Transverse stability of the line soliton with critical frequency for the Nonlinear Schrödinger equations.

BIRS

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Outlines

Introduction

The Schrödinger equations:

We consider the following Nonlinear Schrödinger equations:

$$i\partial_t\psi + \partial_{xx}\psi + \partial_{yy}\psi + |\psi|^{p-1}\psi = 0, \qquad \text{in } \mathbb{R}_t \times \mathbb{R}_x \times \mathbb{T}_y, \qquad \text{(NLS)}$$

where $p > 1$ and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Energy (Hamiltonian)

$$\mathcal{H}(\psi) := \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} \left(|\nabla\psi(x,y)|^2 - \frac{2}{p+1}|\psi(x,y)|^{p+1} \right) dxdy$$

Line soliton

Let R_ω be the unique positive solution to

$$-\partial_{xx} R_\omega + \omega R_\omega - |R_\omega|^{p-1} R_\omega = 0 \qquad \text{in } \mathbb{R},$$

that is,

$$R_\omega(x) = \left(\frac{(p+1)\omega}{2}\right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{(p-1)\omega}{2} x\right).$$

We note that $e^{i\omega t} R_\omega(x)$ becomes the standing waves of the following Schrödinger equations:

$$i\partial_t \psi + \partial_{xx}\psi + |\psi|^{p-1}\psi = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}.$$

Stability

Definition

$$\|\psi_0 - R_\omega\|_{H^1} < \delta \Rightarrow \sup_{t>0} \inf_{\theta\in\mathbb{R}, b\in\mathbb{R}} \|\psi(t,\cdot) - e^{i\theta} R_\omega(\cdot - b)\|_{H^1} < \varepsilon.$$

Stability's results of the line soliton under the 1D NLS flow:

- stable for $1 < p < 5$ (Cazenave and Lions / Grillakis, Shatah and Strauss).
- unstable for $p > 5$ (Berestycki and Cazenave / Grillakis, Shatah and Strauss).
- unstable for $p = 5$ (Weinstein).

Transverse Stability

Remark

The line soliton R_ω is a steady state solution to (NLS) *in the energy space.*

Transverse Stability : Stability of the line solitary wave under the 2D perturbation.

Definition (Transverse Stability)

$$\|\psi_0 - R_\omega\|_{H^1} < \delta \Rightarrow \sup_{t>0} \inf_{\theta\in\mathbb{R}, b\in\mathbb{R}\times\mathbb{T}} \|\psi(t,\cdot) - e^{i\theta} R_\omega(\cdot - b)\|_{H^1} < \varepsilon.$$

Transverse Stability

Literature

- Milewski and Wang : Describe Traveling waves which are localized in the propagation direction and periodic in the transverse direction (Gravity-Capillary).
- Haragus : Transverse stability of those traveling waves for the Euler equation.
- Rousset and Tzvetkov : Linear and nonlinear instability of the line solitary water waves with respect to transverse perturbations.

Transverse Stability

Literature

- Rousset and Tzvetkov : Nonlinear long time instability of the KdV solitary wave under a KP-I flow.

$$u_t + uu_x + u_{xxx} = 0, \tag{KdV}$$

$$u_t + uu_x + u_{xxx} - \partial_x^{-1} u_{yy} = 0, \quad \mathbb{R}_x \times \mathbb{T}_y \tag{KP-I}$$

- Rousset and Tzvetkov : Transverse nonlinear instability of solitary waves for the cubic nonlinear Schrödinger equation.
- Pelinovsky : Instability band of a deep-water soliton of the hyperbolic non-linear Schrödinger equation.
- Yamazaki : Tranverse stability of the line standing waves under the flow of the $2D$ nonlinear Schrödinger equation.

Outlines

2 Classification of the transverse stability with respect to the frequency:
- Sub-critical case.
- Super-critical case.
- The Critical case .

Sub-critical case

Theorem (Yamazaki 2014)

Let $1 < p < 5$ *and* $\omega_p = \frac{4}{(p-1)(p+3)}$.

(i) for $0 < \omega < \omega_p$, *the standing wave* $e^{i\omega t} R_\omega$ *is stable under the flow of (NLS).*

Sub-critical case

From Grillakis, Shatah and Strauss theory, it is sufficient to show

$$\langle \mathcal{S}''_\omega(R_\omega)u, u\rangle \geq \delta \|u\|_X^2,$$

with

$$\mathcal{S}_\omega(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{\omega}{2}\|u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

$$\langle \mathcal{S}''_\omega(R_\omega)u, u\rangle = \sum_{n\in\mathbb{Z}} \left(\langle L_{\omega,+,n}u_n^R, u_n^R\rangle + \langle L_{\omega,-,n}u_n^I, u_n^I\rangle\right),$$

where

$$L_{\omega,+,n} = -\partial_{xx} + \omega + n^2 - pR_\omega^{p-1}, \qquad L_{\omega,-,n} = -\partial_{xx} + \omega + n^2 - R_\omega^{p-1},$$

and

$$u_n^R := \Re u_n, \qquad u_n^I := \Im u_n.$$

Sub-critical case

Spectral properties

- The negative eigenvalue of $L_{\omega,+,0}$ is $-\frac{\omega}{\omega_p}$ and the corresponding eigenfunction is given by $R_\omega^{\frac{p+1}{2}}$.
- We have

$$\langle L_{\omega,+,n} u_n^R, u_n^R \rangle \geq (n^2 - \frac{\omega}{\omega_p}) \| u_n^R \|_{L_x^2}^2.$$

- When $\omega < \omega_p$, we obtain

$$\langle L_{\omega,+,n} u_n^R, u_n^R \rangle \geq \delta \| u_n^R \|_{L_x^2}^2.$$

Super-critical case

Theorem (Yamazaki 2014)

Let $1 < p < 5$ *and* $\omega_p = \frac{4}{(p-1)(p+3)}$.

(ii) for $\omega > \omega_p$, *the standing wave* $e^{i\omega t} R_\omega$ *is unstable under the flow of (NLS).*

Super-critical case

Let $u(t) := e^{i\omega t}\,(R_\omega + v(t))$, so that $v := (\Re v, \Im v)$ is a solution of

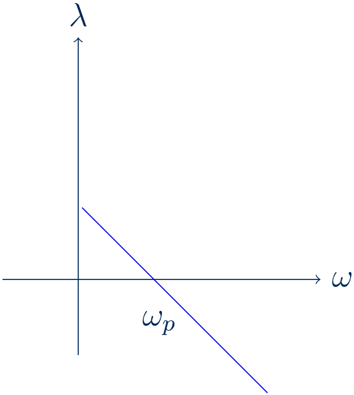
$$v_t = -J(\mathcal{S}''_\omega(R_\omega)v + NL(v, R_\omega)),$$

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For $\omega > \omega_p$, $-J\mathcal{S}''_\omega(R_\omega)$ has at least one positive eigenvalue.

$$\Rightarrow \quad \text{linear instability} \quad \Rightarrow \quad \text{nonlinear instability.}$$

Spectral properties



Spectral properties

The kernel of the linearized operator

$$L_{\omega_p,+} := -\partial_{xx} - \partial_{yy} + \omega_p - pR_\omega^{p-1},$$

is given by

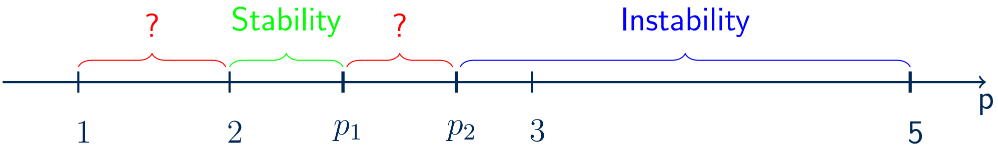
$$\operatorname{Ker} L_{\omega_p,+} = \operatorname{Span}\left\{R'_{\omega_p}, R_{\omega_p}^{\frac{p+1}{2}} \cos y, R_{\omega_p}^{\frac{p+1}{2}} \sin y\right\}.$$

The critical frequency case $\omega = \omega_p$

Theorem (Yamazaki 2015)

There exists $2 < p_1 < p_2 < 3$ satisfying the following two properties:

- *(i) If $2 \leq p \leq p_1$, then the standing wave $e^{i\omega_p t} R_{\omega_p}$ is stable under the flow of (NLS).*
- *(ii) If $p_2 \leq p < 5$, then the standing wave $e^{i\omega_p t} R_{\omega_p}$ is unstable under the flow of (NLS).*


?
Stability
?
Instability
p
1
2
p_1
p_2
3
5

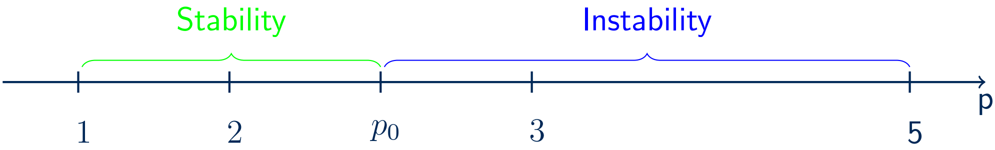
The critical frequency case $\omega = \omega_p$

Theorem (B., Ibrahim and Kikuchi 2019)

There exists $2 < p_0 < 3$ satisfying the following two properties:

(i) *If $1 < p \leq p_0$, then the standing wave $e^{i\omega_p t} R_{\omega_p}$ is stable under the flow of (NLS).*

(ii) *If $p > p_0$, then the standing wave $e^{i\omega_p t} R_{\omega_p}$ is unstable under the flow of (NLS).*


Stability
Instability
p
1
2
p_0
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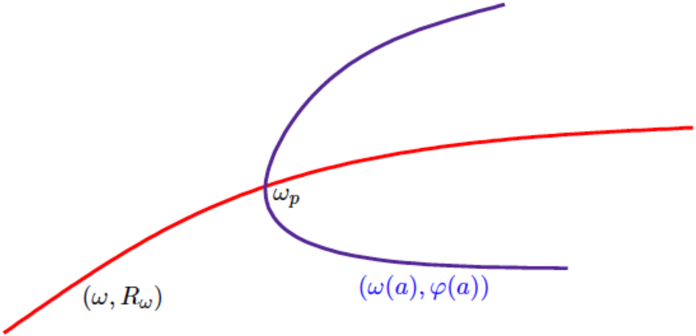
Bifurcation

We construct a steady state to (NLS) which bifurcate from the line solitary wave with the critical frequency ω_p (pitchfork)



Bifurcation

Proposition (Pelinovsky et al. 2011)

Let $p \geq 2$. *There exist* I *a neighborhood of* 0 *and* $a \mapsto \varphi(a) \in C^2(I, H^2)$ *such that* $\varphi(a) > 0$,

$$-\partial_{xx}\varphi(a) - \partial_{yy}\varphi(a) + \omega(a)\varphi(a) - |\varphi(a)|^{p-1}\varphi(a) = 0$$

and

$$\varphi(a) = R_{\omega_p} + aR_{\omega_p}^{\frac{p+1}{2}} \cos y + h(a),$$

where $a \mapsto h(a) \in C^2(I, H^2)$, $\|h(a)\|_{H^2} = O(a^2)$, *and*

$$\omega(a) = \omega_p + \frac{\omega''(0)}{2}a^2 + o(a^2).$$

Lyapounov-Schmidt decomposition

Lemma (Pelinovsky et al. 2011)

There exists a neighborhood $W \subset H^2_{\mathrm{sym}} \times \mathbb{R}$ *of* (R_{ω_p}, ω_p), *a neighborhood* $U \subset \mathbb{R}^2$ *of* $(0, \omega_p)$ *and a unique* C^1 *map* $h : U \mapsto L^2 \cap \{\psi_*\}^\perp$ *such that the function:*

$$\phi = R_{\omega_p} + a\psi_* + h(a,\omega) \qquad (a,\omega) \in U,$$

solves

$$P_\perp F(R_{\omega_p} + a\psi_* + h(a,\omega), \omega) = 0,$$

where

$$\psi_* := R_{\omega_p}^{\frac{p+1}{2}} \cos y,$$

$$F(\phi,\omega) := -\partial_{xx}\phi - \partial_{yy}\phi + \omega\phi - |\phi|^{p-1}\phi,$$

and

$$P_\perp u = u - \frac{\langle u, \psi_* \rangle}{\|\psi_*\|^2_{L^2}} \psi_*.$$

Lyapounov-Schmidt decomposition

Lemma (B., Ibrahim and Kikuchi 2019)

Let $\varepsilon > 0$. There exists $a_0 = a_0(\varepsilon) > 0$ such that for any $a \in (-a_0, a_0)$ and for any ω such that $(0, \omega) \in U$, the solution ϕ satisfies

$$\frac{1}{C} e^{-(\sqrt{\omega}+\varepsilon)|x|} \leq \phi(a, \omega) \leq C e^{-(\sqrt{\omega}-\varepsilon)|x|} \quad \text{for } (x, y) \in \mathbb{R} \times \mathbb{T},$$

where $\varepsilon > 0$ and $C = C(\omega) > 1$.

Corollary

$\phi : U \mapsto H^2$ *is* C^2 *for* $p > 1$.

Lyapounov-Schmidt decomposition

$$\mathcal{F}_{\parallel}(a,\omega) := \langle \psi_*,\ F(\phi,\omega)\rangle\,.$$

Crandall-Rabinowitz transversality argument

$$g(a,\omega) = \begin{cases} \frac{\mathcal{F}_{\parallel}(a,\omega)-\mathcal{F}_{\parallel}(0,\omega)}{a} & \text{if } a \neq 0, \\ \frac{\partial \mathcal{F}_{\parallel}}{\partial a}(0,\omega) & \text{if } a = 0. \end{cases}$$

$$g(a,\omega(a)) = 0 \quad \text{for any} \quad a \in I.$$

Lyapounov-Schmidt decomposition

Proposition

Let $p > 1$. *There exist* I *a neighborhood of* 0 *and* $a \mapsto \varphi(a) \in C^2(I, H^2)$ *such that* $\varphi(a) > 0$,

$$-\partial_{xx}\varphi(a) - \partial_{yy}\varphi(a) + \omega(a)\varphi(a) - |\varphi(a)|^{p-1}\varphi(a) = 0$$

and

$$\varphi(a) = \phi(a, \omega(a)).$$

Transverse Stability

Using a modulation theory, we decompose the solution

$$e^{i\theta(u)}u(\cdot - b(u), \cdot) = \Phi(a(u)) + w(u) + \alpha(u)\varphi(a(u))$$

with

$$\Phi(a(u)) := \varphi(a(u)) + \rho(a(u))\partial_\omega R_{\omega_p},$$

where

$$\rho(a(u)) = O(a(u)^2) \quad \text{and} \quad \|\Phi(a(u))\|_{L^2} = \|R_{\omega_p}\|_{L^2}.$$

We consider the curve $\Phi(a(u))$ in order to capture the degeneracy of the linearized operator.

Transverse Stability

This means that we have

$$\langle \mathcal{S}''_\omega(\Phi(a(u)))w(u), w(u)\rangle \geq \delta\|w(u)\|_X^2,$$

under the orthogonality conditions

$$\langle w(u), i\varphi(a(u))\rangle = \langle w(u), \partial_x\varphi(a(u))\rangle = \langle w(u) + \alpha(u)\varphi(a(u)), \psi_0 \cos y\rangle$$

$$= \langle w(u), \varphi(a(u))\rangle = \langle w(u) + \alpha(u)\varphi(a(u)), \psi_0 \sin y\rangle = 0.$$

From Taylor expansion, we obtain

$$\begin{aligned}\mathcal{S}_{\omega_p}(u) - \mathcal{S}_{\omega_p}(R_{\omega_p}) =& G(p)a(u)^4 + \langle \mathcal{S}''_\omega(\Phi(a(u)))w(u), w(u)\rangle \\ &+ o(a(u)^4) + o(\|w(u)\|_{H^1}^2).\end{aligned}$$

Transverse Stability

Proposition (Yamazaki 2015)

(i) *If $G(p) > 0$, then the standing wave $e^{i\omega_p t} R_{\omega_p}$ is stable under the flow of (NLS).*

(ii) *If $G(p) < 0$, then the standing wave $e^{i\omega_p t} R_{\omega_p}$ is unstable under the flow of (NLS).*

Transverse Stability in the double critical case

Remark

Note that $G(p)$ has the same sign as $\partial_a^2 \|\varphi(a)\|_{L^2}^2 \Big|_{a=0}$.

$$G(p) = 2\lambda'(\omega_p)\|\psi_*\|_{L^2}^2 + \omega''(0)\frac{\partial \|R_\omega\|_{L^2}^2}{\partial \omega}\Big|_{\omega=\omega_p}$$

$$G(p) = \frac{4(p+1)(p^6+18p^5-11p^4-130p^3+13p^2+16p-3)}{(5p-1)(3p+1)(p+3)^2(p-1)(5-p)} + \frac{p^2(p-1)^2\langle R_{\omega_p}^{2p-1}, A_2^{-1}(R_{\omega_p}^{2p-1})\rangle_{L_x^2}}{4\int_{\mathbb{R}} R_{\omega_p}^{p+1}\,dx}.$$

Transverse Stability in the double critical case

Lemma (Yamazaki 2015)

There exist two real numbers $2 < p_1 < p_2 < 3$ *such that*

(i) *If* $2 \le p \le p_1$*, then* $G(p) > 0$*.*

(ii) *If* $p \ge p_2$*, then* $G(p) < 0$*.*

Transverse Stability in the double critical case

Lemma

There exists a real number $2 < p_0 < 3$ *such that*

(i) *If* $1 < p < p_0$, *then* $G(p) > 0$.

(ii) *If* $p > p_0$, *then* $G(p) < 0$.

$A_2^{-1}(R_{\omega_p}^{2p-1})$ is not explicit on p. We compute $\frac{d}{dp}G(p)$ and we show that G is strictly decreasing.

Transverse Stability in the double critical case

For the double critical case i.e. when $p = p_0$. In this case

$$G(p) = 0.$$

- We need to expand to the next order in a!
- We expand $\|\varphi(a)\|_{L^2}^2$ and $\omega(a)$ to the next order.
- We need more regularity of φ in a.

Transverse Stability in the double critical case

$$\phi = R_{\omega_p} + a\psi_* + h(a,\omega) \qquad (a,\omega) \in U,$$

Lemma

(i) h *is* C^5 *on* U.

(ii) $\mathcal{F}_{\|}(a,\omega) := \langle \psi_*,\ F(\phi,\omega) \rangle$ *is* C^5 *on* U.

(iii) g *is* C^4 *on* U.

(iv) $a \mapsto \omega(a)$ *is* C^4 *on* I.

(v) $a \mapsto \varphi(a)$ *is* C^4 *on* I.

Transverse Stability in the double critical case

Claim

Let $\varepsilon > 0$ *and* $l, k \in \{0, \dots, 5\}$, *such that* $l + k \leq 5$. *For any* $(a, \omega) \in U$, *we have*

$$\left| \frac{\partial^{l+k} h}{\partial^l a \partial^k \omega}(a, \omega) \right| \lesssim e^{-(\sqrt{\omega} - \varepsilon)|x|} \quad \text{in} \quad \mathbb{R} \times \mathbb{T}, \tag{4}$$

when $l \neq 0$ *and*

$$\left| \frac{\partial^k h}{\partial^k \omega}(a, \omega) \right| \lesssim e^{-(\sqrt{\omega} - (k+1)\varepsilon)|x|} \quad \text{in} \quad \mathbb{R} \times \mathbb{T}. \tag{5}$$

Transverse Stability in the double critical case

- We have

$$w'''(0) = \partial_a^3 \|\varphi(a)\|_{L^2}^2 \Big|_{a=0} = 0.$$

- We continue to the next order.

Transverse Stability in the double critical case

- We continue to the next order and we compute $\omega^{(4)}(0) \neq 0$ and $\partial_a^4 \|\varphi(a)\|_{L^2}^2 \Big|_{a=0} > 0$.
- The sign of the main term is given by the sign of $\partial_a^4 \|\varphi(a)\|_{L^2}^2 \Big|_{a=0}$.

Thank you for your attention.