# Braid Index Bounds Ropelength From Below

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$\ell(K)$: the minimum length over all possible $\mathcal{K}^c$.

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• (Diao, Ernst and Thistlethwaite) The three-fourth power law does not hold as the upper bound of ropelengths in general. In fact, there exists many families of knots (each containing infinitely many prime knots) with the property that $L(\mathcal{K}_n) = O(Cr(\mathcal{K}_n))$ for $\mathcal{K}_n$ from any of these families.

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• Conjecture $(*)$: If $\mathcal{K}$ is alternating, then $L(\mathcal{K}) \geq O(Cr(\mathcal{K}))$.

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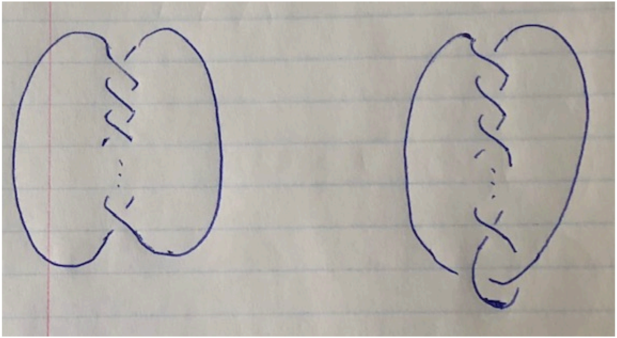

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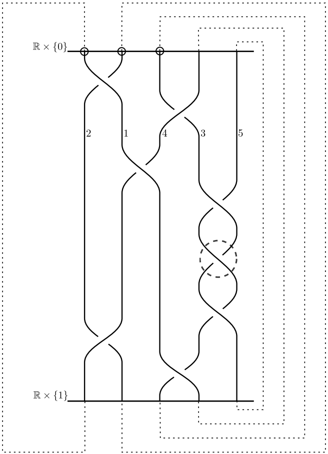

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• (New result!) $a\mathbf{b}(\mathcal{K}) \leq L(\mathcal{K})$ for some constant $a > 0$! (In fact $a \geq 1/14$).

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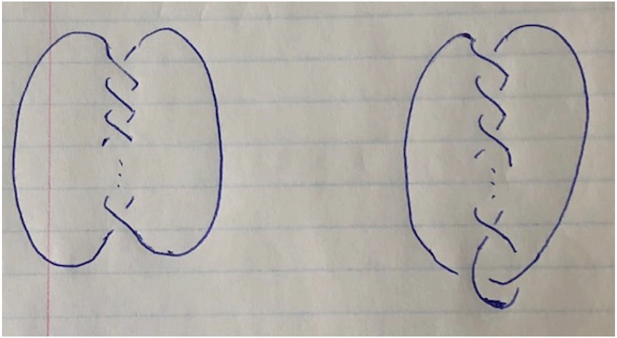

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Example 1. If $\mathcal{K}$ is the $(2, 2n)$ torus link whose components are assigned opposite orientations then $Cr(\mathcal{K}) = 2n$ and $\mathbf{b}(\mathcal{K}) = n + 1$ so $L(\mathcal{K}) > Cr(\mathcal{K})/28$.

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Example 2. If $\mathcal{K}$ is a twist knot with $Cr(\mathcal{K}) = n \geq 4$ crossings, then $\mathbf{b}(\mathcal{K}) = (n+1)/2$ if $n$ is odd, and $\mathbf{b}(\mathcal{K}) = n/2 + 1$ if $n$ is even ($\mathbf{b}(\mathcal{K}) > Cr(\mathcal{K})/2$ in both cases) hence $L(\mathcal{K}) > Cr(\mathcal{K})/28$ as well.

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This conclude s the introductory part of the proof.

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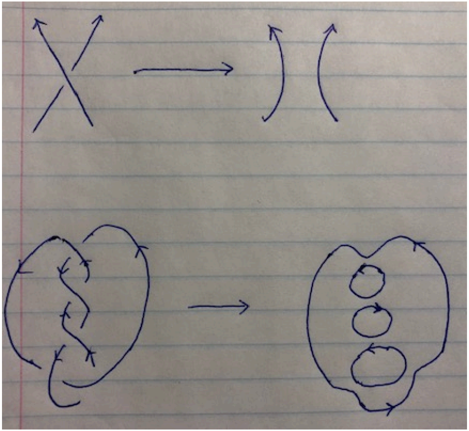
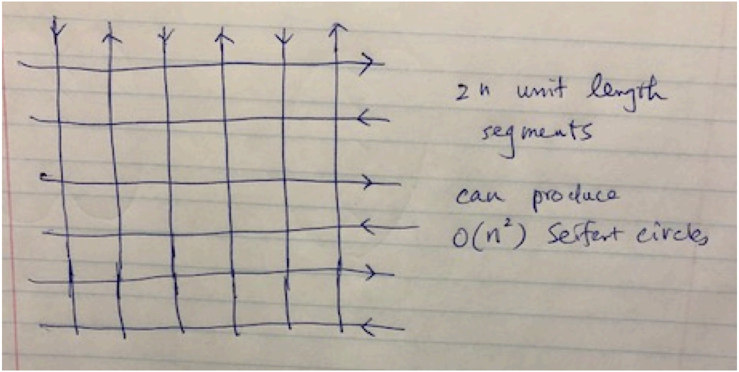
• Theorem (Yamada) $\mathbf{b}(\mathcal{K})$ equals the minimum number of Seifert circles over all possible projections of $\mathcal{K}$.



So if a lattice length minimizer $\mathcal{K}^c$ is such that $\ell(\mathcal{K}^c) \geq s(\mathcal{K}^c)$ where $s(\mathcal{K}^c)$ is the number of Seifert circles in a projection of $\mathcal{K}^c$, then the result would follow trivially since $s(\mathcal{K}^c) \geq \mathbf{b}(\mathcal{K})$.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