On Sum of Squares Representation of Convex Forms and Generalized Cauchy-Schwarz Inequalities

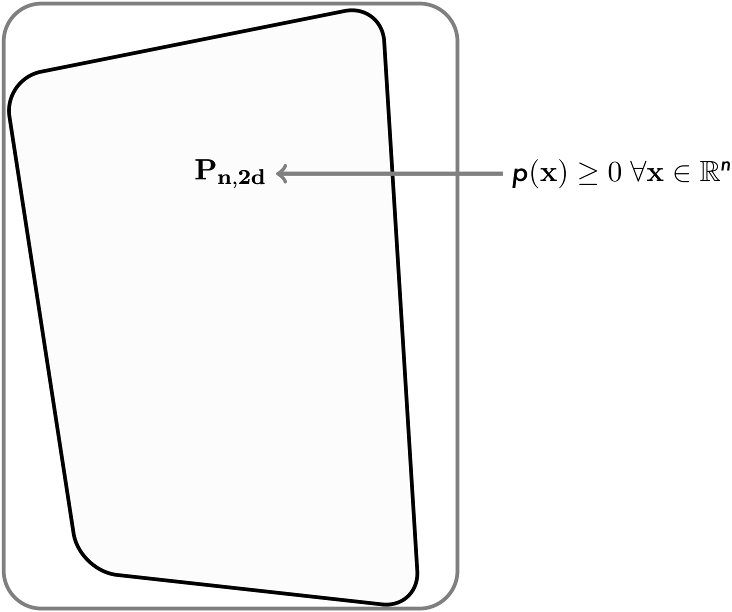
Bachir El Khadir

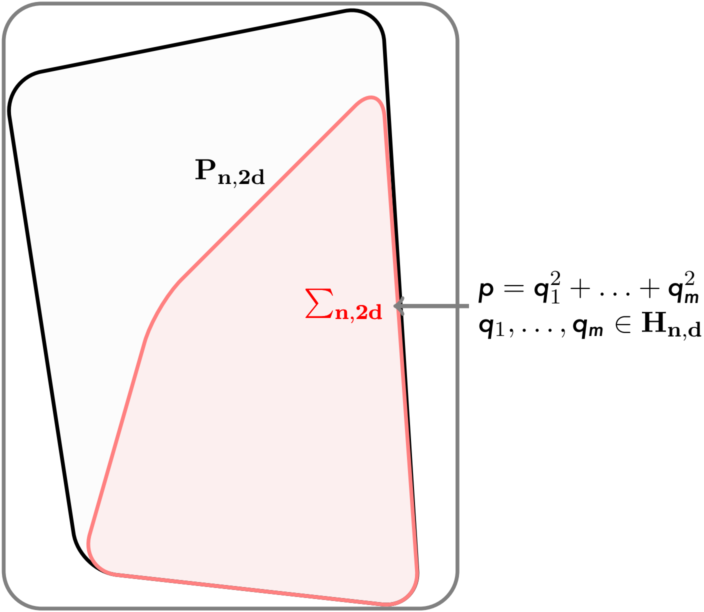
"Geometry of Real Polynomials, Convexity and Optimization"
workshop in Banff, May 28, 2019

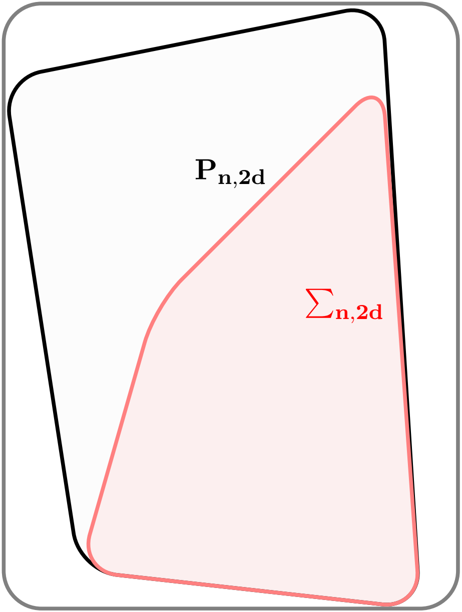
$$\mathbf{H_{n,2d}}$$

Forms of degree $2d$ in n variables

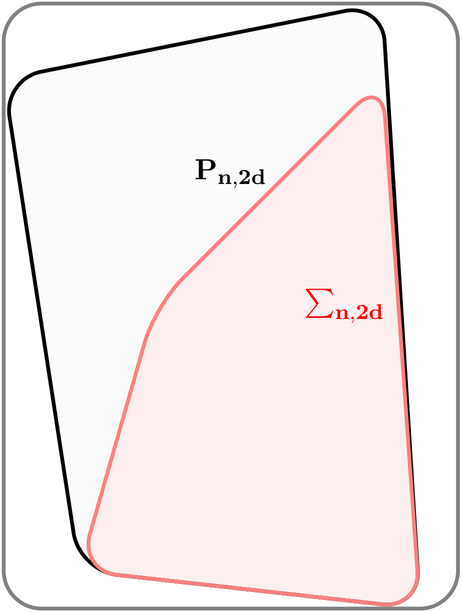
$$\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$$


$\mathbf{P_{n,2d}}$
$p(\mathbf{x}) \geq 0 \ \forall \mathbf{x} \in \mathbb{R}^n$


$\mathbf{P_{n,2d}}$
$\mathbf{\Sigma_{n,2d}}$
$p = q_1^2 + \ldots + q_m^2$
$q_1, \ldots, q_m \in \mathbf{H_{n,d}}$



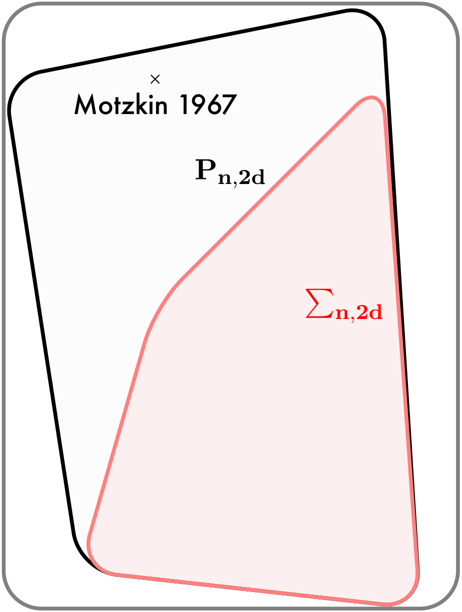
$$\mathbf{P_{n,2d}} = \Sigma_{\mathbf{n,2d}}?$$



$\mathbf{P_{n,2d}} = \Sigma_{\mathbf{n,2d}}$?

n \ 2d	2	4	≥ 6
1	✓	✓	✓
2	✓	✓	✓
3	✓	✓	✗
≥ 4	✓	✗	✗

[Hilbert 1888]



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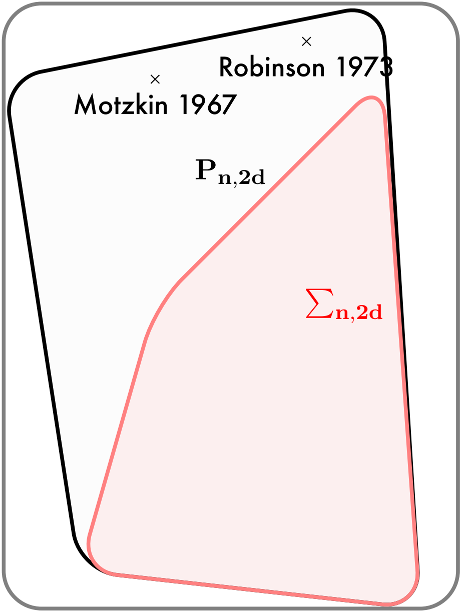

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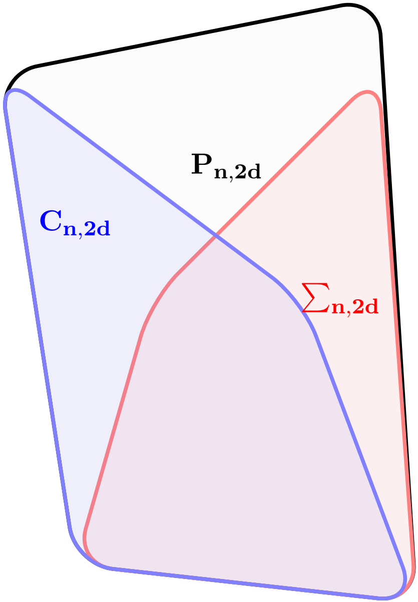
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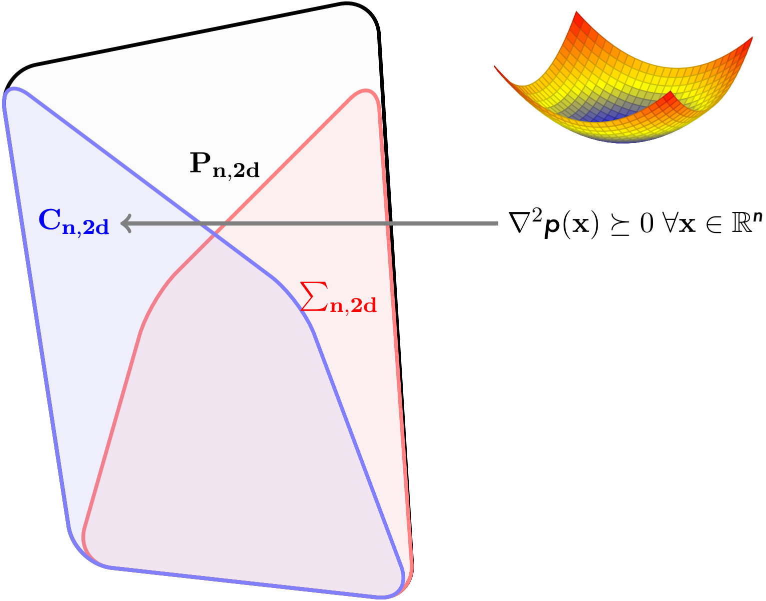
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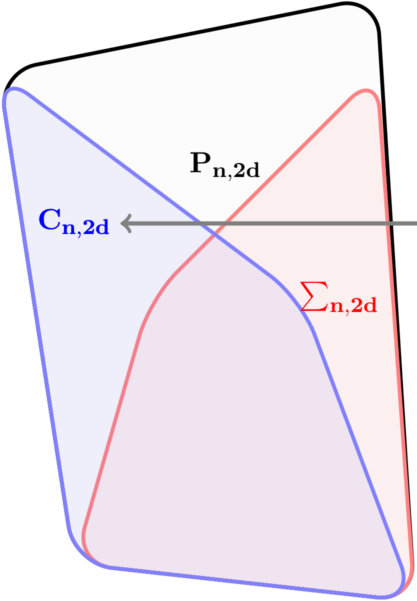
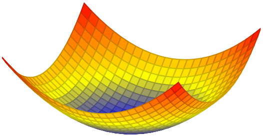
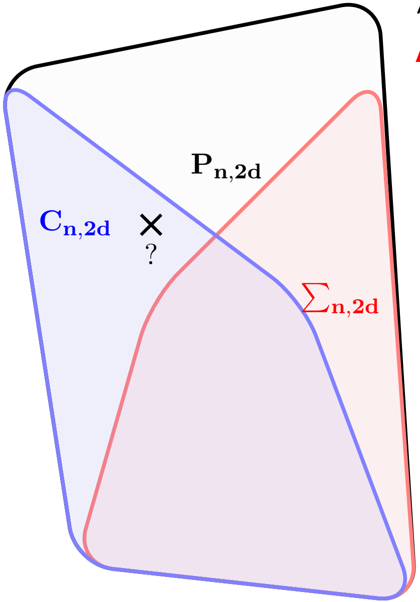




$\nabla^2 p(\mathbf{x}) \succeq 0 \ \forall \mathbf{x} \in \mathbb{R}^n$

Convex $\implies$ Nonnegative?

Yes! Euler:
$2d(2d-1)p(\mathbf{x}) = \mathbf{x}^T \nabla^2 p(\mathbf{x}) \mathbf{x}$



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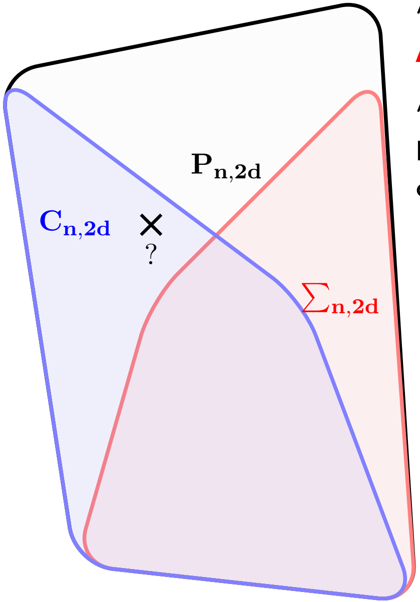
Are all convex forms sos?



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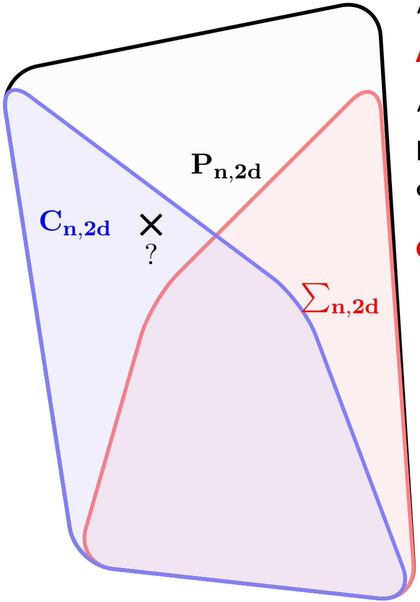

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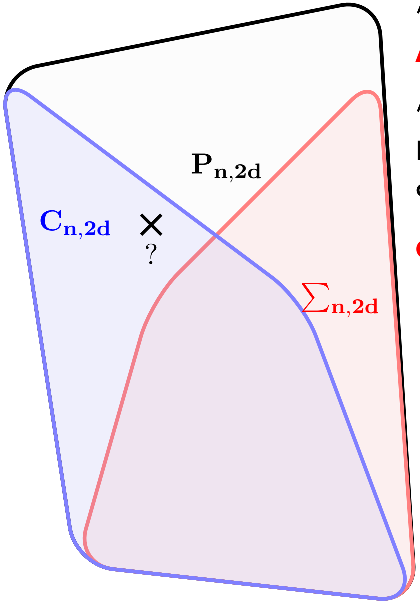

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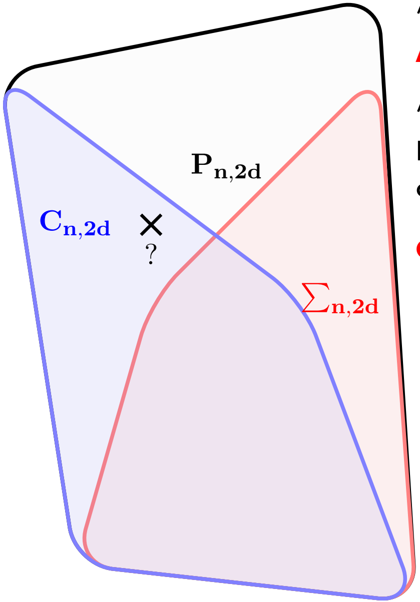
Find any element in $\mathbf{C_{n,2d}} \setminus \mathbf{\sum_{n,2d}}$



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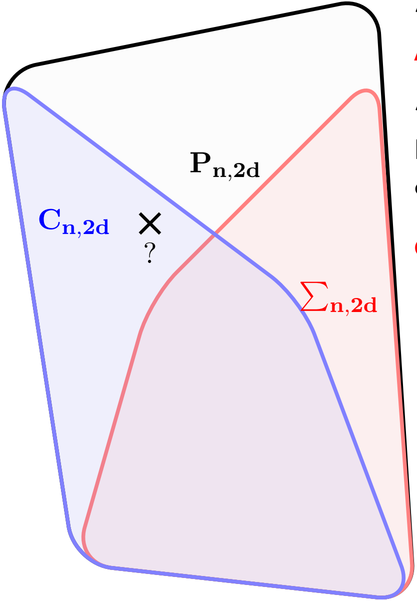
What is the smallest $(n, 2d)$ for which $\mathbf{C_{n,2d}} \not\subseteq \Sigma_{\mathbf{n,2d}}$?



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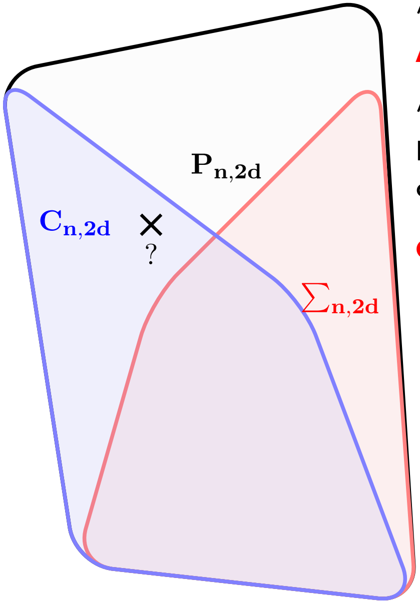
($(n, 2d)$ should be $\geq (3, 6), (4, 4)$)



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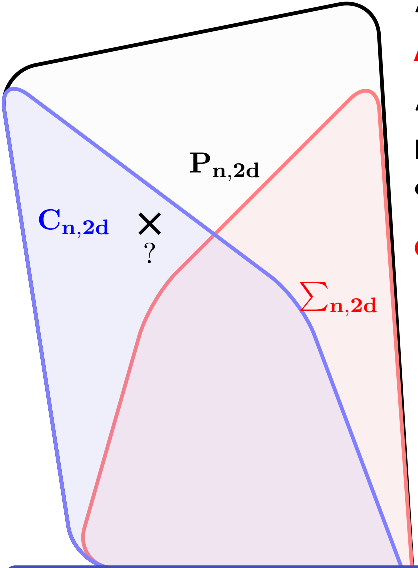

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Today: Convex quaternary quartic forms are sos, i.e. $\mathbf{C_{4,4}} \subseteq \mathbf{\Sigma_{4,4}}$

Outline

1. A curious generalization of the Cauchy-Schwarz inequality

2. Proof that all convex quaternary quartics are sos.

Generalized Cauchy-Schwarz Inequalities

Augustin-Louis Cauchy
(1789-1857)

Hermann Schwarz
(1843-1921)

Generalized Cauchy-Schwarz inequalities (1/3)

$\mathbf{x}^T Q \mathbf{x}$ psd $\longleftrightarrow$ $\mathbf{x}^T Q \mathbf{y}$
deg 2 in $\mathbf{x}$ — deg 1 in $\mathbf{x}$ and 1 in $\mathbf{y}$

Generalized Cauchy-Schwarz inequalities (1/3)

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$p(\mathbf{x})$ convex $\longleftrightarrow$ $B(\mathbf{x}, \mathbf{y})$ bi-form
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We want:

$$B(\mathbf{x}, \mathbf{y}) \leq K_{n,d}\ \sqrt{p(\mathbf{x})}\ \sqrt{p(\mathbf{y})}$$

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$$= \binom{2d}{d}^{-1} \times \text{coefficient of } \alpha^d \beta^d \text{ in } p(\alpha \mathbf{x} + \beta \mathbf{y})$$

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For instance:

- $2d = 2 \rightarrow B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T Q \mathbf{y}$
- $2d = 4 \rightarrow B(\mathbf{x}, \mathbf{y}) = \frac{1}{12}\ \mathbf{y}^T \nabla^2 p(\mathbf{x}) \mathbf{y}$

Generalized Cauchy-Schwarz inequality (2/3)

Thm:
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K_d	1	1	1	1.01	1	1.06	1	...

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Complex variant:
$\exists$ a constant L_d s.t. every convex form $p(\mathbf{x})$ of deg. $2d \leq 16$ satisfies

$$\sqrt{p(\mathbf{z})p(\bar{\mathbf{z}})} \leq L_d\, B(\mathbf{z},\bar{\mathbf{z}}) \quad \forall \mathbf{z} \in \mathbb{C}^n$$

$2d$	2	4	6	8	10	12	14	16	...
L_d	1	1	2	5	14	42	132	429	...

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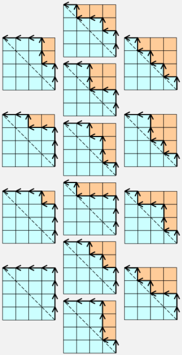

Catalan numbers describe the number of:

- ways a polygon with $n+2$ sides can be cut into n triangles.
- ways to use n rectangles to tile a stairstep shape $(1, 2, \ldots, n-1, n)$.
- ways in which parentheses can be placed in a sequence of numbers to be multiplied, two at a time
- planar binary trees with $n+1$ leaves
- paths of length $2n$ through an n-by-n grid that do not rise above the main diagonal

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Thm: For any convex form $p(\mathrm{x})$ of degree 4 in n variables

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Why? For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, set $\tilde{\mathbf{x}} := \dfrac{\mathbf{x}}{p(\mathbf{x})^{\frac{1}{4}}}, \tilde{\mathbf{y}} := \dfrac{\mathbf{y}}{p(\mathbf{y})^{\frac{1}{4}}}$ + use homogeneity.

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Why? For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, set $\tilde{\mathbf{x}} := \dfrac{\mathbf{x}}{p(\mathbf{x})^{\frac{1}{4}}}, \tilde{\mathbf{y}} := \dfrac{\mathbf{y}}{p(\mathbf{y})^{\frac{1}{4}}}$ + use homogeneity.

It is enough to prove:

$$\max_{\substack{q \text{ of deg } 4\\ \text{in } 2 \text{ vars}}} \quad \frac{1}{12}\,\mathbf{e}_2{}^T\,\nabla^2 q(\mathbf{e}_1)\,\mathbf{e}_2 \;\le 1$$

s.t. $\quad q$ convex

$$q(\mathbf{e}_1) \;=\; q(\mathbf{e}_2) \;= 1.$$

Generalized Cauchy-Schwarz inequality (3/3)

Thm: For any convex form $p(\mathbf{x})$ of degree 4 in n variables

$$\frac{1}{12}\,\mathbf{y}^T\nabla^2 p(\mathbf{x})\mathbf{y} \leq K_4\,\sqrt{p(\mathbf{x})}\sqrt{p(\mathbf{y})} \text{ with } K_4 = 1$$

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Generalized Cauchy-Sc

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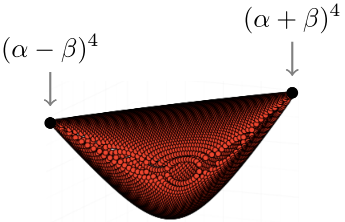
s.t. $\quad q$ convex $=$ "sos-convex"

$$q(\mathbf{e}_1)\ =\ q(\mathbf{e}_2)\ = 1.$$

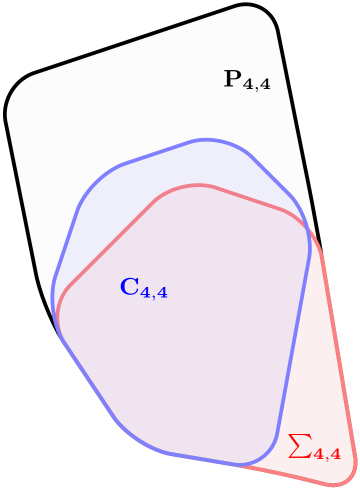
Why? Fix $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, set $q(\alpha, \beta) := p(\alpha\mathbf{x} + \beta\mathbf{y})$.

```
d = 2 # 2d = 4
model = SOSModel(...)

# declare a convex polynomial
@polyvar x[1:2]
@variable model q Poly(monomials(x, 2d))
@constraint model q in SOSConvexCone()

# q(1, 0) = q(0, 1) = 1
@constraint model q(x => [1, 0]) == 1
@constraint model q(x => [0, 1]) == 1

# objective
@objective model Max coefficients(q)[d+1]

# solve
optimize!(model)
```



Convex Quaternary Quartics are SOS

Linear inequalities that $\Sigma_{4,4}$ satisfies



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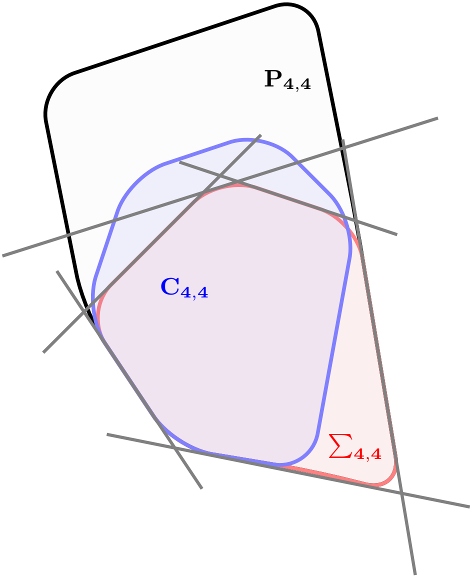

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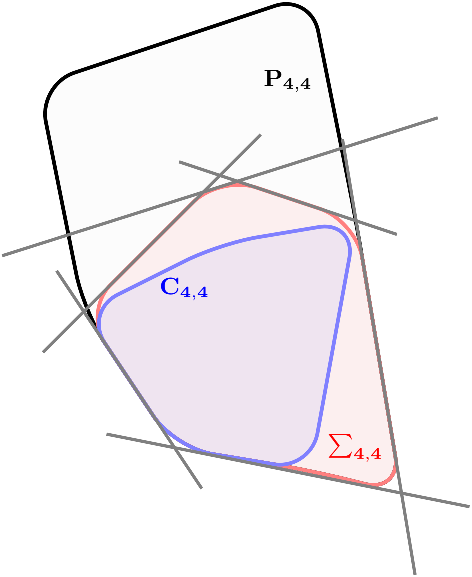

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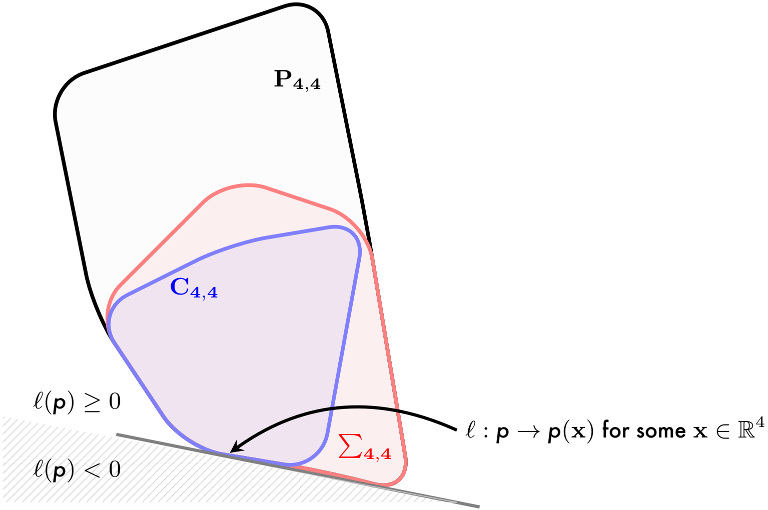

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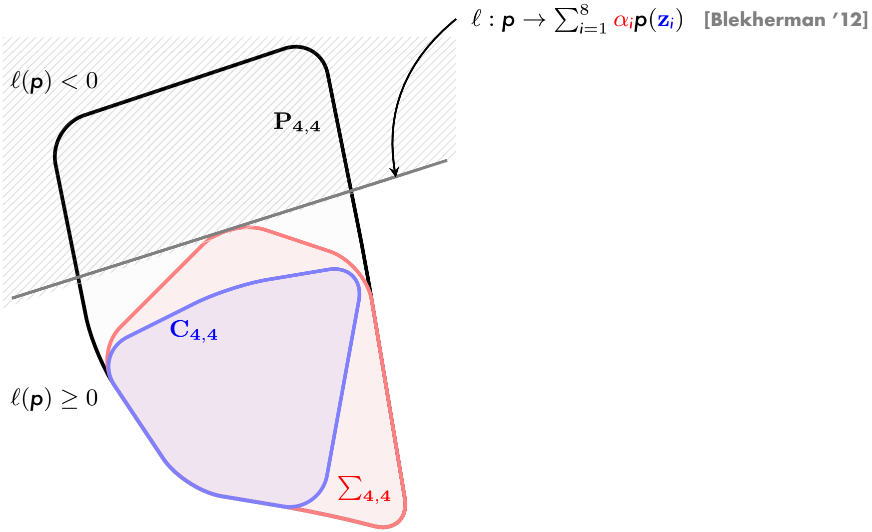

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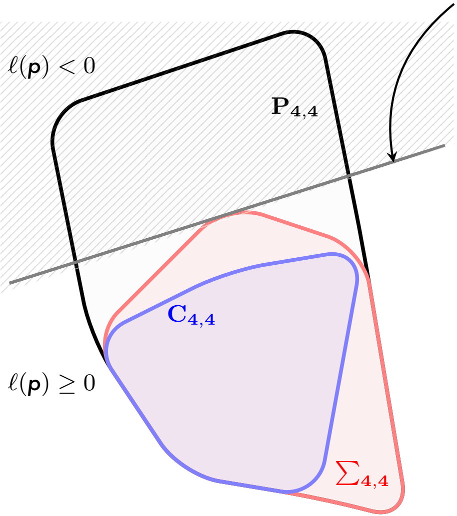
$\ell : p \to \sum_{i=1}^{8} \alpha_i p(\mathbf{z}_i)$ **[Blekherman '12]**



$\sum_{i=1}^{8} \pm \mathbf{z}_i \mathbf{z}_i^T = 0$ and other cond. on the α_i

(Cayley-Bacharach)

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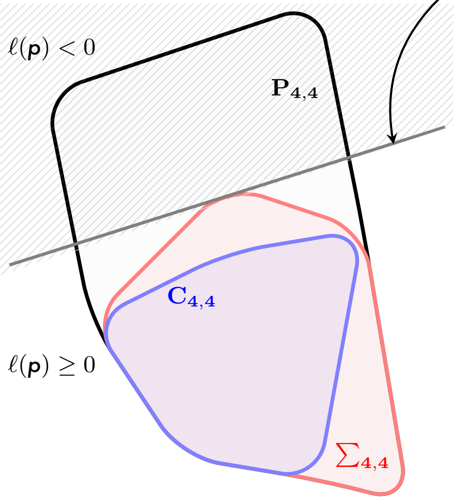

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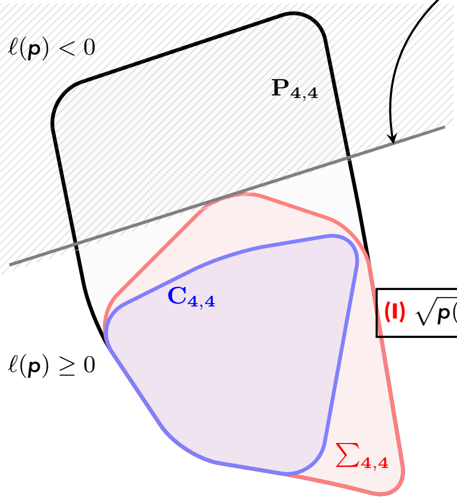

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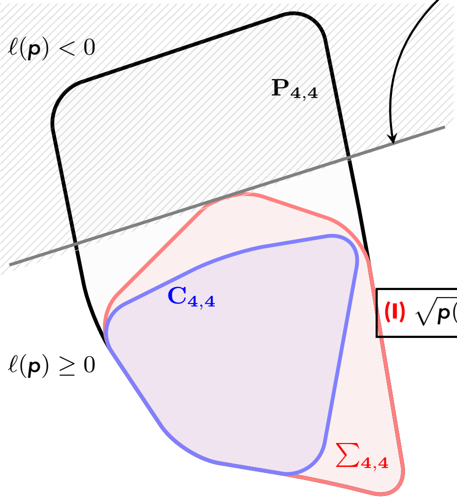

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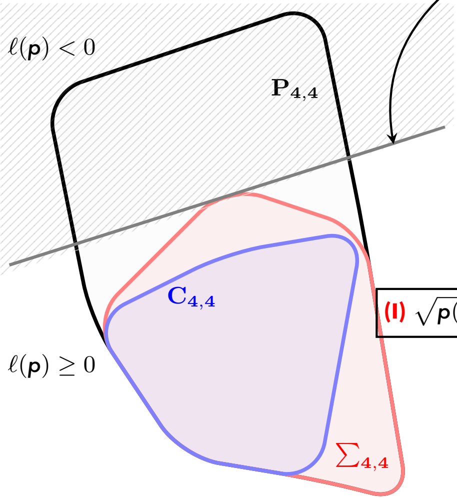

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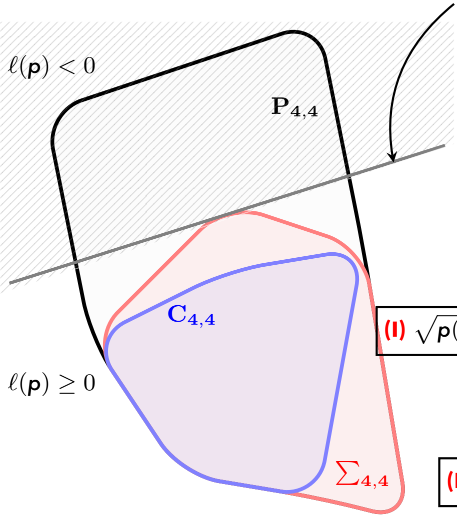

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Convex forms satisfy inequality (I)

Fix $p \in \mathbf{C_{4,4}}$. Let $\mathbf{z}_1, \ldots, \mathbf{z}_8 \in \mathbb{R}^4$ such that

$$\mathbf{z}_1\mathbf{z}_1^T = \sum_{i=2}^{8} \pm \mathbf{z}_i\mathbf{z}_i^T.$$

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$$B(\mathbf{z}_1, \mathbf{z}_1) + B(\mathbf{z}_2, \mathbf{z}_2) + 2\,B(\mathbf{z}_1, \mathbf{z}_2) = \sum_{i,j=3}^{8} \pm \quad B(\mathbf{z}_i, \mathbf{z}_j)$$

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All in all:

$$2\,\mathrm{Re}\,(p(\mathbf{z}_1)) + 2|p(\mathbf{z}_1)| \leq \sum_{i,j=3}^{8} \sqrt{p(\mathbf{z}_i)}\sqrt{p(\mathbf{z}_j)}.$$

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Squaring both sides and applying $B(\cdot,\cdot)$ leads to

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All in all:

$$2\,\mathrm{Re}\,(p(\mathbf{z}_1)) + 2|p(\mathbf{z}_1)| \leq \sum_{i,j=3}^{8} \sqrt{p(\mathbf{z}_i)}\sqrt{p(\mathbf{z}_j)}.$$

$$\textbf{WTS:}\sqrt{2}\sqrt{|p(\mathbf{z}_i)| + \mathrm{Re}\,(p(\mathbf{z}_i))} \leq \sum_{k\geq 3}^{8} \sqrt{p(\mathbf{z}_k)}$$

We have just proved that $\mathbf{C}_{4,4} \subseteq \sum_{4,4}$

What about convex ternary sextics?

The case of ternary sextics, i.e. $(n, 2d) = (3, 6)$

Very similar to the case of quaternary quartics!

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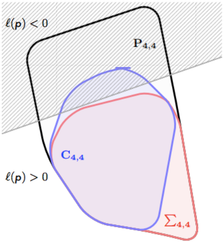
the extreme rays of $\Sigma^*_{6,3}$

[Blekherman '12]



Type (I)

Cauchy-Schwarz

$$\frac{1}{6\cdot5\cdot4}\nabla^3 p(\mathbf{x}) \cdot (\mathbf{y},\mathbf{y},\mathbf{y}) \le \sqrt{p(\mathbf{x})p(\mathbf{y})}$$

✓

Type (II)

Complex Cauchy-Schwarz

$$|p(\mathbf{z})| \le \frac{\mathbf{2}}{6\cdot5\cdot4}\nabla^3 p(\mathbf{z}) \cdot (\bar{\mathbf{z}},\bar{\mathbf{z}},\bar{\mathbf{z}})$$

✗

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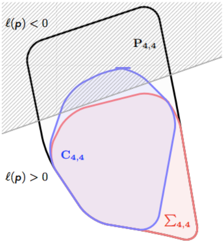

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$$\frac{1}{6\cdot 5\cdot 4}\nabla^3 p(\mathbf{x})\cdot(\mathbf{y},\mathbf{y},\mathbf{y}) \le \sqrt{p(\mathbf{x})p(\mathbf{y})}$$

✓

Type (II)

Complex Cauchy-Schwarz

$$|p(\mathbf{z})| \le \frac{2}{6\cdot 5\cdot 4}\nabla^3 p(\mathbf{z})\cdot(\bar{\mathbf{z}},\bar{\mathbf{z}},\bar{\mathbf{z}})$$

✗

Conjecture: Separating hyperplanes of Type (II) are not needed

[Blekherman '12]

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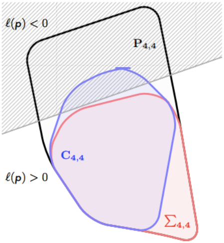

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Conjecture: Separating hyperplanes of Type (II) are not needed

[Blekherman '12]

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$$|p(\mathbf{z})| \le \frac{2}{6\cdot5\cdot4}\nabla^3 p(\mathbf{z})\cdot(\bar{\mathbf{z}},\bar{\mathbf{z}},\bar{\mathbf{z}})$$

✗

In which case, $\mathbf{C_{6,3}} \subseteq \sum_{\mathbf{6,3}}$!

Conclusion

It is hard to find a convex form that is not sos.

Generalized Cauchy-Schwarz inequalities are one of the main obstacles for small degree/number of variables.

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It is hard to find a convex form that is not sos.

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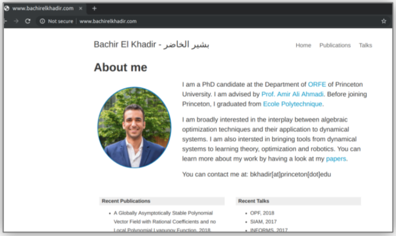

bachirelkhadir.com

Thanks!