1. Multiscale Decomposition of Diffeomorphisms in Image Registration
2. A Nonlinear Plancherel Theorem, and Reconstruction Method for the Inverse Conductivity Problem

Adrian I. Nachman

University of Toronto

1. Joint work with Klas Modin and Luca Rondi
2. Joint work with Idan Regev and Daniel I. Tataru

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The The Calderón Inverse Conductivity Problem

Let Ω be a simply connected domain in $\mathbb{R}^2 \simeq \mathbb{C}$

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 \text{ in } \Omega \\ u\Big|_{\partial\Omega} = g. \end{cases} \tag{1}$$

The Dirichlet-to-Neumann map is defined as

$$\Lambda_\sigma f := \sigma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

A.P. Calderón (1980) posed the problem: does Λ_σ uniquely determine σ?

- N. (1996) - Unique reconstruction for $\sigma \in W^{2,p}(\Omega)$ for some $p > 1$
- R. Brown. G. Uhlman (1997) - $\sigma \in W^{1,p}(\Omega)$, for some $p > 2$.
- K. Astala, L. Päivärinta (2006) - $\sigma \in L^\infty$
- K. Astala, M. Lassas, L. Päivärinta (2016) - Larger class of conductivities which includes some unbounded ones.
- C.Carstea J.-N. Wang $\log \sigma \in L^2(\Omega)$ with small norm (2018)

Reconstruction via Inversion of the Scattering Transform

Assume $\nabla log \sigma \in L^2(\Omega)$ and (for simplicity) $\sigma = 1$ on $\partial\Omega$.

Let $v = \sigma^{\frac{1}{2}} \partial u$ then for u real valued, v is pseudoanalytic i.e. $\bar{\partial} v = q \bar{v}$ with $q = -\frac{1}{2} \partial \log \sigma \in L^2$.

We'll use a nonlinear transform of q, the Scattering Transform $\mathcal{S} q$, which can be calculated from Λ_σ.

The main result of Part 1 is a Plancherel and Inversion Theorem for the Scattering Transform.

The Scattering Transform

Given $q(z)$, we solve for $m_\pm(z,k)$ satisfying the pseudo-analytic equations

$$\begin{cases} \frac{\partial}{\partial \bar{z}} m_\pm = \pm e_{-k} q \overline{m_\pm} \\ m_\pm \to 1 \text{ as } |z| \to \infty \end{cases}$$

where

$$z = x_1 + ix_2; \quad k = k_1 + ik_2; \quad \bar{\partial} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big); \quad e_k(z) = e^{i(zk+\overline{zk})}.$$

The Scattering Transform - first introduced by Ablowitz and Fokas (1982) to solve a nonlinear PDE - is defined as

$$\mathbf{s}(k) = \mathcal{S}q(k) = \frac{1}{2\pi i}\int_{\mathbb{R}^2} e_k(z)\overline{q(z)}\Big(m_+(z,k) + m_-(z,k)\Big)dz,$$

where $dz = dx_1 dx_2$. When $q = 0$, then $m_\pm = 1$ and $\mathbf{s}(k) = \overline{\hat{q}(k)}$.

Nonlinear Plancherel Identity

Beals and Coifman (1998) proved that for q in Schwartz class $\mathbf{s}$ is in Schwartz class and :

$$\int_{\mathbb{R}^2} |\mathbf{s}(k)|^2 dk = \int_{\mathbb{R}^2} |q(z)|^2 dz.$$

Open Problem: true for all q in L^2 ?

- R. Brown (2001) - q in L^2 with small norm
- P. Perry (2014) - q in weighted Sobolev space $H^{1,1}$
- K. Astala, D. Faraco and K. Rogers (2015) - q in weighted Sobolev space $H^{\varepsilon,\varepsilon}$, $\varepsilon > 0$
- R. Brown, K. Ott and P. Perry (2016) - $q \in H^{\alpha,\beta}$ iff $\mathbf{s} \in H^{\beta,\alpha}$, $\alpha, \beta > 0$

Plancherel Theorem

Theorem (N-Regev-Tataru)

The nonlinear scattering transform $\mathcal{S} : q \mapsto \mathbf{s}$ *is a* C^1 *diffeomorphism* $\mathcal{S} : L^2 \to L^2$, *satisfying:*

1. *The Plancherel Identity:* $\|\mathcal{S}q\|_{L^2} = \|q\|_{L^2}$
2. *The pointwise bound:* $|\mathcal{S}q(k)| \leq C(\|q\|_{L^2}) M\hat{q}(k)$ *for a.e.* k
3. *Locally uniform bi-Lipschitz continuity:*

$$\frac{1}{C}\|\mathcal{S}q_1 - \mathcal{S}q_2\|_{L^2} \leq \|q_1 - q_2\|_{L^2} \leq C\|\mathcal{S}q_1 - \mathcal{S}q_2\|_{L^2}$$

where

$$C = C(\|q_1\|_{L^2})C(\|q_2\|_{L^2}).$$

4. *Inversion Theorem:* $\mathcal{S}^{-1} = \mathcal{S}$.

Using the Scattering Transform to solve DSII

The (integrable, defocusing) DSII (Davey Stewartson) equations

$$\begin{cases} i\partial_t q + 2(\bar{\partial}^2 + \partial^2)q + q(g+\overline{g}) = 0 \\ \bar{\partial} g + \partial(|q|^2) = 0 \\ q(0,z) = q_0(z) \end{cases} \tag{2}$$

arise in the study of water waves, ferromagnetism, plasma physics, and nonlinear optics. Analogous to Fourier transform for linear PDEs:

$$\begin{cases} \mathbf{s}_0(k) &= \mathcal{S} q_0(k) \\ \mathbf{s}(t,k) &= e^{2i(k^2+\overline{k}^2)t}\mathbf{s}_0(k) \\ q(t,z) &= \mathcal{I}\big(\mathbf{s}(t,k)\big)(z). \end{cases} \tag{3}$$

$$\begin{array}{ccc} q_0(z) & \xrightarrow{\text{nonlin}} & q(t,z) \\ \big\downarrow \mathcal{S} & & \mathcal{I} \big\uparrow \\ \mathbf{s}_0(k) & \xrightarrow{\text{linear}} & \mathbf{s}(t,k). \end{array}$$

A bit about the Proof

We need to solve

$$\begin{cases} \frac{\partial}{\partial \bar{z}} m_\pm = \pm e_{-k} q \overline{m_\pm} \\ m_\pm \to 1 \text{ as } |z| \to \infty. \end{cases}$$

In integral form,

$$m_\pm - 1 = (\bar{\partial} \mp e_{-k} q \bar{\cdot})^{-1} \bar{\partial}^{-1} (e_{-k} q).$$

1. For $q \in L^2$, we need new bounds on $\bar{\partial}^{-1}(e_{-k}q)$ which allow us to capture the large k decay without assuming any smoothness on q.
2. We need bounds on $(\bar{\partial} \mp e_{-k} q \bar{\cdot})^{-1}$ which depend only on the L^2 norm of q.

New Estimate on Fractional Integrals

Lemma

For $q \in L^2(\mathbb{C})$,

$$\|\bar{\partial}^{-1}(e_{-k}q)\|_{L^4} \lesssim \|q\|_{L^2}^{\frac{1}{2}} \Big(M\hat{q}(k)\Big)^{\frac{1}{2}}.$$

M is the Hardy-Littlewood Maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

which yields a bounded operator on L^p for $1 < p \leq \infty$.

Theorem

For $0 < \alpha < n$, $f \in L^p(\mathbb{R}^n)$, $1 < p \leq 2$

$$\left|(-\Delta)^{-\frac{\alpha}{2}} f(x)\right| \leq c_{n,\alpha} \Big(M\hat{f}(0)\Big)^{\frac{\alpha}{n}} \Big(Mf(x)\Big)^{1-\frac{\alpha}{n}}$$

Sketch of Proof - Fractional Integrals

Proof.

Using Littlewood-Paley decomposition,

$$(-\Delta)^{-\frac{\alpha}{2}} f(x) = \frac{1}{(2\pi)^n} \sum_{j=-\infty}^{j_0} \int_{\mathbb{R}^n} \psi_j(\xi) \frac{e^{ix\cdot\xi}}{|\xi|^\alpha} \hat{f}(\xi) d\xi + \sum_{j_0+1}^{\infty} \ldots$$

with $\psi_j(\xi) = \psi(\xi/2^j)$ supported in $2^{j-1} < |\xi| < 2^{j+1}$. For $j \leq j_0$ use

$$\int_{|\xi|<r} |\hat{f}(\xi)| d\xi \leq c_n r^n M\hat{f}(0)$$

...

$$\left|(-\Delta)^{-\frac{\alpha}{2}} f(x)\right| \lesssim 2^{j_0(n-\alpha)} M\hat{f}(0) + 2^{-j_0\alpha} Mf(x)$$

optimize over j_0. □

Key Theorem - bounds in terms of $\|q\|_{L^2}$

Theorem

Let $q \in L^2$. Then for each $f \in \dot{H}^{-\frac{1}{2}}$ there exists a unique solution $u \in \dot{H}^{\frac{1}{2}}$ of

$$L_q u := \bar{\partial} u + q\bar{u} = f \tag{4}$$

with

$$\|u\|_{\dot{H}^{\frac{1}{2}}} \leq C(\|q\|_{L^2})\|f\|_{\dot{H}^{-\frac{1}{2}}}. \tag{5}$$

In particular, for $f \in L^{\frac{4}{3}}$ the same holds, with $\|u\|_{L^4} \leq C(\|q\|_{L^2})\|f\|_{L^{\frac{4}{3}}}$.

Idea of the proof: use Kenig and Merle method of Induction on Energy and Gerard Profile Decompositions to study the static problem.

Construction of the Jost Solutions for $q \in L^2$

As a result of the new estimates on fractional integrals and the Key Theorem, we can now establish

Theorem (Jost Solutions)

Suppose $q \in L^2$, then for almost every k there exist unique Jost solutions $m_\pm(z,k)$ with $m_\pm(\cdot,k) - 1 \in L^4$ and moreover

$$\|m(\cdot,k)_\pm - 1\|_{L^4} \leq C(\|q\|_{L^2})\big(M\hat{q}(k)\big)^{\frac{1}{2}}$$

$$\|m_\pm - 1\|_{L^4_z L^4_k} \leq C(\|q\|_{L^2}).$$

$$\|\bar{\partial} m^1(\cdot,k)\|_{L^{\frac{4}{3}}} \leq C(\|q\|_{L^2})\big(M\hat{q}(k)\big)^{\frac{1}{2}}.$$

Scattering Transform as a ΨDO

Recall

$$\mathbf{s}(k) = \hat{\overline{q}}(k) - \frac{i}{\pi}\int_{\mathbb{R}^2} e_k(z)\overline{q(z)}a(k,z)dz,$$

where $a(k,z) = \Big(m_+(z,k) + m_-(z,k)\Big)$. Replace $\overline{q}$ by the Fourier transform of some function in L^2. Then the above becomes a pseudo-differential operator with symbol $a(k,z)$. We'd like to prove it is a bounded operator on L^2.

Theorem

Let $0 \leq \alpha < n$. *Suppose* $a(x,\xi)$ *satisfies*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |a(x,\xi)|^{\frac{2n}{n-\alpha}} dx d\xi < \infty \qquad \text{and} \qquad \|(-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\xi)\|_{L_\xi^{\frac{2n}{n+\alpha}}} \in L_x^{\frac{2n}{n-\alpha}}.$$

Then the pseudo-differential operator

$$a(x,D)f(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi) d\xi \tag{6}$$

is bounded on L^2. *Moreover, we have the pointwise bound*

$$|a(x,D)f(x)| \leq c_{\alpha,n}(Mf(x))^{\alpha/n} \|(-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\cdot)\|_{L^{\frac{2n}{n+\alpha}}} \|f\|_{L^2}^{1-\frac{\alpha}{n}} \tag{7}$$

for a.e. x.

This completes the sketch of the proof of the Plancherel Theorem.

GWP for Defocusing DSII on L^2

Theorem

Given $q_0 \in L^2$, there exists a unique solution to the Cauchy Problem for defocusing DSII such that:

1. *Regularity:*
$$q(t,z) \in C(\mathbb{R}, L^2_z(\mathbb{C})) \cap L^4_{t,z}(\mathbb{R} \times \mathbb{C}).$$
2. *Uniform bounds:* $\|q(t,\cdot)\|_{L^2} = \|q_0\|_{L^2}$ *for all* $t \in \mathbb{R}$ *and*
$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |q(t,z)|^4 dzdt \leq C(\|q_0\|_{L^2}).$$
3. *Stability: if* $q_1(t,\cdot)$ *and* $q_2(t,\cdot)$ *are two solutions corresponding to initial data* $q_1(0,\cdot)$ *and* $q_2(0,\cdot)$ *with* $\|q_j(0,\cdot)\|_{L^2} \leq R$ *then*
$$\|q_1(t,\cdot) - q_2(t,\cdot)\|_{L^2} \leq C(R)\|q_1(0,\cdot) - q_2(0,\cdot)\|_{L^2} \qquad \text{for all } t \in \mathbb{R}.$$

Back to The Calderón Inverse Conductivity Problem

Theorem

Suppose $\sigma > 0$ is such that $\nabla \log \sigma \in L^2(\Omega)$ and $\sigma = 1$ on $\partial\Omega$, then we can reconstruct σ from knowledge of Λ_σ.

Start of first step: from Λ_σ to $\mathbf{s}(k) = \mathcal{S}q(k)$

Let $v = \sigma^{\frac{1}{2}}\partial u$ then for u real valued, $\bar{\partial} v = q\overline{v}$ where $q = -\frac{1}{2}\partial \log \sigma \in L^2$.

$$\begin{aligned}
\mathbf{s}(k) &= \frac{1}{2\pi i}\int_{\mathbb{R}^2} e_k(z)\overline{q(z)}\Big(m_+(\cdot,k) + m_-(\cdot,k)\Big) \\
&= \frac{1}{2\pi i}\int_{\Omega} \partial\Big(\overline{m_+}(\cdot,k) - \overline{m_-}(\cdot,k)\Big) \\
&= \frac{1}{4\pi i}\int_{\partial\Omega} \overline{\nu}\Big(\overline{m_+}(\cdot,k) - \overline{m_-}(\cdot,k)\Big)
\end{aligned}$$

Proof consists in showing that Λ_σ determines the traces of $m_\pm(\cdot,k)$ on $\partial\Omega$.

Thank You!