Polynomial processes for modelling energy commodity prices

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Polynomial processes

Polynomial processes

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A polynomial process has the property that any conditional expectation of the form $\mathbb{E}[\Psi(X_t)|X_s]$, where Ψ is a polynomial, is *itself* a polynomial function of X_s, with degree at most that of Ψ.

Some familiar examples…

OU $dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$

GBM $dX_t = X_t\big(\mu dt + \sigma dW_t\big)$

IGBM $dX_t = \kappa(\theta - X_t)dt + \sigma X_t dW_t$

CIR $dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t$

Polynomial processes: more examples

Jacobi $dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t(1-X_t)}dW_t$

In this case, $X_t \in (0,1)$ a.s. if $2\kappa \min\{\theta, 1-\theta\} \geq \sigma^2$.

Exponential Lévy If $X_t = x\,\mathrm{e}^{L_t}$, where L is a Lévy process with triplet (b, c, μ), and if $\int_{|y|>1} \mathrm{e}^{my}\,\mu(dy) < \infty$, then X is m-polynomial.

Lévy-driven SDEs Here $dX_t = \sum_i V_i(X_{t-})dL_t^i$, where the functions V_i are affine. (This is m-polynomial if m moments of the Lévy measure are defined.)

Polynomial processes

How does it work?

If $\mathcal{G}$ is the infinitesimal generator of a 1-D polynomial diffusion X_t, then the action of $\mathcal{G}$ on a polynomial function

$$\Psi(x) = \sum_{n=0}^{N} p_n x^n$$

can be represented by a matrix multiplication of the coefficient vector $\mathbf{p} = (p_0, p_1, \ldots, p_N)'$, i.e.

$$[\mathcal{G}\Psi](x) = \sum_{n=0}^{N} (G\mathbf{p})_n x^n.$$

Polynomial processes

How does it work?

Using this matrix representation, for $s \le t$,

$$\mathbb{E}[\Psi(X_t)|X_s] = \sum_{n=0}^{N} \left(\mathrm{e}^{G(t-s)}\, \mathbf{p} \right)_n X_s^n = \mathbf{H}(X_s)\mathrm{e}^{G(t-s)}\mathbf{p},$$

where $\mathbf{H}(x) = (1, x, x^2, \ldots, x^n)$ is the vector of basis functions.

An example

For the OU process, if Ψ is a polynomial of degree 4,

$$G = \begin{bmatrix} 0 & \kappa\theta & \sigma^2 & 0 & 0 \\ 0 & -\kappa & 2\kappa\theta & 3\sigma^2 & 0 \\ 0 & 0 & -2\kappa & 3\kappa\theta & 6\sigma^2 \\ 0 & 0 & 0 & -3\kappa & 4\kappa\theta \\ 0 & 0 & 0 & 0 & -4\kappa \end{bmatrix}.$$

Polynomial processes: applications in finance

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- Moment estimation
- Valuation
  - Bond markets
  - Credit risk
  - Stochastic volatility
  - Energy markets
- Variance reduction

In an arbitrage-free market, with a state price density (a positive semimartingale ζ), the model price $\Pi(t,T)$ of a cash flow C_T is given by

$$\Pi(t,T) = \frac{1}{\zeta_t}\mathbb{E}[\zeta_T C_T | \mathcal{F}_t].$$

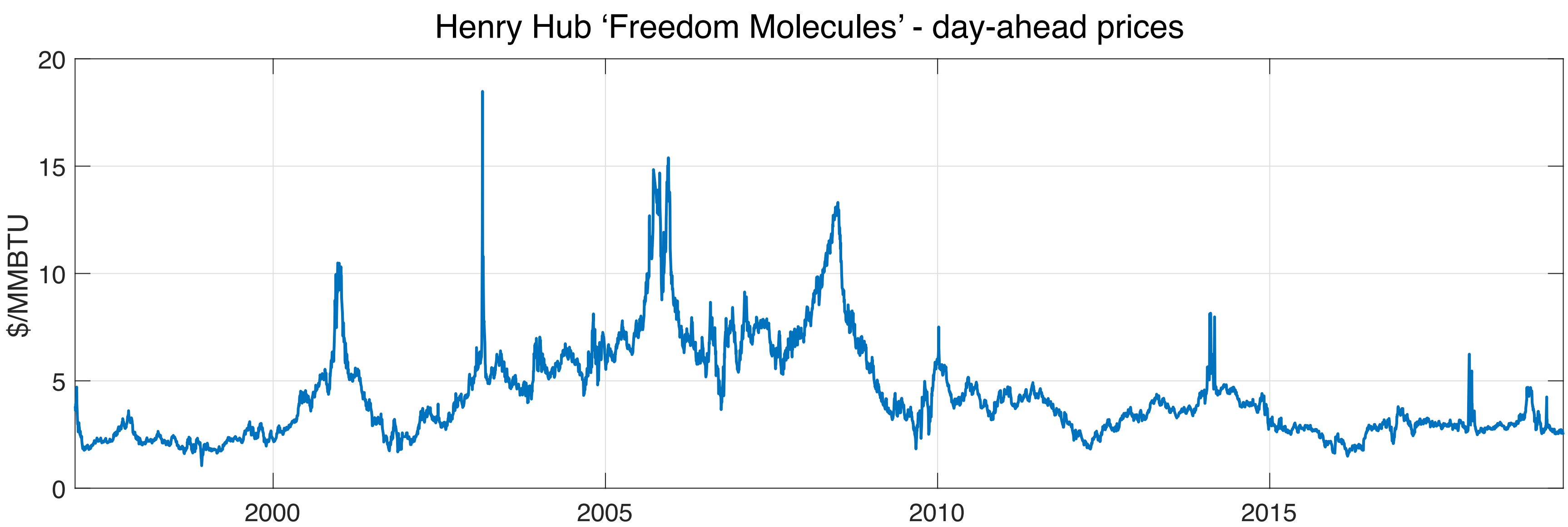
If $\zeta_t = \mathrm{e}^{-\alpha t}\, p(X_t)$, and $C_T = q(X_T)$ for polynomials p and q, where X_t is a polynomial diffusion, then $\Pi(t,T)$ is rational in X_t.

✓ C. Cuchiero, M. Keller-Ressel, and J. Teichmann. *Polynomial processes and their applications to mathematical finance*. Finance and Stochastics, 2012.

✓ D. Filipović and M. Larsson. *Polynomial diffusions and applications in finance*. Finance and Stochastics, 2016.

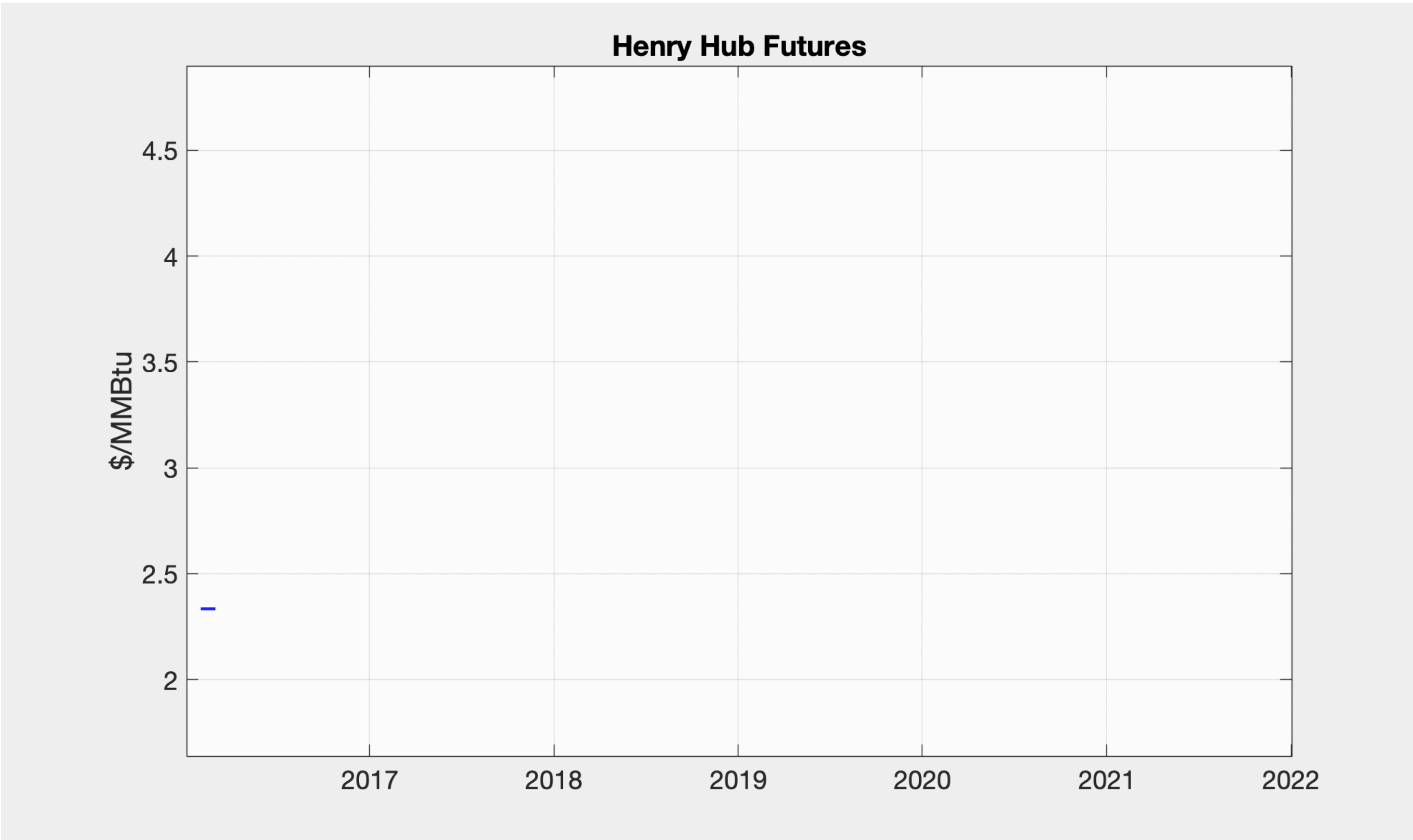
Energy commodity markets

Energy commodity markets: natural gas



- High volatility (sometimes)
- Occasional extreme spikes
- Mean reversion
- Seasonality?

Energy commodity markets: natural gas



Source: NRGStream.com

Energy commodity markets: emissions

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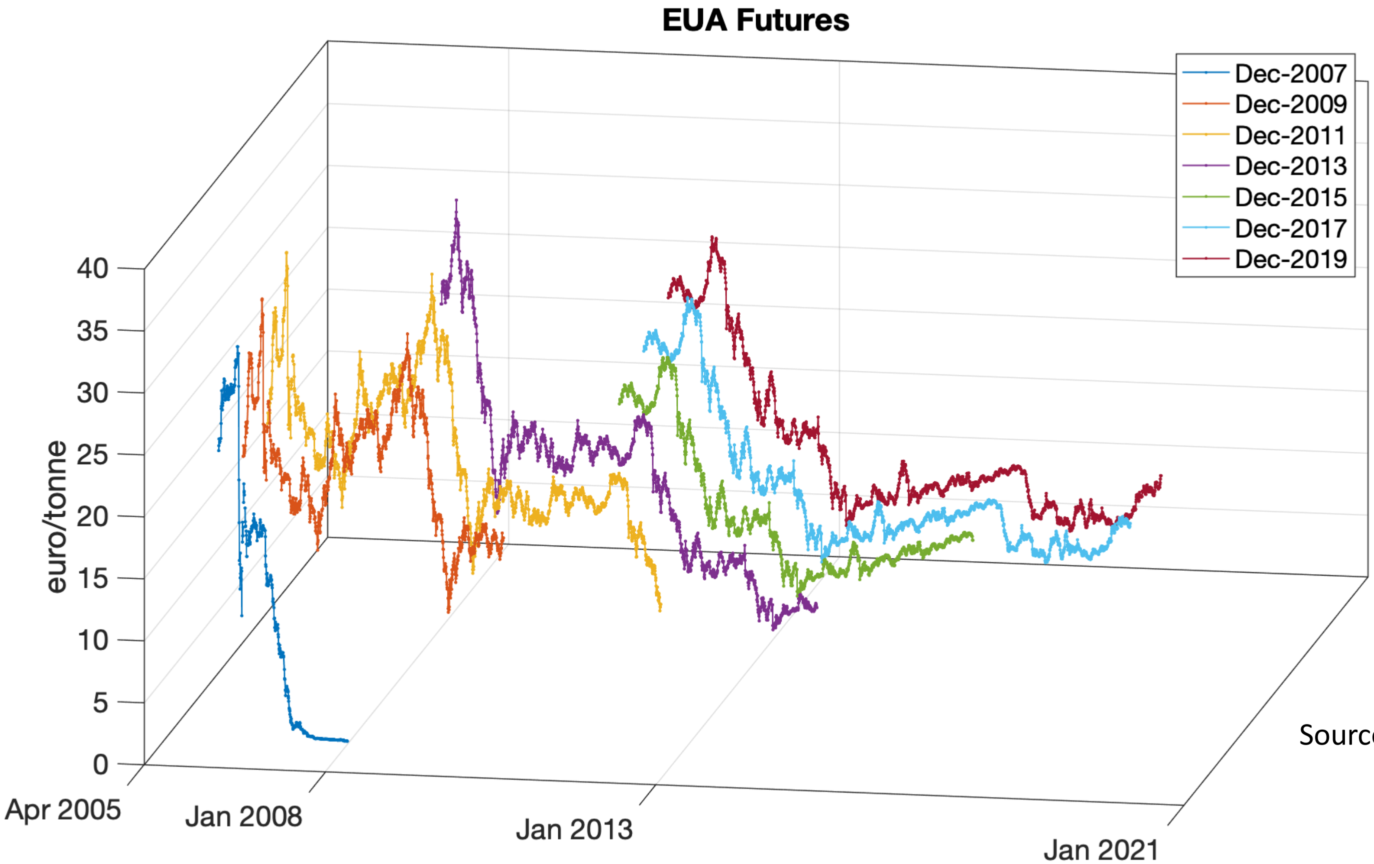
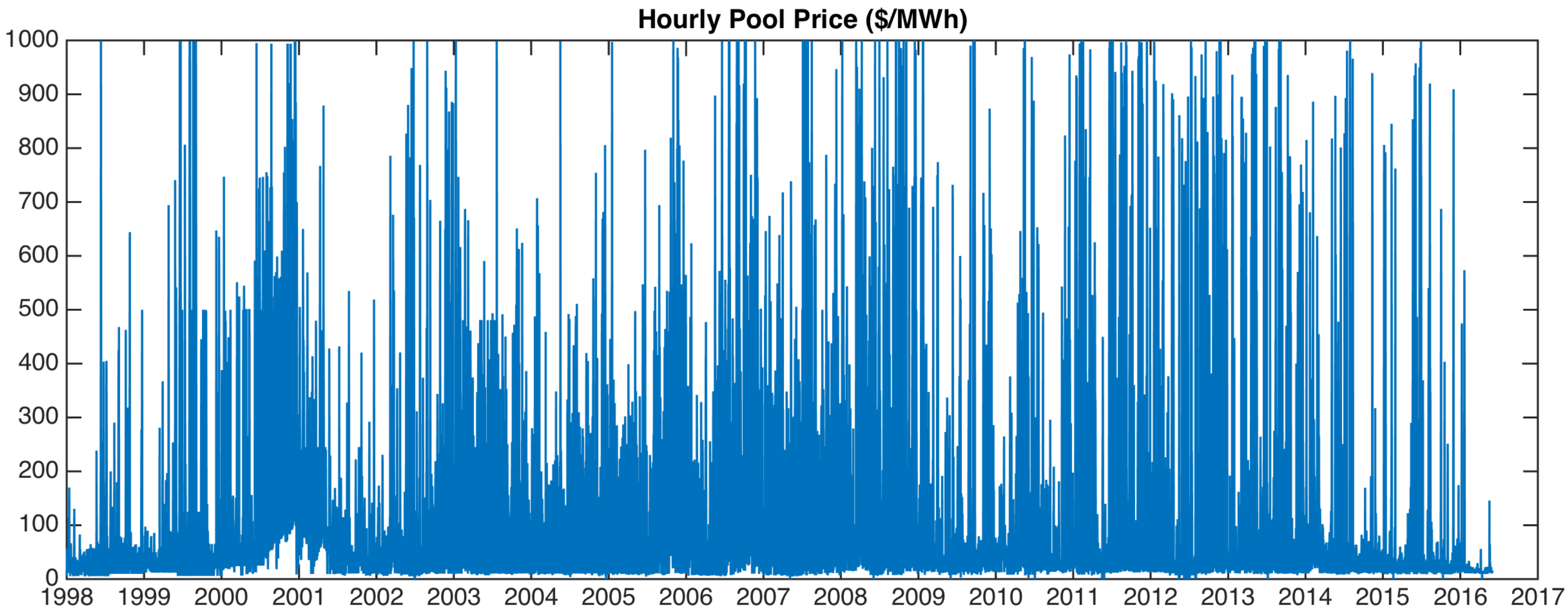


Source: NRGStream.com

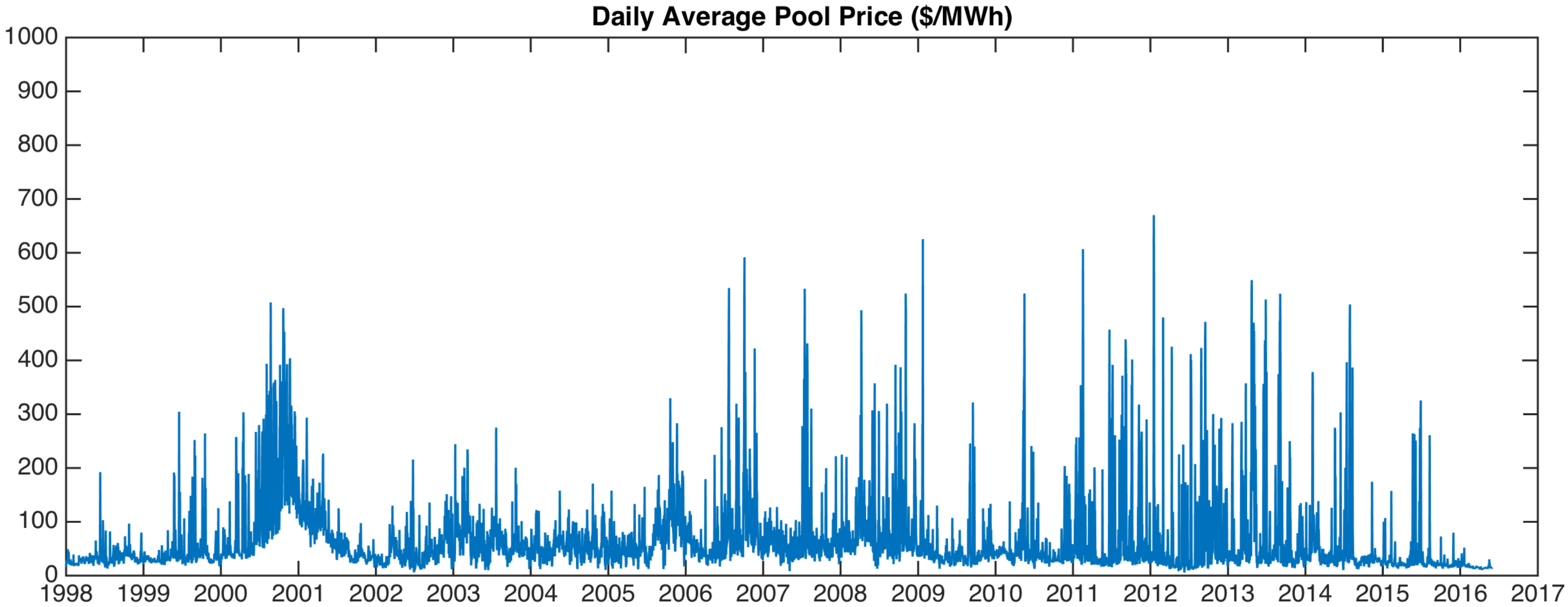
- ➢ Futures prices with different maturities highly correlated
- ➢ Prices constrained to lie in a bounded interval

Energy commodity markets: power



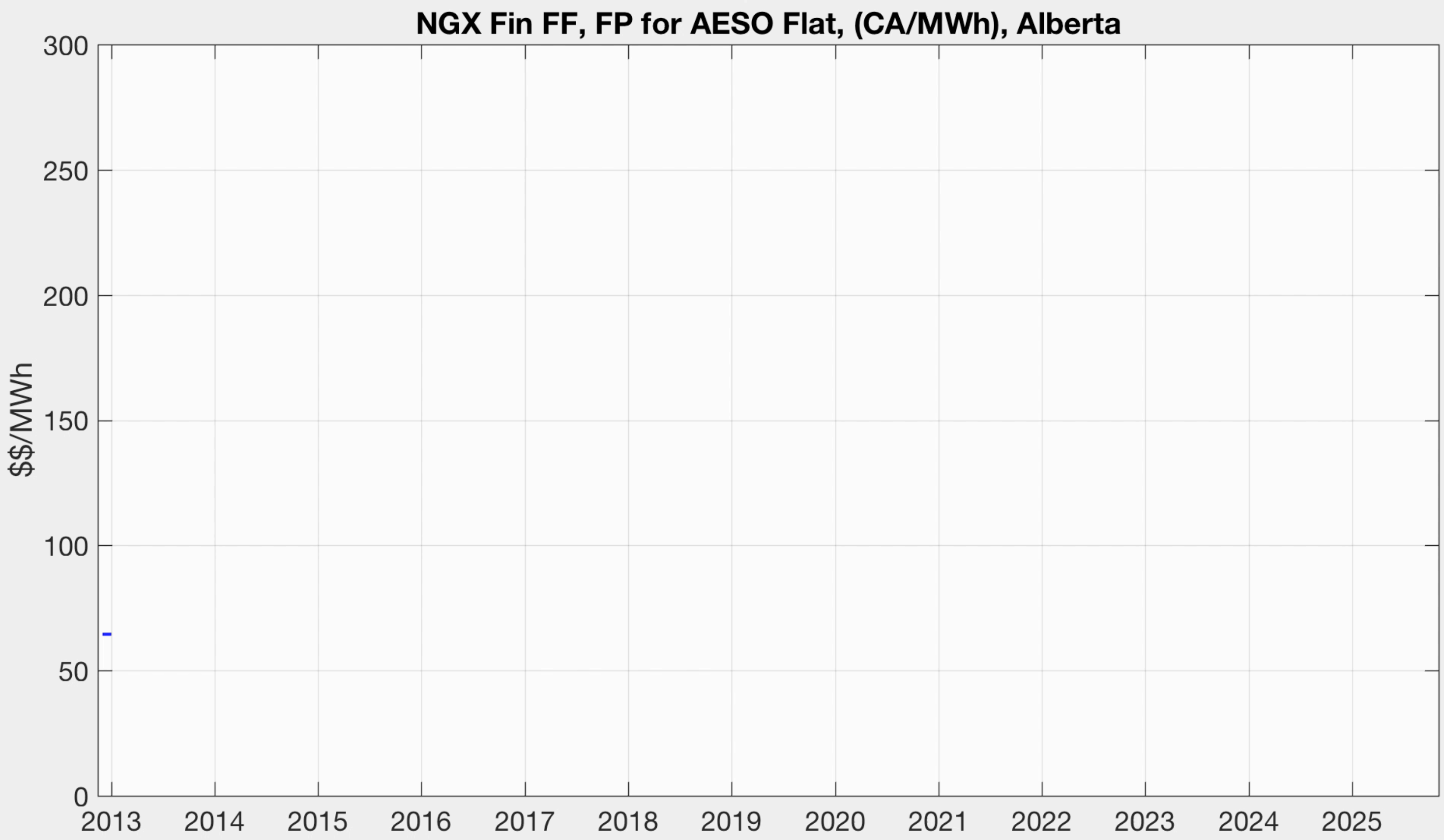
- Frequent extreme peaks
- Prices constrained to lie in a bounded interval

Energy commodity markets: power



- Frequent extreme peaks (even in daily averages)
- Prices constrained to lie in a bounded interval

Energy commodity markets: AB power



Source: NRGStream.com

Modelling energy prices

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Commonly, energy price models take the form $S_t = \Psi(\mathbf{X}_t)$, where $\mathbf{X}_t$ is a vector of underlying factors (and often Ψ is an *exponential* map).

- In Schwartz-Smith (2000), $\boxed{S_t = \mathrm{e}^{\xi_t + \chi_t}}$, with ξ and χ following correlated OU processes.
- For Barlow (2002), Ψ is conceived of as representing a demand-price curve. Demand is a linear function of a factor X_t which is modelled as an OU process:

$$dX_t = -\kappa(X_t - \theta)dt + \sigma dW_t,$$

and (for some $\alpha < 0$) $\Psi(x) = \begin{cases} \left(1 + \alpha x\right)^{\frac{1}{\alpha}} & \text{if } 1 + \alpha x > \epsilon_0, \\ \epsilon_0^{\frac{1}{\alpha}} & \text{otherwise.} \end{cases}$

Modelling energy prices

… using polynomial processes

- If the map Ψ is a polynomial, then we can exploit the freedom in the choice of Ψ to generate extreme dynamics even if $\mathbf{X}_t$ is relatively tame.

- If $\mathbf{X}_t$ follows a *polynomial process* under a pricing measure $\mathbb{Q}$, then futures prices can be computed explicitly:

$$F(t,T) = \mathbf{H}(\mathbf{X}_t)\,\mathrm{e}^{(T-t)G}\,\mathbf{p}(t),$$

  where G is the matrix representation of the generator of $\mathbf{X}_t$ with respect to the basis $\mathbf{H}$.

Modelling energy prices

… using polynomial processes

Collaborators

- ❖ Martin Larsson and Damir Filipović
- ❖ Clémence Alasseur and Thomas Deschatre
- ❖ Zuming Sun, Wenning Wei, Alex Dreher

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- ✓ France-Canada Research Fund

Example: IGBM and natural gas

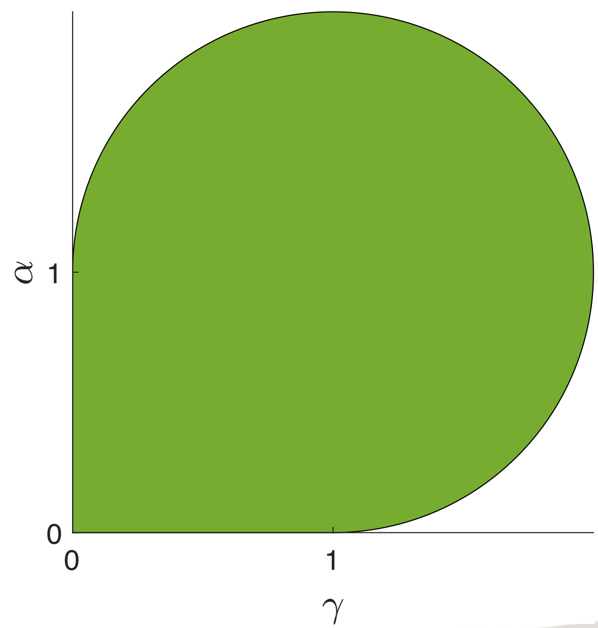
Recall the IGBM process:

$$dX_t = \kappa(\theta - X_t)dt + \sigma X_t dW_t.$$

Here $X_t > 0$ a.s. if $\kappa, \theta > 0$.
For $S_t = \Psi(X_t)$, Ψ should be increasing on $\mathbb{R}_+$.

- We search over all increasing cubic maps Ψ on $[0, \infty)$, normalized so that $\Psi(0) = 0$, and $\int_0^\infty \mathrm{e}^{-x}\,\Psi(x)dx = 1$.
- These are characterised by

  $\Psi'(x) = \frac{\alpha}{2}x^2 + (1 - \alpha - \gamma)x + \gamma$, with (γ, α) in the green region.
- For higher degree maps, we can represent Ψ' as a *product* of such factors.



Example: IGBM and natural gas

If we want to look at discretizations of the IGBM process, it makes sense to use a semi-implicit method such as *split step backward Euler* (Higham, Mao, Stuart 2002):

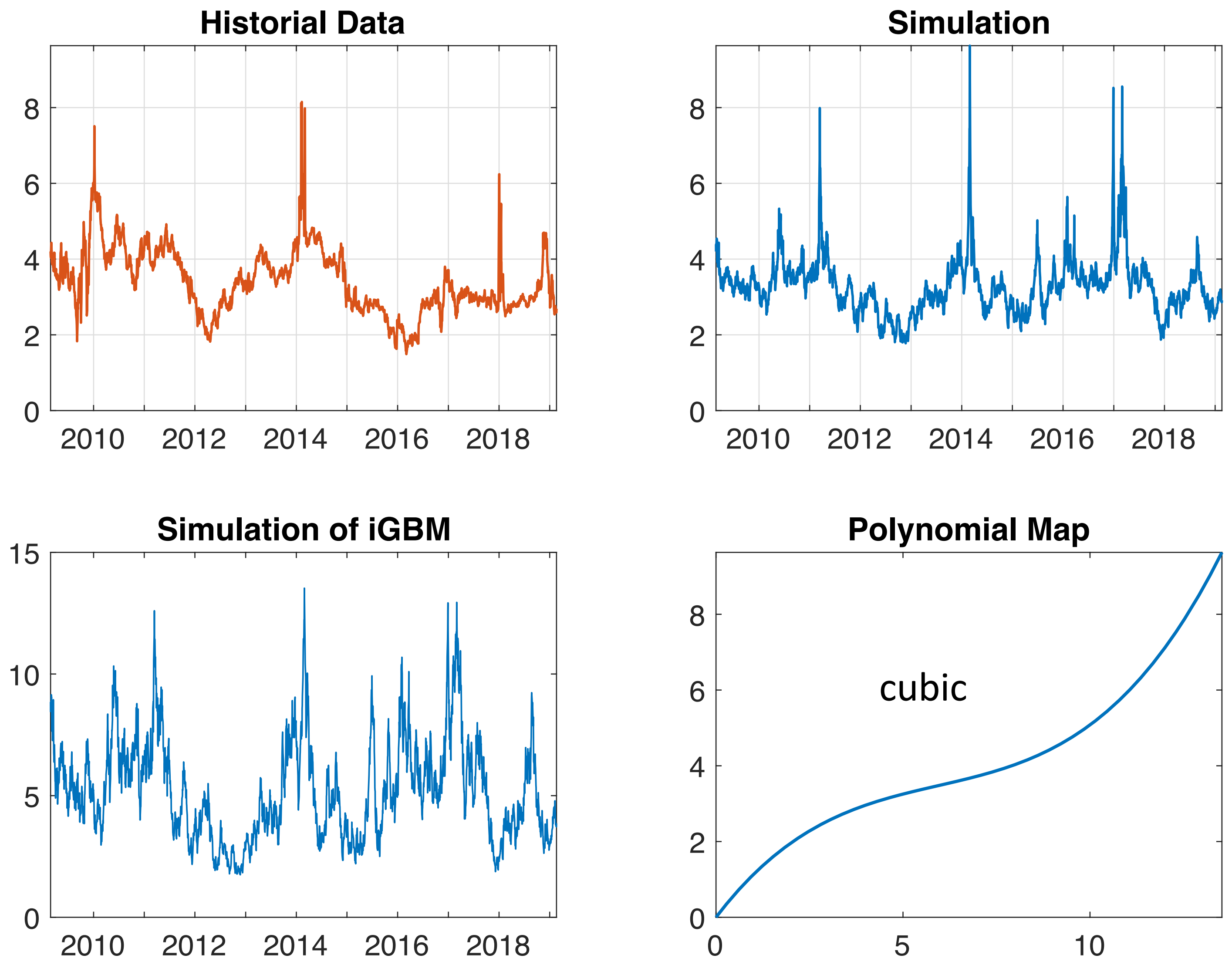
$$X^* = X_n + h\kappa(\theta - X^*)$$

$$X_{n+1} = X^* + \sigma X^* \sqrt{h} Z_n,$$

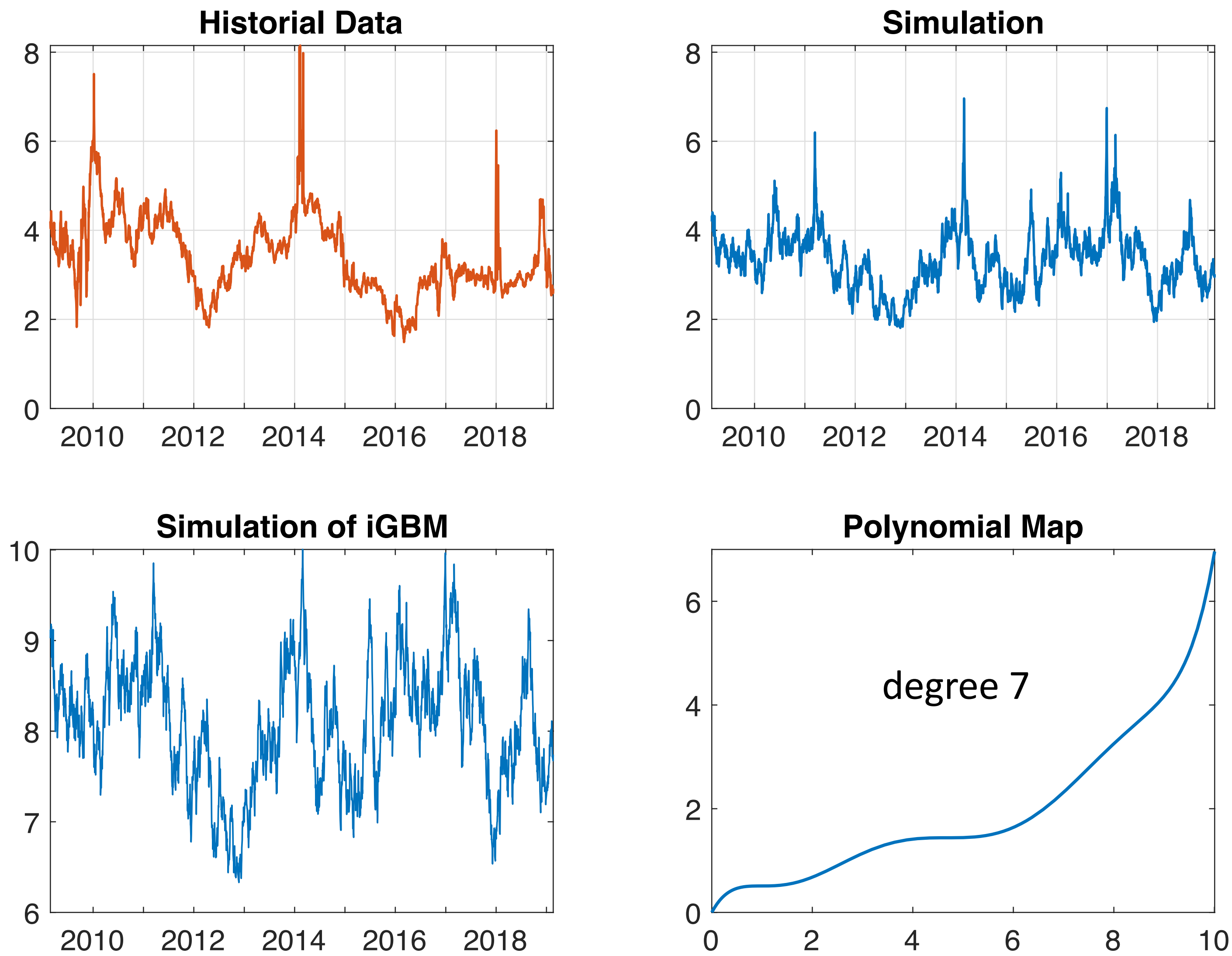
with $Z_n \sim N(0,1)$.
This has the property that it is a *discrete polynomial process*.

To estimate the model—including the polynomial map Ψ—we used MLE, and we used the discrete process to define the conditional transition likelihoods.

Example: IGBM and natural gas



Example: IGBM and natural gas



Alberta power

Alberta power

- Power prices in Alberta must stay between \$0 and \$1000 per MWh, and so we need to have a process that lives in this interval.
- One possible approach (taken by Carmon, Fehr and Hinz, 2009, in the context of emissions prices) is to use a normal CDF to map between the real line and the bounded interval.
- But we want something that will – naturally – allow us to express futures prices (almost) explicitly, and so we use a polynomial process that lives on a bounded interval – the ***Jacobi process***.

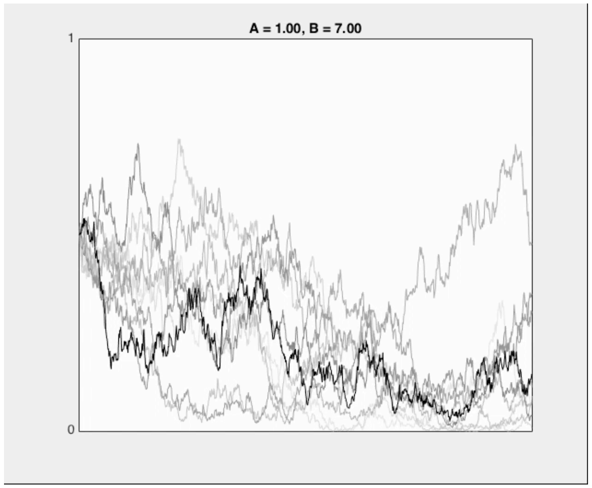
Jacobi process

The characteristics of the Jacobi process

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t(1-X_t)}dW_t$$

are determined by the dimensionless quantities

$$A = \frac{2\kappa\theta}{\sigma^2} \text{ and } B = \frac{2\kappa(1-\theta)}{\sigma^2}.$$



In terms of these quantities, the SDE takes the form

$$dX_t = \frac{\sigma^2}{2}\big(A(1-X_t) - BX_t\big)dt + \sigma\sqrt{X_t(1-X_t)}dW_t,$$

and, as already noted, $X_t \in (0,1)$ almost surely if $\min\{A, B\} \geq 1$.

Jacobi process: some facts

Conditional on $X_{t_0} = x_0$, the density of X_t is given by

$$p(x,t;x_0,t_0) = \sum_{n=0}^{\infty} k_n \psi_n(x_0)\psi_n(x)w(x)\mathrm{e}^{-\lambda_n(t-t_0)},$$

where

$$\lambda_n = \frac{\sigma^2}{2}(A+B-1+n)n,$$

$$k_n = \frac{(A+B-1+2n)(A)_n(A+B)_n}{n!(A+B-1+n)(B)_n},$$

$$w(x) = \frac{\Gamma(A+B)}{\Gamma(A)\Gamma(B)}x^{A-1}(1-x)^{B-1},$$

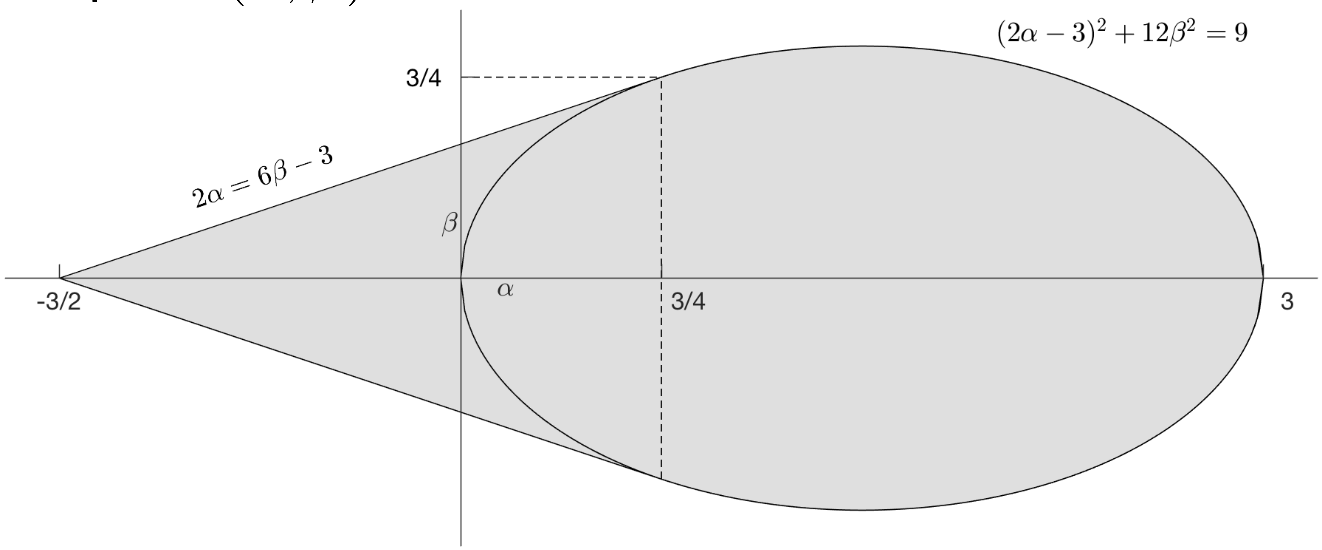
$$\psi_n(x) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(A+B-1+n)_k}{B_k} x^k,$$

and we have used the notation $(\cdot)_k := \Gamma(\cdot + k)/\Gamma(\cdot)$.

Jacobi process: polynomial maps

- Look for a polynomial ψ that is non-negative on $[0,1]$, and set $\Psi(x) = \dfrac{\int_0^x \psi(y)dy}{\int_0^1 \psi(y)dy}$, so that Ψ is increasing and onto as a map from $[0,1] \to [0,1]$.

- We construct ψ as a product of quadratic factors of the form $\psi(x) = q_{\alpha,\beta}(2x-1)$, where $q_{\alpha,\beta}(x) = \alpha x^2 + 2\beta x + 1 - \frac{1}{3}\alpha$, and the point (α, β) lies within the shaded area below.



Spot model estimation

- Given the freedom to choose the form of the polynomial map, we seek to determine both the parameters of the Jacobi process and the optimal polynomial map Ψ together, using MLE and Hamilton-style filtering.

- The maximum likelihood used initial parameter estimates generated from optimal lower-degree models, exploiting the nested construction of the polynomial maps:

- The map Ψ depends on a sequence of polynomial parameters

$$(\beta_1, \alpha_1, \beta_2, \alpha_2, \ldots),$$

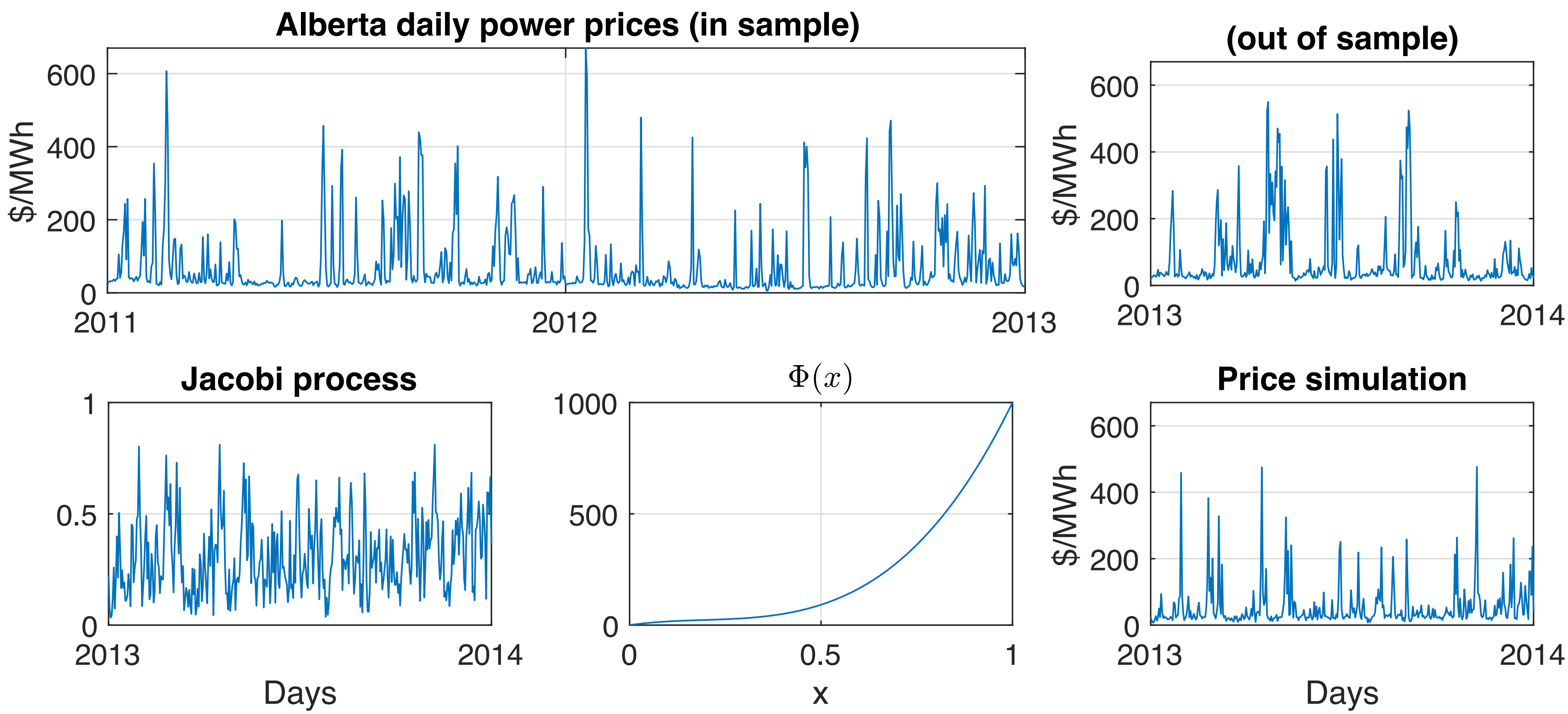
  with the degree being the number of parameters plus two.

- An initial estimate for a model can be formed from the lower-degree model by adding a zero to the polynomial parameters.

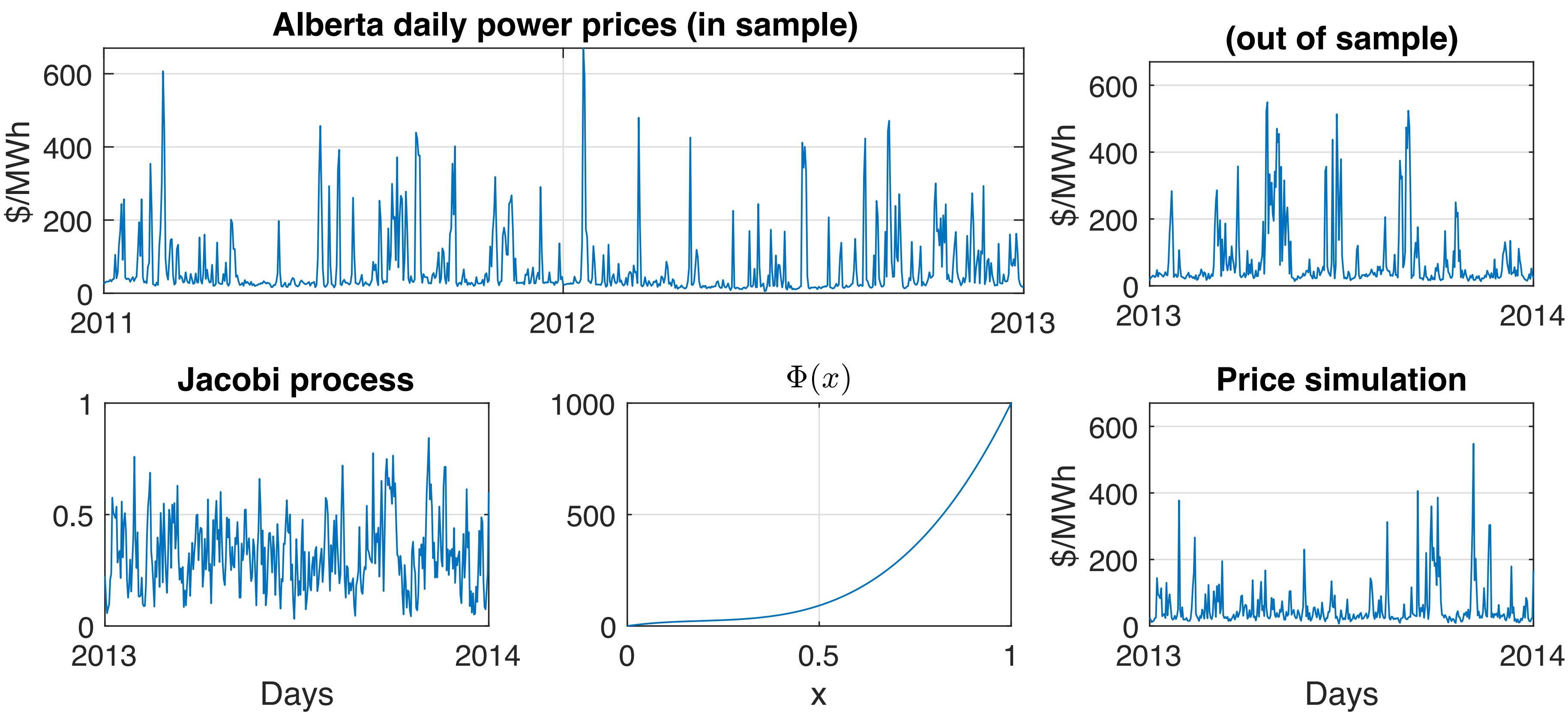
Spot model: estimation

deg Ψ	1	2	3	4	5	6	7	8
A	1.00	1.61	1.70	2.12	1.95	1.86	1.71	2.48
B	8.62	6.17	3.05	2.84	1.98	2.21	2.83	3.05
σ	11.93	10.66	10.55	10.03	10.48	10.22	9.49	8.70
c_0	0.50	0.40	0.30	0.26	0.21	0.24	0.28	0.26
c_1	0.50	0.50	0.44	0.42	0.36	0.39	0.43	0.41
c_2		0.10	0.20	0.22	0.23	0.24	0.21	0.22
c_3			0.06	0.08	0.13	0.11	0.06	0.08
c_4				0.01	0.05	0.03	0.00	0.00
c_5					0.01	0.00	0.01	-0.00
c_6						-0.00	0.01	0.01
c_7							0.01	0.01
c_8								0.00
LL	1870.30	1925.34	2148.11	2154.10	2186.09	2186.38	2188.26	2190.50
BIC	-3720.81	-3824.29	-4263.24	-4268.63	-4326.02	-4319.99	-4317.16	-4315.03
OS LL	909.88	944.40	1060.96	1062.46	1062.64	1062.94	1063.87	1064.60
OS BIC	-1802.06	-1865.21	-2092.42	-2089.53	-2083.97	-2078.69	-2074.65	-2070.19

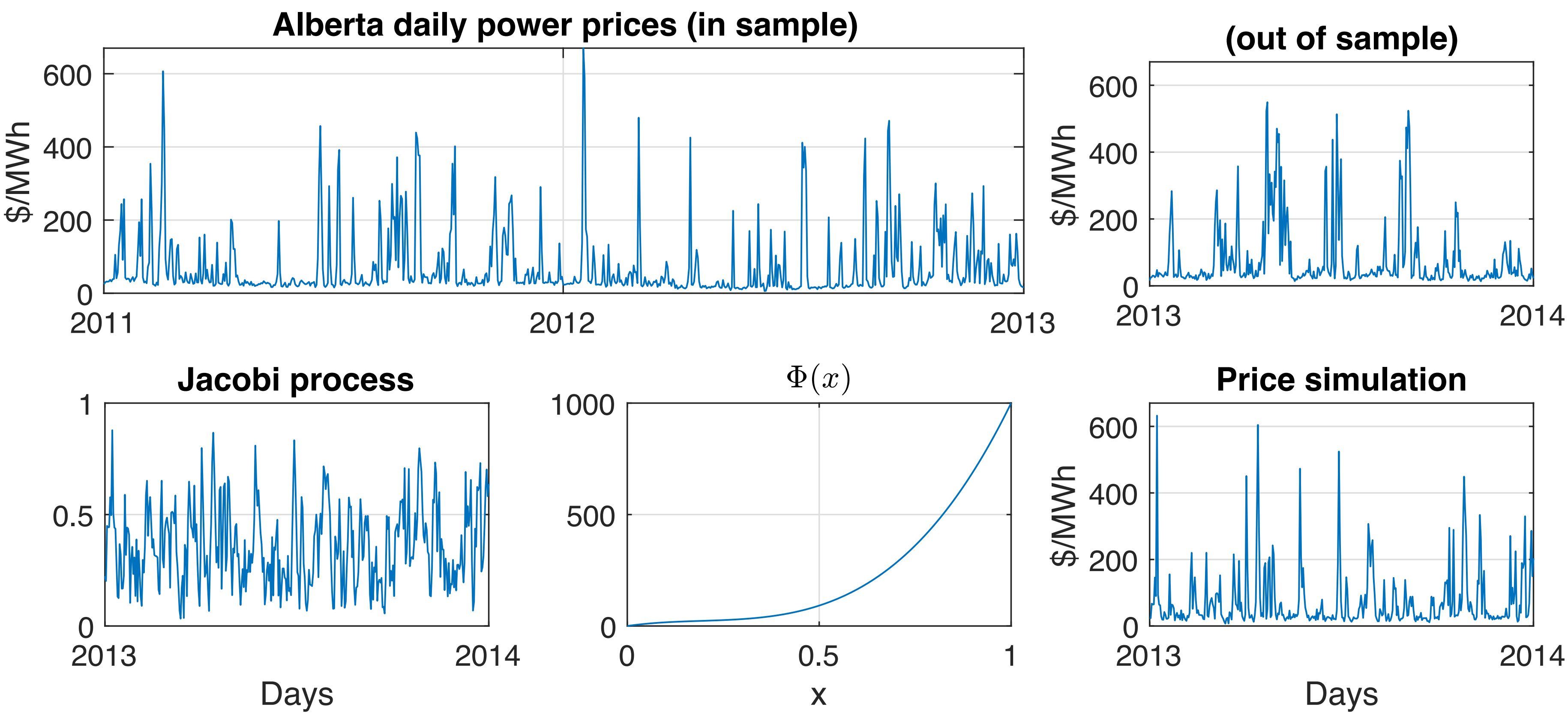
Spot model: results



Spot model: results



Spot model: results



Multi-factor models

Several multifactor models on the interval are possible. For example,

- Feedback from X_t into the drift of Y_t:

$$dX_t = (b_1 + B_{11}X_t + B_{12}Y_t)dt + \sigma\sqrt{X_t(1 - X_t)}dW_{1t}$$
$$dY_t = (b_2 + B_{21}X_t + B_{22}Y_t)dt + \rho\sqrt{Y_t(1 - Y_t)}dW_{2t}.$$

- The range of X_t depending on Y_t:

$$dX_t = (b_1 + B_{11}X_t + B_{12}Y_t)dt + \sigma\sqrt{X_t(\mu + \nu Y_t - X_t)}dW_{1t}$$
$$dY_t = (b_2 + B_{21}X_t + B_{22}Y_t)dt + \rho\sqrt{Y_t(1 - Y_t)}dW_{2t}.$$

for suitable parameters $\mu \geq 0$ and $\nu \geq 0$. Here X_t takes values in $[0, \mu + \nu Y_t]$ and Y_t takes values in $[0, 1]$.

Multi-factor models

If we have a polynomial process on the unit simplex:

$$Z_t \in \left\{[0,1]^N | Z_{1t} + \ldots Z_{Nt} = 1\right\} \quad \text{(c.f. Filipović, Larsson 2016)},$$

we can construct a finite number of (possibly time-dependent) polynomial maps Ψ_n for $n = 1, \ldots, N$, with coefficients $\mathbf{p}_n(t)$ and set

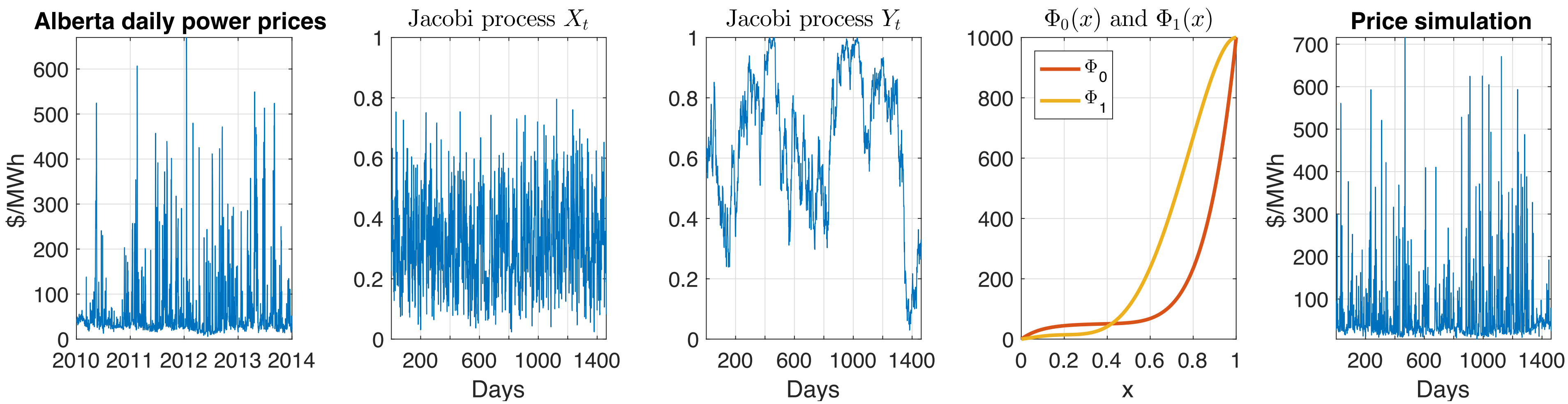
$$S_t = H(X_t) \sum_n Z_{nt} \mathbf{p}_n(t).$$

For a two-factor model, we can take $Z_t \in [0,1]$ to also be a Jacobi process. In this case we write

$$S_t = H(X_t)\big[(1 - Z_t)\mathbf{p}_0(t) + Z_t \mathbf{p}_1(t)\big],$$

and now we are dealing with a continuum of potential polynomial maps, indexed by Z_t at any given moment.

Multi-factor models: spot estimation



deg Ψ	2	3	4	5	6
A_X	2.54	2.52	2.83	3.10	3.03
B_X	10.72	4.33	4.03	5.54	5.23
σ_X	5.70	7.16	7.09	6.38	6.53
A_Y	1.03	7.27	10.40	2.04	2.20
B_Y	4.08	2.72	4.34	1.22	1.36
σ_Y	0.95	1.06	1.02	1.38	1.36
c_1^0	0.81	1.00	1.00	0.72	0.76
c_2^0		0.79	0.81	0.89	0.89
c_3^0			0.05	0.77	0.77
c_4^0				0.61	0.57
c_5^0					-0.08
c_1^1	1.00	0.99	0.97	0.99	0.99
c_2^1		0.61	0.66	0.66	0.66
c_3^1			0.71	-1.00	-1.00
c_4^1				-0.79	-0.59
c_5^1					0.34
LL	2915.7	3633.5	3637.1	3648.8	3650.5
BIC	-5773.1	-7194.1	-7186.7	-7195.7	-7184.3

Optimal parameters, log-likelihoods and Bayesian Information Criterion (BIC) scores

Futures prices

Changes of measure for Jacobi processes

Under the assumption that $\min\{A, B\} \geq 1$, a pricing measure $\mathbb{Q}$ can be specified via the market price of risk

$$\chi_t = \frac{\nu_A X_t - \nu_B(1 - X_t)}{\sqrt{X_t(1 - X_t)}},$$

where ν_A and ν_B are constants satisfying $\nu_A \geq -\frac{\sigma}{2}(A - 1)$ and $\nu_B \geq -\frac{\sigma}{2}(B - 1)$. The factor dynamics become

$$dX_t = \frac{\sigma^2}{2}\big(A_{\mathbb{Q}}(1 - X_t) - B_{\mathbb{Q}} X_t\big)dt + \sigma\sqrt{X_t(1 - X_t)}\, dW_t^{\mathbb{Q}},$$

where $A_{\mathbb{Q}} = A + 2\nu_A/\sigma$ and $B_{\mathbb{Q}} = B + 2\nu_B/\sigma$.

Futures prices

- If $G_{\mathbb{Q}}^X$ denotes the matrix representation of the infinitesimal generator of the process followed by X_t in the basis of Jacobi polynomials with parameters $A_{\mathbb{Q}}$ and $B_{\mathbb{Q}}$ (and similarly for Y_t), then the corresponding matrix for the joint process is

$$G_{\mathbb{Q}} = G_{\mathbb{Q}}^X \otimes G_{\mathbb{Q}}^Y.$$

- Similarly, we can express the joint basis of Jacobi polynomials as

$$\mathbf{H}(x, y) = \mathbf{H}^X(x) \otimes \mathbf{H}^Y(y).$$

- Then futures prices (given X_t, Y_t) are given by

$$F(t, T) = \mathbf{H}(X_t, Y_t)\, \mathrm{e}^{(T-t)G_{\mathbb{Q}}}\, \mathbf{p},$$

where the coefficients $\mathbf{p}$ represent the map Ψ in this basis.

Futures prices

- Unless we observe (X_t, Y_t) directly, we need to compute the expected value of this conditional on $\mathcal{F}_t$. **The density needed for this is directly available from the filtering procedure.**

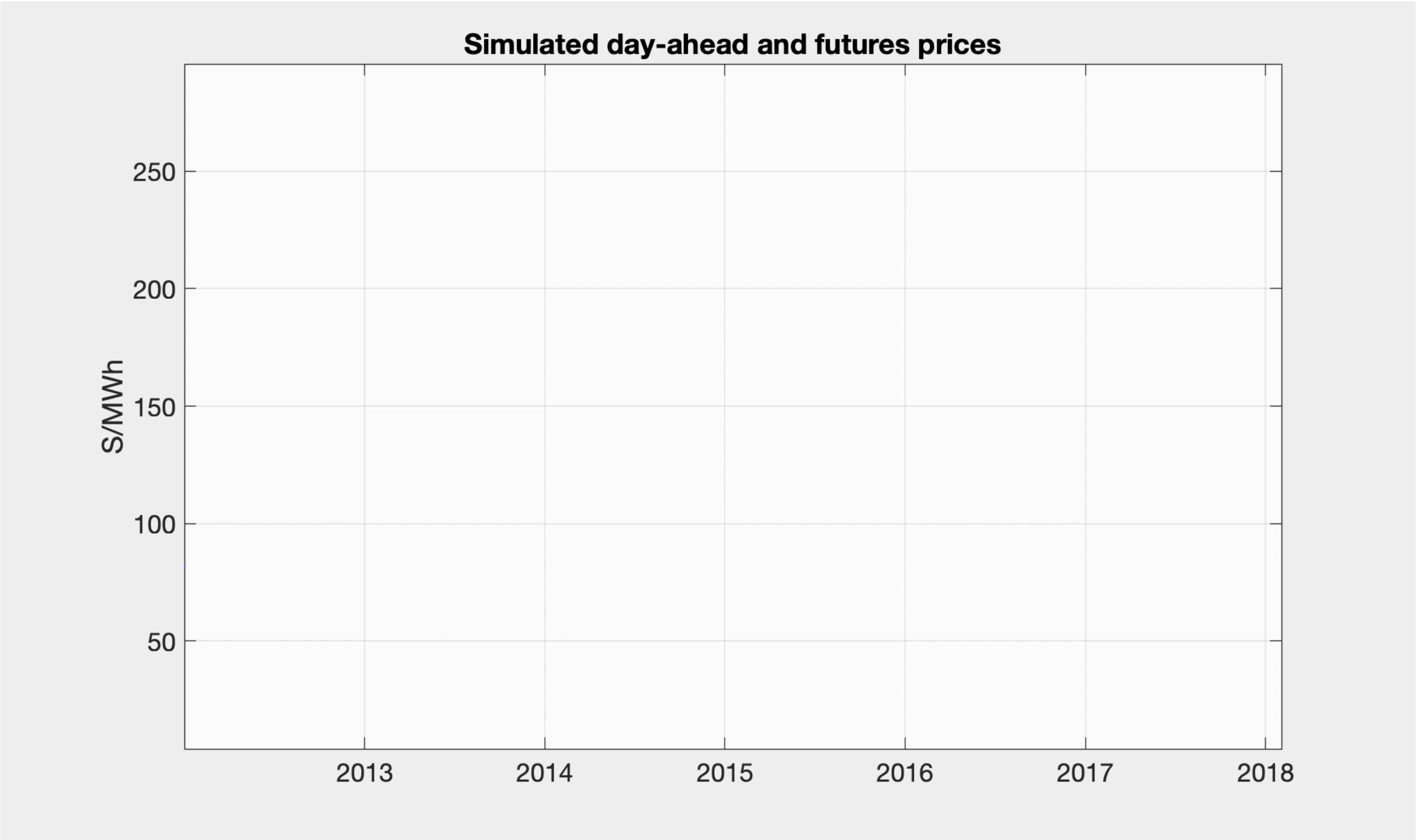
- There is also the problem of *seasonality*. In these computations we used a seasonally-varying market price of risk to calibrate to market futures prices.

- This results in the formula

$$F(t_0, t_N) = \mathbf{q}_0^T \, \mathrm{e}^{h_1 G_{\mathbb{Q}}^1} \, I_1^{-1} I_2 \, \mathrm{e}^{h_2 G_{\mathbb{Q}}^2} \ldots I_{N-1}^{-1} I_N \, \mathrm{e}^{h_N G_{\mathbb{Q}}^N} \, \mathbf{p},$$

where $\mathbf{q}_0 = \mathbb{E}[\mathbf{H}(X_0, Y_0)|\mathcal{F}_0]$, $h_n = t_n - t_{n-1}$, the t_n denote times where the prices of risk change, and the matrices $I_{n-1}^{-1} I_n$ are used to implement the changes of basis.

Futures prices: model simulation



Valuation of other derivatives

- The formula

$$F(t_0, t_N) = \left[\mathbf{q}_0^T\, \mathrm{e}^{h_1 G_{\mathbb{Q}}^1}\, I_1^{-1} I_2\, \mathrm{e}^{h_2 G_{\mathbb{Q}}^2} \ldots I_{N-1}^{-1} I_N\, \mathrm{e}^{h_N G_{\mathbb{Q}}^N}\right] \mathbf{p},$$

  is the basis for valuation of other cash flows.

- To value a payoff $\Lambda(S_T) = \Lambda(\Psi(X_T, Y_T))$, we must find the coefficients of the vector $\mathbf{p}$ that represent the approximation of this in the polynomial basis at time $T = t_N$.

- In practice, the vector multiplying $\mathbf{p}$ is **rapidly decaying**, and so only a few coefficients need to be calculated (but the need to be calculated accurately).

- This is entirely analogous to (for example) the COS method of Fang and Oosterlee.

Thank for your attention!

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Selected references

- ✓ C. Cuchiero, M. Keller-Ressel, and J. Teichmann. *Polynomial processes and their applications to mathematical finance*. Finance and Stochastics, 2012.
- ✓ D. Filipović and M. Larsson. *Polynomial diffusions and applications in finance*. Finance and Stochastics, 2016.
- ✓ L. Chen, D. Filipović, and H.V. Poor. *Quadratic term structure models for risk-free and defaultable rates*. Math. Finance, 2004.
- ✓ J. L. Forman and M. Sørensen. *The Pearson diffusions: A class of statistically tractable diffusion processes.* Scandinavian Journal of Statistics, 2008.
- ✓ F. Fang and C. W. Oosterlee. A novel pricing method for European options based on Fourier-cosine series expansions. SIAM Journal on Scientific Computing, 2008.