Limitations on the expressive power of convex cones without long chains of faces

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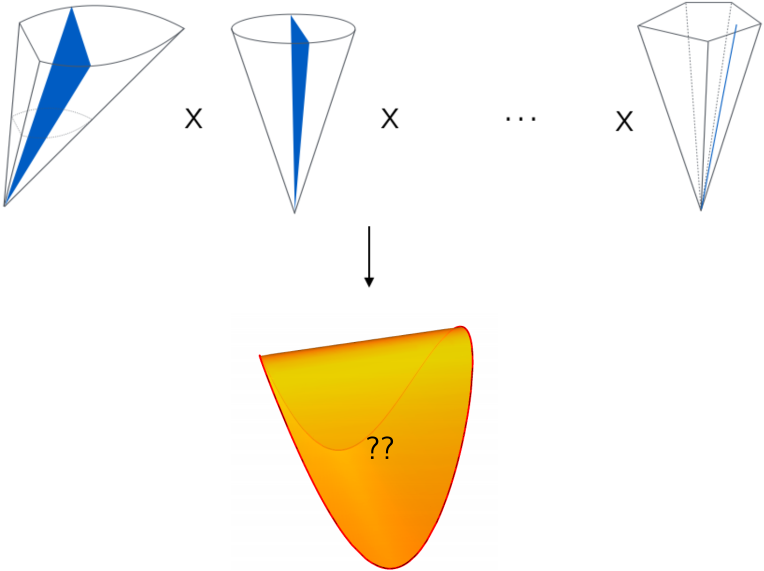
x
x
...
x
??

Conic optimization

$$\text{minimize}_x \ \ \langle c, x\rangle \quad \text{subject to} \quad x \in K \cap L$$

where L is an affine subspace, K a convex cone

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Linear programming: $K = \mathbb{R}^m_+ =$ nonnegative orthant

Second-order cone programming:

$$K = \mathcal{Q}^m \quad \text{where} \quad \mathcal{Q} = \{(x, y, z) \ : \ \sqrt{x^2 + y^2} \leq z\}$$

Semidefinite programming:

$$K = \mathcal{S}^d_+ = d \times d \text{ positive semidefinite matrices}$$

Beyond: relative entropy cone, hyperbolicity cones, power cones, cones of nonnegative polynomials,...

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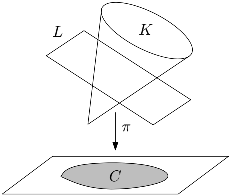
What are the relationships between different families?

Lifts of convex sets

Definition: A convex set C has a K-lift if there is a subspace L and linear map π such that

$$C = \pi(K \cap L)$$



If C has a K-lift then conic programs over C can be reformulated as conic programs over K.

Finite Cartesian products

$$K = K_1 \times K_2 \times \cdots \times K_m$$

Some benefits:

- Membership, separation, projection, etc. are separable
- Easier to exploit sparsity

Examples:

- Anything with SOCP representation has a $(\mathcal{S}_+^2)^m$-lift
    - SDSOS, SONC, power cone, etc.
- SAGE cone has a K_{ent}^m-lift where

$$K_{\mathrm{ent}} = \mathrm{cl}\{(x, y, z) \; : \; x, z > 0, \; z \log(z/x) \leq y\}$$

- G chordal graph with maximal cliques of size $k_1, \ldots, k_m$:

$$\{\text{PSD and sparse w.r.t. } G\} \;\text{ has }\; \mathcal{S}_+^{k_1} \times \cdots \times \mathcal{S}_+^{k_m}\text{-lift}$$

This talk

What are obstructions to representability with products of ‘low-complexity’ cones?

Outline:

- What does ‘low-complexity’ mean?
- An obstruction: infinite k-neighborly
- Ingredients of proof

Related work

If proper convex cone C ...

- has infinitely many extreme rays

then C has no...

- $\mathbb{R}^m_+$-lift (follows from Fourier-Motzkin)

Related work

If proper convex cone C ...

- has infinitely many extreme rays
- is infinite 2-neighborly

then C has no...

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- $(\mathcal{S}_+^2)^m$-lift (Fawzi 2018)

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If proper convex cone C ...

- has infinitely many extreme rays
- is infinite 2-neighborly
- is infinite k-neighborly

then C has no...

- $\mathbb{R}_+^m$-lift (follows from Fourier-Motzkin)
- $(\mathcal{S}_+^2)^m$-lift (Fawzi 2018)
- $(\mathcal{S}_+^k)^m$-lift (Averkov 2019)

Infinite k-neighborly

Proper convex cone C is

k-neighborly w.r.t. subset V of extreme rays if

- for each k element subset $S \subset V$
- there exists a linear functional ℓ_S such that
  - $\ell_S(x) \geq 0$ for all $x \in C$
  - $\ell_S(x) = 0$ for $x \in S$ and
  - $\ell_S(x) > 0$ for $x \in V \setminus S$

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Special case: k-neighborly polyhedral cone

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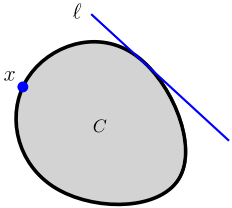
C infinite k-neighborly if k-neigborly w.r.t. infinite set V

Examples



Inifinite 1-neighborly: infinitely many (exposed) extreme points

PSD cone: $\mathcal{S}_+^{k+1}$ infinite k-neighborly with

- $V = \{v_i v_i^T \ : \ v_i = \begin{bmatrix} 1 & i & i^2 & \cdots & i^k \end{bmatrix}^T, \ \ i \in \mathbb{N}\}$
- $\ell_S(v_t v_t^T) = \prod_{i \in S}^k (t-i)^2$

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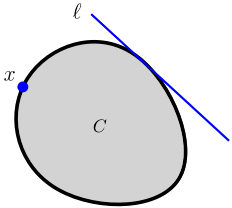

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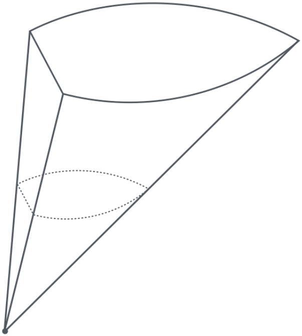
Averkov (2019): If $X \subseteq \mathbb{R}^n$ has non-empty interior and

$$\mathrm{PSD}_{n,2d}(X)^* \subseteq C \subseteq \mathrm{SOS}_{n,2d}^*$$

then C is infinite $\left(\binom{n+d}{d} - 1\right)$-neighborly.

Chains of faces



Chain of faces:

$$\{0\} \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_{\ell-1}$$

$$\boxed{\ell(K) = \text{length of longest chain of non-empty faces of } K}$$

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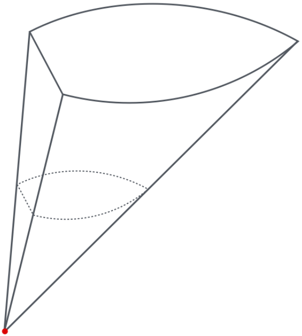

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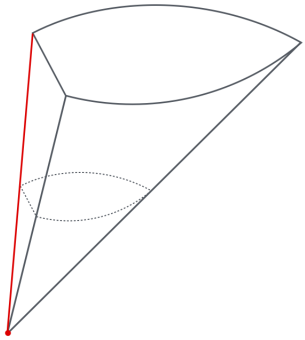

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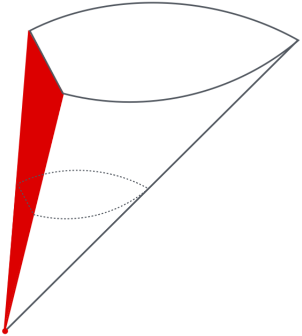

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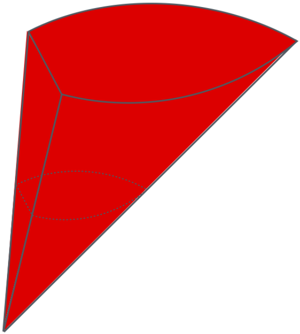

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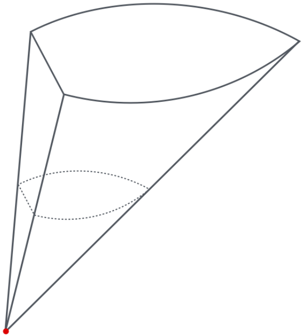

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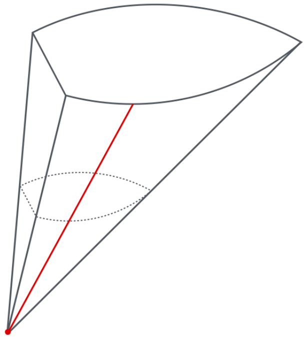

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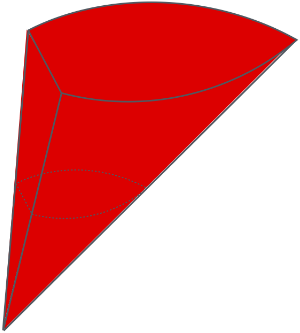

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Examples

k-dimensional convex cone: $\ell(K) \leq k + 1$

Halfspace: $\ell(K) \leq 2$

Smooth cone: $\ell(K) \leq 3$

$k \times k$ PSD cone: $\ell(K) \leq k + 1$

Hyperbolicity cone: $\ell(K) \leq \deg(p) + 1$

Main result

Theorem (S. 2019)
If C is infinite k-neighborly proper convex cone then

$$C \text{ does not have a } K_1 \times \cdots \times K_m\text{-lift}$$

whenever

- m is positive integer and
- $\ell(K_i) \leq k + 1$ for $i = 1, 2, \ldots, m$

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Special cases:

- infinite 1-neighborly implies no polyhedral lift
- infinite 2-neighborly implies no $(\mathcal{S}_+^2)^m$-lift (Fawzi)
- infinite k-neighborly implies no $(\mathcal{S}_+^k)^m$-lift (Averkov)

Corollary: Infinite k-neighborly $\implies$
no lift using hyperbolicity cone where all irreducible components of p have degree at most k

Origin of proof

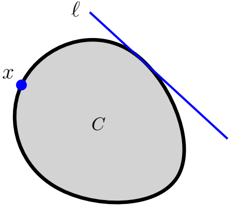
- Modify Averkov's proof for ruling out $(\mathcal{S}_+^k)^m$-lifts
- Essentially: 'rank' $\mapsto$ 'length of longest chain of faces'

$\boxed{\text{Main contribution: algebraic} \mapsto \text{convex geometric}}$

Slack matrix

Associate slack matrix with convex cone C

$$S_{\ell,x} = \ell(x)$$



where

- ℓ linear functional non-negative on C
- x an element of C

The slack matrix is entry-wise nonnegative.

cone C is k-neighborly w.r.t. V

$$\longleftrightarrow$$

$\binom{V}{k} \times V$ submatrix of slack with certain zero/non-zero pattern

Factorization theorem

Special case of Gouveia-Parrilo-Thomas (2013)

If C has a proper $K_1 \times K_2 \times \cdots \times K_m$-lift then

$$S_{\ell,x} = \langle b_1(\ell), a_1(x)\rangle + \langle b_2(\ell), a_2(x)\rangle + \cdots + \langle b_m(\ell), a_m(x)\rangle$$

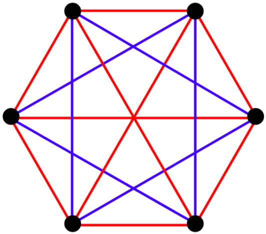
where $a_i(x) \in K_i^*,\ b_i(\ell) \in K_i$ for all $i = 1, 2, \ldots, m$

Obstructions to factorization $\implies$ obstructions to lifts

Ramsey's theorem for (hyper)graphs

Ramsey (1930): There is a positive integer $R_2(3; c)$ such that

> If $n \geq R_2(3; c)$ then any coloring of the edges of the complete graph on n vertices with c colors has a monochromatic triangle.



Ramsey (1930) also extends to complete uniform hypergraphs

Outline of argument

Suppose infinite k-neighborly C has $K_1 \times K_2 \times \cdots \times K_m$-lift

- Choose finite $V' \subset V$ with $|V'| \geq R_k(k+1; (k+1)^m)$

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Since V is infinite, can do this for any finite m.

Conclusion

Expressivity of finite products of 'low-complexity' cones?

Main technical conclusion:

- infinite k-neighborly is obstruction
  to having $K_1 \times \cdots \times K_m$-lifts where
  each K_i only has chains of faces of length at most $k+1$

Questions:

- Quantitative results?
- Other limitations on lifts using
  hyperbolicity cones (beyond quantifier elimination)

For more information

Preprint

J. Saunderson, 'Limitations on the expressive power of convex cones without long chains of faces',
`https://arXiv.org/abs/1902.06401`

Fawzi's paper

H. Fawzi, 'On representing the positive semidefinite cone using the second-order cone', Mathematical Programming, 2018

Averkov's paper

G. Averkov, 'Optimal size of linear matrix inequalities in semidefinite approaches to polynomial optimization' SIAM Journal on Applied Algebra and Geometry, 2019

Thank you!