$, we have, for some $0 < \beta < 1$ and $C_0 > 0$,

$$\|\Lambda(\sigma_1) - \Lambda(\sigma_2)\|_{L^2\text{-}L^2} \leqslant C_0 \|\sigma_1 - \sigma_2\|^{\beta}_{L^1(\Omega)}.$$

Remark: the L^2-L^2 norm controls the error on the so-called experimental measurements introduced by Somersalo, Cheney, & Isaacson (1992).
R. (2015): if $R(\sigma)$ is the resistance matrix associated to σ, we have

$$\|R(\sigma_1) - R(\sigma_2)\| \leqslant C \|\Lambda(\sigma_1) - \Lambda(\sigma_2)\|_{L^2\text{-}L^2}$$

Multiscale approach for nonlinear inverse problems

The Calderón inverse problem

Functional setting

- X Banach space with norm $\| \cdot \|_X$

$$X = L^1(\Omega, \mathbb{M}^{N\times N}_{sym}(\mathbb{R})) \text{ or } X = L^1(\Omega) \text{ with norm } \| \cdot \|_{L^1(\Omega)}$$

- $E \subset X$ suitable closed subset

$$E = \mathcal{M} \text{ with } \mathcal{M} = \mathcal{M}_{sym}(c_0, c_1) \text{ or } \mathcal{M} = \mathcal{M}_{scal}(c_0, c_1)$$

- Y metric space with distance d_Y

$$Y = \mathcal{L}(L^2_*(\partial\Omega), L^2_*(\partial\Omega))$$

with d_Y induced by its norm $\| \cdot \| = \| \cdot \|_{L^2\text{-}L^2}$

Functional setting and starting point

- $\Lambda : E \to Y$ continuous and $\widetilde{\Lambda} \in Y$

$$\begin{array}{rcl} \Lambda : \mathcal{M} & \to & \mathcal{L}(L^2_*(\Omega), L^2_*(\Omega)) \\ \sigma & \mapsto & \Lambda(\sigma) \end{array}$$

$\widetilde{\Lambda} \in \mathcal{L}(L^2_*(\partial\Omega), L^2_*(\partial\Omega))$ is the measured Neumann-to-Dirichlet map

Regularisation operator

$R = |\cdot| : X \to [0, +\infty]$

$$R = |\cdot| = |\cdot|_{BV(\Omega)} : L^1(\Omega, \mathbb{M}^{N\times N}_{sym}(\mathbb{R})) \to [0, +\infty]$$

Solve, for $\lambda_0 > 0$ and $a_0 \geqslant 0$,

$$\min \left\{ \lambda_0 \left[\|\widetilde{\Lambda} - \Lambda(\sigma)\|^2 + a_0|\sigma| \right] + |\sigma| : \ \sigma \in \mathcal{M} \right\}$$

$\sigma_0 = u_0$ is a minimiser

Multiscale procedure: iteration

- Take $0 < \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$ and $0 \leqslant \ldots \leqslant a_n \leqslant \ldots \leqslant a_1 \leqslant a_0$. By induction, for any $n \geqslant 1$ define

$$\sigma_{n-1} = \sum_{i=0}^{n-1} u_i$$

and solve

$$\min \left\{ \lambda_n \left[\| \widetilde{\Lambda} - \Lambda(\sigma_{n-1} + u) \|^2 + a_n |\sigma_{n-1} + u| \right] + |u| : \ (\sigma_{n-1} + u) \in \mathcal{M} \right\}$$

u_n is a minimiser

and

$$\sigma_n = \sum_{i=0}^{n} u_i$$

Main remark and notation

$$\min\left\{\lambda_n\left[\|\widetilde{\Lambda}-\Lambda(\sigma_{n-1}+u)\|^2+a_n|\sigma_{n-1}+u|\right]+|u|:\ (\sigma_{n-1}+u)\in\mathcal{M}\right\}$$

By taking $u=0$ and using $a_n \leqslant a_{n-1}$, we observe that for any $n \geqslant 1$

$$\|\widetilde{\Lambda}-\Lambda(\sigma_n)\|^2+a_n|\sigma_n| \leqslant \|\widetilde{\Lambda}-\Lambda(\sigma_{n-1})\|^2+a_{n-1}|\sigma_{n-1}|$$

Let

$$\delta_0=\lim_n\left[\|\widetilde{\Lambda}-\Lambda(\sigma_n)\|^2+a_n|\sigma_n|\right]^{1/2}$$

and

$$\varepsilon_0=\inf\left\{\|\widetilde{\Lambda}-\Lambda(\sigma)\|:\ \sigma\in\mathcal{M}\right\}$$

Clearly

$$\varepsilon_0 \leqslant \delta_0$$

Convergence in the data space

Theorem: convergence of $\Lambda(\sigma_n)$

Assume

$$a_n \leqslant a_{n-1} \text{ for any } n \geqslant 1, \quad \lim_n a_n = 0 \quad \text{and} \quad \limsup_n \frac{2^n}{\lambda_n} < +\infty.$$

Then

$$\varepsilon_0 = \delta_0$$

and

$$\lim_n \|\widetilde{\Lambda} - \Lambda(\sigma_n)\| = \varepsilon_0 = \inf\left\{\|\widetilde{\Lambda} - \Lambda(\sigma)\| : \ \sigma \in \mathcal{M}\right\}$$

Remark: it is enough to take $\lambda_0 > 0$ fixed parameter and let

$$a_n = 0 \quad \text{and} \quad \lambda_n = 2^n \lambda_0 \quad \text{for any } n \geqslant 0$$

Multiscale decomposition in a general setting

Additive case

General abstract setting

- X Banach space with norm $\|\cdot\|_X$; $E \subset X$ suitable closed subset
- Y metric space with distance d_Y
- $\Lambda : E \to Y$ continuous and $\widetilde{\Lambda} \in Y$

Regularisation operator

$R = |\cdot| : X \to [0, +\infty]$ such that

- $|0| = 0$ and $|-u| = |u| \quad \forall\, u \in X$
- $|u_1 + u_2| \leqslant |u_1| + |u_2| \quad \forall\, u_1, u_2 \in X$
- $\{u \in X : \ |u| < +\infty\}$ dense in X
- $|\cdot|$ sequentially lower semicontinuous on X, with respect to the convergence in X
- $\{u \in X : \ |u| \leqslant b\}$ sequentially compact in $X \quad \forall\, b \in \mathbb{R}$

Examples of admissible regularisations

$\Omega \subset \mathbb{R}^N$ ($N \geqslant 1$) fixed bounded domain; $\partial\Omega$ Lipschitz

- BV regularisation:
  $X = L^1(\Omega)$, with norm $\|\cdot\|_{L^1(\Omega)}$; $E \subset X$ suitable closed subset

$$R(u) = |u| = \|u\|_{BV(\Omega)} \quad \forall\, u \in L^1(\Omega)$$

- $W^{1,2}$ regularisation:
  $X = L^2(\Omega)$, with norm $\|\cdot\|_{L^2(\Omega)}$; $E \subset X$ suitable closed subset

$$R(u) = |u| = \|u\|_{W^{1,2}(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \quad \forall\, u \in L^2(\Omega)$$

- $C^{0,\alpha}$ regularisation, $0 < \alpha \leqslant 1$:
  $X = C^0(\overline{\Omega})$, with the sup norm; $E \subset X$ suitable closed subset

$$R(u) = |u| = \|u\|_{C^{0,\alpha}(\Omega)} = \|u\|_{L^\infty(\Omega)} + |u|_{C^{0,\alpha}(\Omega)} \quad \forall\, u \in C^0(\overline{\Omega})$$

Example: denoising of images or signals

Setting:

- X Banach space with norm $\|\cdot\|_X$; $E \subset X$ suitable closed subset
- Y metric space with distance d_Y
- $\Lambda : E \to Y$ continuous and $\widetilde{\Lambda} \in Y$
- Regularisation $R = |\cdot| : X \to [0, +\infty]$

Denoising of images or signals:

$\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, fixed bounded domain; $\partial\Omega$ Lipschitz

- $X = L^2(\Omega)$, with norm $\|\cdot\|_{L^2(\Omega)}$; $E = X = L^2(\Omega)$
- $Y = X = L^2(\Omega)$ with distance induced by its norm
- $\Lambda = \mathrm{Id} : L^2(\Omega) \to L^2(\Omega)$ and $\widetilde{\Lambda} = f \in L^2(\Omega)$
- As regularisation, with small modifications,

$$R = |\cdot| = |\cdot|_{BV(\Omega)} : L^2(\Omega) \to [0, +\infty]$$

The T-N-V multiscale decomposition: reprise

$\lambda_0 > 0$ fixed parameter. Let

$$\lambda_n = 2^n \lambda_0 \quad \text{and} \quad a_n = 0 \quad \text{for any } n \geqslant 0.$$

For any $n \geqslant 0$ we have

$$f = u_0 + u_1 + \ldots + u_n + v_n = \sigma_n + v_n.$$

Theorem — Modin, Nachman & R. (2019)

If $f \in L^2(\Omega)$ then $\lim_n v_n = 0$ in $L^2(\Omega)$,
that is, f has the following multiscale decomposition

$$f = \lim_n \sigma_n = \sum_{i=0}^{\infty} u_i \quad \text{in the } L^2(\Omega) \text{ sense}$$

Remark: it holds for any dimension $N \geqslant 1$

Multiscale approach for the Calderón problem

Convergence in the unknowns space

Convergence in the unknowns space

We know that

$$\lim_n \|\widetilde{\Lambda} - \Lambda(\sigma_n)\| = \varepsilon_0 = \inf\left\{\|\widetilde{\Lambda} - \Lambda(\sigma)\| : \ \sigma \in \mathcal{M}\right\}$$

Remark: if $\lim_n \sigma_n = \sigma_\infty$ in L^1 or in the G-convergence sense, then

$$\|\widetilde{\Lambda} - \Lambda(\sigma_\infty)\| = \varepsilon_0 = \min\left\{\|\widetilde{\Lambda} - \Lambda(\sigma)\| : \ \sigma \in \mathcal{M}\right\}$$

Necessary condition:

$$\exists \min\left\{\|\widetilde{\Lambda} - \Lambda(\sigma)\| : \ \sigma \in \mathcal{M}\right\}$$

Question: is this a sufficient condition?

Main properties of G-convergence

$$\mathcal{M} = \mathcal{M}_{sym} = \mathcal{M}_{sym}(c_0, c_1)$$

Properties of G-convergence

- $\mathcal{M}_{sym}$ is (sequentially) compact with respect to G-convergence
- Λ is (sequentially) continuous with respect to G-convergence

Consequence: the necessary condition

$$\exists \min \left\{ \|\widetilde{\Lambda} - \Lambda(\sigma)\| : \ \sigma \in \mathcal{M}_{sym} \right\}$$

is satisfied.

Remark: $\mathcal{M}_{scal}$ is not (sequentially) compact with respect to G-convergence

G-convergence result

Theorem: G-convergence of the decomposition

Let $\mathcal{M} = \mathcal{M}_{sym} = \mathcal{M}_{sym}(c_0, c_1)$.
Assume

$$a_n \leqslant a_{n-1} \text{ for any } n \geqslant 1, \quad \lim_n a_n = 0 \quad \text{and} \quad \limsup_n \frac{2^n}{\lambda_n} < +\infty.$$

By the multiscale procedure, we construct

$$\sigma_n = \sum_{i=0}^{n} u_i$$

Then $\exists$ a subsequence $\{\sigma_{n_k}\}_k$ and $\exists$ $\sigma_\infty \in \mathcal{M}_{sym}$ such that

σ_{n_k} G-converges to σ_∞ as $k \to \infty$ and

$$\Lambda(\sigma_\infty) = \min\left\{ \|\widetilde{\Lambda} - \Lambda(\sigma)\| : \ \sigma \in \mathcal{M}_{sym} \right\}$$

Assumption for convergence in L^1

$$\mathcal{M} = \mathcal{M}_{scal} = \mathcal{M}_{scal}(c_0, c_1), \qquad R = |\cdot| = |\cdot|_{BV(\Omega)}$$

Remark: the necessary condition is NOT sufficient, we need a stronger assumption

Crucial assumptions

- Assume $\exists\, \widetilde{\sigma} \in \mathcal{M}_{scal} \cap BV(\Omega)$ (i.e. with $|\widetilde{\sigma}|_{BV(\Omega)} < +\infty$) such that

$$\|\widetilde{\Lambda} - \Lambda(\widetilde{\sigma})\| = \min\left\{\|\widetilde{\Lambda} - \Lambda(\sigma)\| : \ \sigma \in \mathcal{M}_{scal}\right\} = \varepsilon_0$$

that is

$$\exists \min \{ |\sigma|_{BV(\Omega)} : \ \sigma \in \mathcal{M}_{scal} \text{ and } \|\widetilde{\Lambda} - \Lambda(\sigma)\| = \varepsilon_0 \} = R_0 < +\infty$$

- $a_n \leqslant a_{n-1}$ for $n \geqslant 1$, $\quad \lim_n a_n = 0 \quad$ and $\quad \limsup_n \dfrac{2^n}{a_n \lambda_n} < +\infty.$

The main theorem: convergence of σ_n

Let S be the set of optimal solutions

$$S = \{ \sigma \in \mathcal{M}_{scal} : \ \|\widetilde{\Lambda} - \Lambda(\sigma)\| = \varepsilon_0 \text{ and } |\sigma|_{BV(\Omega)} = R_0 \}$$

Remark: S is sequentially compact in $L^1(\Omega)$

Theorem

Under the crucial assumptions, $\exists$ a subsequence $\{\sigma_{n_k}\}_k$ and $\exists$ $\sigma_\infty \in S$ such that

$$\sigma_\infty = \lim_k \sigma_{n_k} \quad \text{in } L^1(\Omega),$$

$$\lim_n |\sigma_n|_{BV(\Omega)} = |\sigma_\infty|_{BV(\Omega)} = R_0 \quad \text{and} \quad \lim_n \operatorname{dist}(\sigma_n, S) = 0$$

Finally, if $S = \{ \widetilde{\sigma} \}$ then

$$\widetilde{\sigma} = \lim_n \sigma_n = \sum_{i=0}^{\infty} u_i \quad \text{in } L^1(\Omega)$$

(e.g. $N = 2$ and $\widetilde{\Lambda} = \Lambda(\sigma_0)$ with $\sigma_0 = \widetilde{\sigma} \in \mathcal{M}_{scal} \cap BV(\Omega)$)