From finite to infinite dimensional moment problems

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Workshop: Geometry of Real Polynomials, Convexity and Optimization

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Outline

1. Motivation and Framework
   - The classical full $K-$Moment Problem (KMP)
   - A general formulation of the full KMP
2. Our strategy for solving the general KMP
   - The character space as a projective limit
   - Extending cylindrical quasi-measures
3. Outcome of our "projective limit" approach
   - Old and new results for the KMP
   - Final remarks and open questions

The classical moment problem in one dimension

Let μ be a nonnegative Radon measure on $\mathbb{R}$. The $n-$**th moment of** μ is:

$$m_n^\mu := \int_{\mathbb{R}} x^n \mu(dx)$$

If all moments of μ exist and are finite, then $(m_n^\mu)_{n=0}^\infty$ is the **moment sequence** of μ.

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Let $N \in \mathbb{N} \cup \{\infty\}$ and $K \subseteq \mathbb{R}$ closed.

The one-dimensional $K-$Moment Problem (KMP)

Given a sequence $m = (m_n)_{n=0}^N$ of real numbers and a closed $K \subseteq \mathbb{R}$, does there exist a nonnegative Radon measure μ supported on K s.t. for any $n = 0, 1, \ldots, N$ we have

$$m_n = \underbrace{\int_K x^n \mu(dx)}_{n\text{-th moment of } \mu} \quad ?$$

If yes, μ is called $K-$**representing (Radon) measure** for m.

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$$\begin{array}{rlcl} L_m : & \mathbb{R}[x] & \to & \mathbb{R} \\ & p(x) := \sum_{n=0}^{N} a_n x^n & \mapsto & L_m(p) := \sum_{n=0}^{N} a_n m_n. \end{array}$$

Note:
If μ is a $K-$representing Radon measure for m, then

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- What if K is an infinite dimensional $\mathbb{R}-$vector space?
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Infinite dimensional K-Moment Problem

A general formulation of the full KMP

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Results' types for the classical KMP

Let $L : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$ be linear.

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Riesz-Haviland

Let $K \subseteq \mathbb{R}^d$ be closed.

$L(\mathrm{Psd}(K)) \subseteq [0, \infty) \Leftrightarrow \exists$ K–representing measure for L.

where $\mathrm{Psd}(K) := \{p \in \mathbb{R}[\mathbf{x}] : p \geq 0 \text{ on } K\}$.

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Nussbaum type

(i) $L(p^2) \geq 0$ for all $p \in \mathbb{R}[\mathbf{x}]$.

(ii) $\forall i = 1, \ldots, d : \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(X_i^{2n})}} = \infty$

Carleman Condition

$\Downarrow$

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- $K_M := \{y \in \mathbb{R}^d : q(y) \geq 0, \forall q \in M\}$ **basic closed semi-algebraic set**.

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Closure type

(i) τ is a topology on $\mathbb{R}[\mathbf{x}]$ s.t. $\overline{M}^{\tau} \supseteq \operatorname{Psd}(K_M \cap \operatorname{sp}(\tau))$ for some M quadratic module of $\mathbb{R}[\mathbf{x}]$

(ii) L is $\tau-$continuous

(iii) $L(M) \geq 0$

$\Downarrow$

$\exists!\; K_M \cap \operatorname{sp}(\tau)-$representing measure for L.

What about infinite dimensional settings?

Riesz-Haviland type

- $A = S(V)$
  -$V = \mathcal{C}_c^+(\mathbb{R}^d)$ (Lenard 1975
  (Krein, Nudelmann 1977)
  -V nuclear space, L continuous
  e.g. Berezansky; Borchers, Yngvason; Challifour, Slinker; Hergerfeldt; Schmüdgen 1975-90
- **A general and K s.t. (*)**

$(*)$ $\exists$ $a \in A$: $\hat{a} \geq 0$ on K & $\{\alpha \in X(A) : \hat{a}(\alpha) \leq n\}$ is compact $\forall n \in \mathbb{N}$ (Marshall 2003)

- $A = \mathbb{R}[x_i, i \in \Omega]$
  -Ω countable (Alpay, Jorgensen, Kimsey 2015)
  -Ω arbitrary, K described by countably many ineq. (Ghasemi, Kuhlmann, Marshall 2016)
- **A generated by $T \cup \{1\}$ lc s.t. $T' \subseteq X(A)$**
  (Schmüdgen 2017)

Nussbaum type

- **$A = S(V)$ with V nuclear**
  -$K = V'$ (Berezansky, Kondratiev, Šifrin 1970's)
  -$K = K_M$, $V = \mathcal{C}_c^\infty(\mathbb{R}^d)$(Infusino, Kuna, Rota 2014)
- $A = \mathbb{R}[x_i, i \in \Omega]$
  Ω arbitrary, $K = K_M$, M countably generated (Ghasemi, Kuhlmann, Marshall 2016)

Mixed type

- **A =poly of point processes, K =point configurations**,
  (e.g. Berezansky, Kondratiev, Kuna, Lytvynov, Oliveira, Lebowitz, Speer, 1999–2011.)
- **A =polynomials of Brownian motion, K =Wiener space of $\mathbb{R}$** (Albeverio, Herzberg 2008)

Archimedean property

- **A general**, $K = K_M$ with M Archimedean quadratic mod. (Putinar, Jacobi, Prestel 93-01)
- **A general**, $K = K_M$ with T weakly torsion preprime, M archimedean $T-$module (Marshall 01)
- $A = \mathbb{R}[x_i, i \in \Omega]$ and Ω countable, K_P compact and P preordering (Alpay, Jorgensen, Kimsey 2015)

Closure type

- **(A, τ) topological algebras with involution**, L continuous (e.g. Berg, Christiansen, Ressel, Schmüdgen 1970's)
- **(A, τ) lmc algebra**, L continuous, $K = K_M \cap sp(\tau)$, M 2d-power module (Ghasemi, Kuhlmann, Marshall 2014)
- **$A = S(V)$ and V lc**, L continuous, $K = K_M \cap B \subseteq V'$, M 2d-power module (Infusino, Ghasemi, Kuhlmann, Marshall 2018)

Urge for a general approach to the infinite dimensional KMP

PROBLEM

finding a unifying framework in which compare these results

developing an approach general enough to encompass all those special cases and beyond

...beyond:

- analysis of interacting particle systems $\rightsquigarrow$ K =point configuration spaces
- solving KMP for $\rightsquigarrow$ K =space of solutions of PDEs or SDEs,
- computation of the ground state energy of systems of non-relativistic electrons KMP for wave functions
- random packing and heterogeneous materials

Our strategy for solving the general KMP

The $K-$moment problem for unital commutative $\mathbb{R}-$algebras

Given a linear functional $L : A \to \mathbb{R}$ and $K \subseteq X(A)$ closed, does there exist a nonnegative Radon measure μ supported on K s.t.

$$L(a) = \int_{X(A)} \hat{a}(\alpha)\mu(d\alpha), \ \forall \ a \in A?$$

If yes, μ is called $K-$**representing (Radon) measure** for L.

Our idea

construct $X(A)$ as a projective limit of all $(X(S), \mathcal{B}_S)$

- S finitely generated subalgebra of A with $1 \in S$
- $\mathcal{B}_S$ Borel σ-algebra on $X(S)$ w.r.t. ω_S.

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The character space as a projective limit of topological spaces

A =unital commutative $\mathbb{R}-$**algebra**

For any $S \subseteq A$ subalgebra

- $X(S)$=character space of A=Hom$(S;\mathbb{R})$
- For $a \in S$ the Gelfand transform $\hat{a}_S : X(S) \to \mathbb{R}$ is $\hat{a}_S(\alpha) := \alpha(a)$, $\forall \alpha \in X(S)$.
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$$\equiv \quad \text{topology induced by the embedding} \begin{cases} X(S) & \hookrightarrow & \mathbb{R}^S \\ \alpha & \mapsto & (\hat{a}(\alpha))_{a\in S}\,. \end{cases}$$

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Consider the directed partially ordered set:

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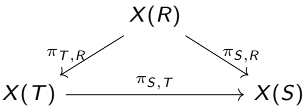
$$\pi_{S,T} : \begin{matrix} X(T) & \to & X(S) \\ \alpha & \mapsto & \alpha\restriction_S \end{matrix}$$

then $\forall S \subseteq T \subseteq R$ in J:



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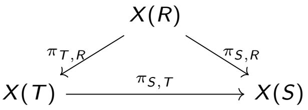

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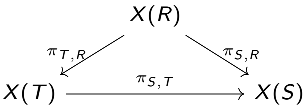

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The character space as a projective limit of topological spaces

Proposition

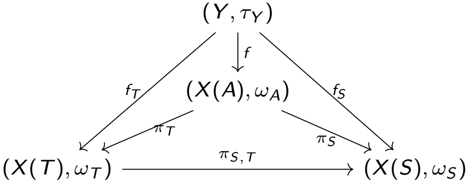
Let $J := \{S \subseteq A : S$ finitely generated subalgebra of $A, 1 \in S\}$ and for any $S \in J$

$$\pi_S := \pi_{S,A} : X(A) \to X(S), \alpha \mapsto \alpha \restriction_S$$

Then $\{(X(A), \omega_A), \pi_S, J\}$ is a **projective limit** of $\{(X(S), \omega_S), \pi_{S,T}, J\}$

Proof

- $\pi_{S,T} \circ \pi_T = \pi_S$ for all $S \subseteq T$ in J
- ω_A coincides with the weakest topology w.r.t. which all π_S's are *continuous*
- For any *topological space* (Y, τ_Y) and any *continuous* $f_S : Y \to X(S)$ with $S \in J$ and $f_S = \pi_{S,T} \circ f_T, \forall\, S \subseteq T$, $\exists!$ *continuous* $f : Y \to X(A)$ s.t. $\pi_S \circ f = f_S\ \forall\, S \in J$.



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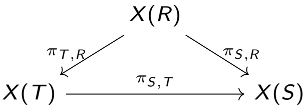

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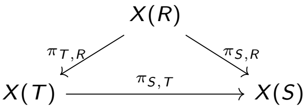

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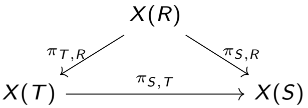

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The character space as a projective limit of measurable spaces

Let $J := \{S \subseteq A : S$ finitely generated subalgebra of $A, 1 \in S\}$.

$\{(X(S), \omega_S), \pi_{S,T}, J\}$ projective system of Hausdorff top. spaces

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$$\begin{array}{ccccc} & & (X(A), \omega_A) & & \\ & \swarrow^{\pi_T} & & \searrow^{\pi_S} & \\ (X(T), \omega_T) & & \xrightarrow{\pi_{S,T}} & & (X(S), \omega_S) \end{array}$$

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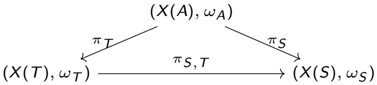

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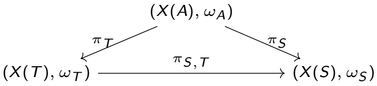

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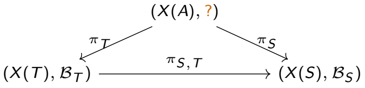

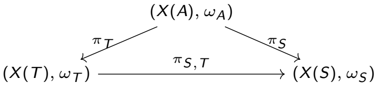

$\Sigma_J :=$ the smallest σ–algebra on $X(A)$ s.t. all the $\pi_S, S \in J$ are measurable.

then

Proposition

$\{(X(A), \Sigma_J), \pi_S, J\}$ is a **projective limit** of $\{(X(S), \mathcal{B}_S), \pi_{S,T}, J\}$

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Let $J := \{S \subseteq A : S$ finitely generated subalgebra of $A, 1 \in S\}$.

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$\{(X(S), \mathcal{B}_S), \pi_{S,T}, J\}$ associated projective system of Borel measurable spaces

$$\begin{array}{ccc} & (X(A), ?) & \\ \swarrow^{\pi_T} & & \searrow^{\pi_S} \\ (X(T), \mathcal{B}_T) & \xrightarrow{\pi_{S,T}} & (X(S), \mathcal{B}_S) \end{array} \qquad \begin{array}{ccc} & (X(A), \omega_A) & \\ \swarrow^{\pi_T} & & \searrow^{\pi_S} \\ (X(T), \omega_T) & \xrightarrow{\pi_{S,T}} & (X(S), \omega_S) \end{array}$$

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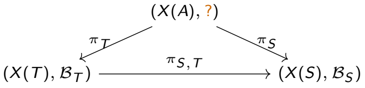

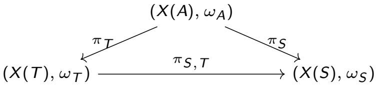

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- **cylinder set** in $X(A)$: $\pi_S^{-1}(M)$ for some $S \in J$ and $M \in \mathcal{B}_S$
- **cylinder algebra** on $X(A)$: $\mathcal{C}_J := \{\pi_S^{-1}(M) : M \in \mathcal{B}_S, \forall\ S \in J\}$
- **cylinder $\sigma-$algebra** on $X(A)$: $\sigma(\mathcal{C}_J) \equiv \Sigma_J$

$$\sigma(\mathcal{C}_J) \equiv \Sigma_J \subseteq \mathcal{B}_A$$

The character space as a projective limit

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Proposition

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where for any $S \in J$

- $\pi_S := \pi_{S,A} : X(A) \to X(S), \alpha \mapsto \alpha \restriction_S$
- Σ_J the smallest σ–algebra on $X(A)$ s.t. all the $\pi_S, S \in J$ are measurable

$$(X(A), \Sigma_J) \xrightarrow{\pi_T} (X(T), \mathcal{B}_T), \quad (X(A), \Sigma_J) \xrightarrow{\pi_S} (X(S), \mathcal{B}_S), \quad (X(T), \mathcal{B}_T) \xrightarrow{\pi_{S,T}} (X(S), \mathcal{B}_S)$$

$$(X(A), \omega_A) \xrightarrow{\pi_T} (X(T), \omega_T), \quad (X(A), \omega_A) \xrightarrow{\pi_S} (X(S), \omega_S), \quad (X(T), \omega_T) \xrightarrow{\pi_{S,T}} (X(S), \omega_S)$$

Representing measure on $(X(A), \Sigma_J)$ $\hookrightarrow$ Representing measure on $(X(A), \mathcal{B}_A)$

Cylindrical quasi-measures vs Measures

$\mathcal{P} := \{(X(S), \mathcal{B}_S), \pi_{S,T}, J\}$ = projective system of Borel measurable spaces

$\{(X(A), \Sigma_J), \pi_S, J\}$ projective limit of $\mathcal{P}$

$$\begin{array}{ccc} & (X(A), \Sigma_J) & \\ \swarrow^{\pi_T} & & \searrow^{\pi_S} \\ (X(T), \mathcal{B}_T) & \xrightarrow{\pi_{S,T}} & (X(S), \mathcal{B}_S) \end{array} \qquad \begin{array}{ccc} & (X(A), \omega_A) & \\ \swarrow^{\pi_T} & & \searrow^{\pi_S} \\ (X(T), \omega_T) & \xrightarrow{\pi_{S,T}} & (X(S), \omega_S) \end{array}$$

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$$\left(\forall S \in J, \mu_S \text{ measure on } (X(S), \mathcal{B}_S)\right) \Rightarrow \left(\begin{array}{l} \mu : \mathcal{C}_J \to \mathbb{R}^+ s.t. \pi_{S\#}\mu = \mu_S, \forall S \in J \\ i.e. \mu(\pi_S^{-1}(E)) := \mu_S(E), E \in \mathcal{B}_S \end{array} \right)$$

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Cylindrical quasi-measure

A *cylindrical quasi-measure* μ *w.r.t.* $\mathcal{P}$ is a set function $\mu : \mathcal{C}_J \to \mathbb{R}^+$ s.t.

$$\pi_{S\#}\mu \text{ is a measure on } (X(S), \mathcal{B}_S) \text{ for all } S \in J.$$

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NB: Cylindrical quasi-measures are NOT measures!

Cylindrical quasi-measures vs Measures

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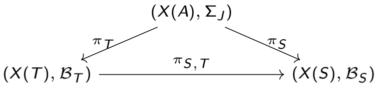

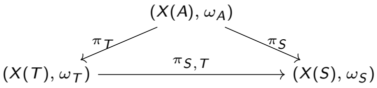

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Question 1

When can a cylindrical quasi-measure w.r.t. $\mathcal{P}$ be extended to a measure on Σ_J?

Cylindrical quasi-measures vs Measures

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A *cylindrical quasi-measure* μ *w.r.t.* $\mathcal{P}$ is a set function $\mu : \mathcal{C}_J \to \mathbb{R}^+$ s.t.

$$\pi_{S\#}\mu \text{ is a measure on } (X(S), \mathcal{B}_S) \text{ for all } S \in J.$$

Question 1

When can a cylindrical quasi-measure w.r.t. $\mathcal{P}$ be extended to a measure on Σ_J?

Question 2

When can a cylindrical quasi-measure be extended to a Radon measure on $\mathcal{B}_A$?

Extension theorems à la Prokhorov

$\mathcal{P} := \{(X(S), \mathcal{B}_S), \pi_{S,T}, J\}$ = projective system of Borel measurable spaces

Cylindrical quasi-measure on $(X(A), \mathcal{C}_J)$	$\hookrightarrow$	measure on $(X(A), \Sigma_J)$	$\hookrightarrow$	Borel measure on $(X(A), \mathcal{B}_A)$

An **exact projective system of measures** w.r.t. $\mathcal{P}$ is a family $\{\mu_S, S \in J\}$ s.t.

- μ_S measure on $\mathcal{B}_S$ for all $S \in J$
- ${\pi_{S,T}}_{\#}\mu_T = \mu_S$ for all $S \subseteq T$ in J

Extension theorems à la Prokhorov

$\mathcal{P} := \{(X(S), \mathcal{B}_S), \pi_{S,T}, J\}$ = projective system of Borel measurable spaces

Cylindrical quasi-measure on $(X(A), \mathcal{C}_J)$	$\hookrightarrow$	measure on $(X(A), \Sigma_J)$	$\hookrightarrow$	Borel measure on $(X(A), \mathcal{B}_A)$

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Extension theorems à la Prokhorov

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Cylindrical quasi-measure on $(X(A), \mathcal{C}_J)$ $\hookrightarrow$ measure on $(X(A), \Sigma_J)$ $\hookrightarrow$ Borel measure on $(X(A), \mathcal{B}_A)$

An **exact projective system of measures** w.r.t. $\mathcal{P}$ is a family $\{\mu_S, S \in J\}$ s.t.

- μ_S measure on $\mathcal{B}_S$ for all $S \in J$
- ${\pi_{S,T}}_{\#}\mu_T = \mu_S$ for all $S \subseteq T$ in J

Answer to Question 1

If $\{\mu_S, S \in J\}$ is an exact projective system of Radon probabilities w.r.t. $\mathcal{P}$, then $\exists!$ **probability** ν on $(X(A), \Sigma_J)$ such that ${\pi_S}_{\#}\nu = \mu_S$ for all $S \in J$.

Answer to Question 2

If $\{\mu_S, S \in J\}$ is an exact projective system of Radon probabilities w.r.t. $\mathcal{P}$,, then $\exists!$ **Radon probability** μ on $(X(A), \Sigma_J)$ such that ${\pi_S}_{\#}\nu = \mu_S$ for all $S \in J$ iff

$\forall \varepsilon > 0\ \exists\ K \subset X(A)$ compact s.t. $\forall S \in J,\ \mu_S(\pi_S(K)) \geq 1 - \varepsilon$ (ε-K)

Existence of representing measures on the cylinder $\sigma-$alg.

Theorem* (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and

$$J := \{S \subseteq A : S \text{ finitely generated subalgebra of } A, 1 \in S\}.$$

$$\left(\begin{array}{l} \forall S \in J, \exists\ X(S)\text{−representing} \\ \text{Radon measure for } L\restriction_S \end{array} \right) \Longleftrightarrow \left(\begin{array}{l} \exists\ X(A)\text{−representing} \\ \text{measure } \mu \text{ on } \Sigma_J \text{ for } L \end{array} \right)$$

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Sketch of the proof

$\mathcal{P} := \{(X(S), \mathcal{B}_S), \pi_{S,T}, J\}$ projective system

$\Downarrow$ EXISTENCE HP

$\forall S \in J,\ \exists\ \mu_S$representing Radon measure for $L\restriction_S$
s.t. $\forall S \subseteq T$, $\mu_S = {\pi_{S,T}}_{\#}\mu_T$

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Existence of representing measures on the cylinder $\sigma-$alg.

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$\Downarrow$ THM 1(Prokhorov)

$\exists! \nu$ measure on $(X(A), \Sigma_J)$ s.t. ${\pi_S}_{\#}\nu = \mu_S, \quad \forall\, S \in J$

Existence of representing measures on the cylinder σ−alg.

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$\exists! \nu$ measure on $(X(A), \Sigma_J)$ s.t. ${\pi_S}_{\#}\nu = \mu_S, \quad \forall S \in J$

Hence, for any $a \in A$ we have $a \in S$ for some $S \in J$ and so

$$L(a) = L\restriction_S (a) = \int_{X(S)} \hat{a}(\beta) d\mu_S(\beta) = \int_{X(A)} \hat{a}(\pi_S(\beta)) d\nu(\beta) = \int_{X(A)} \hat{a}(\alpha) d\nu(\alpha).$$

Existence of representing measures on the cylinder $\sigma-$alg.

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$\Downarrow$ THM 1

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Sketch of the proof

μ is a constructibly Radon measure [Ghasemi-Kulmann-Marshall, '16]

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Existence of representing measures on the cylinder $\sigma-$alg.

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and if

(I.) $L(a^2) \geq 0$ for all $a \in A$.

(II.) For each $a \in A$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(a^{2n})}} = \infty$.

then $\exists!$ $X(A)-$representing measure ν on $(X(A), \Sigma_J)$ for L.

Existence of representing measures on the cylinder $\sigma-$alg.

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Proof makes use of the following well-known finite-dimensional result:

Theorem (Nussbaum, 1965)

Let $L : \mathbb{R}[X_1, \ldots, X_d] \to \mathbb{R}$ be linear s.t. $L(1) = 1$. If

(i) $L(p^2) \geq 0$ for all $p \in \mathbb{R}[X_1, \ldots, X_d]$.

(ii) $\forall i = 1, \ldots, d : \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(X_i^{2n})}} = \infty$ **Carleman Condition**

then $\exists!$ $\mathbb{R}^d-$representing Radon measure for L.

Existence of representing measures on the cylinder $\sigma-$alg.

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Special case: $A = \mathbb{R}[X_i, i \in \Omega]$ with Ω arbitrary index set.

Theorem (Ghasemi, Kuhlmann, Marshall, 2016)

Let $L : \mathbb{R}[X_i, i \in \Omega] \to \mathbb{R}$ be linear s.t. $L(1) = 1$. If

(i) $L(p^2) \geq 0$ for all $p \in \mathbb{R}[X_i, i \in \Omega]$.

(ii) $\forall i \in \Omega : \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(X_i^{2n})}} = \infty$.

then $\exists!$ $\mathbb{R}^{\Omega}-$representing constructibly Radon measure for L.

Existence of representing measures on the cylinder $\sigma-$alg.

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

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(I.) $L(a^2) \geq 0$ for all $a \in A$.

(II.) For each $a \in A$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(a^{2n})}} = \infty$.

then $\exists!$ $X(A)-$representing measure ν on $(X(A), \Sigma_J)$ for L.

Special case: $A = \mathbb{R}[X_i, i \in \Omega]$ with Ω arbitrary index set.

Theorem (Ghasemi, Kuhlmann, Marshall, 2016)

Let $L : \mathbb{R}[X_i, i \in \Omega] \to \mathbb{R}$ be linear s.t. $L(1) = 1$. If Ω is countable and

(i) $L(p^2) \geq 0$ for all $p \in \mathbb{R}[X_i, i \in \Omega]$.

(ii) $\forall i \in \Omega : \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(X_i^{2n})}} = \infty$.

then $\exists!$ $\mathbb{R}^{\Omega}-$representing ~~constructibly~~ Radon measure for L.

Existence of representing measures on the cylinder $\sigma-$alg.

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and if

(I.) $L(a^2) \geq 0$ for all $a \in A$.

(II.) For each $a \in A$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(a^{2n})}} = \infty$.

then $\exists!$ $X(A)-$representing measure ν on $(X(A), \Sigma_J)$ for L.

Special case: $A = S(V) :=$ symmetric tensor algebra of V real vector space.

Theorem (Schmüdgen, 2018)

Let $L : S(V) \to \mathbb{R}$ be linear s.t. $L(1) = 1$. If

(i) $L(p^2) \geq 0$ for all $p \in S(V)$.

(ii) $\forall p \in S(V) : \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(p^{2n})}} = \infty$.

then $\exists! V^*-$representing measure on Σ_J for L, where $J := \{S(W) : W \subseteq V, \dim(W) < \infty\}$.

Existence of Radon representing measures

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)**

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and

$$J := \{S \subseteq A : S \text{ finitely generated subalgebra of } A, 1 \in S\}.$$

$$\left(\begin{array}{l} \forall S \in J, \exists\ X(S)-\text{representing} \\ \text{Radon measure for } L\restriction_S + (\varepsilon\text{-K}) \end{array} \right) \Longleftrightarrow \left(\begin{array}{l} \exists\ X(A)-\text{representing} \\ \text{Radon measure } \nu \text{ for } L \end{array} \right)$$

Sketch of the proof

$\mathcal{P} := \{(X(S), \mathcal{B}_S), \pi_{S,T}, J\}$ projective system

$\Downarrow$ EXISTENCE HP+ (ε-K)

$\forall S \in J,\ \exists\ \mu_S$representing Radon measure for $L\restriction_S$
s.t. $\forall S \subseteq T$, $\mu_S = {\pi_{S,T}}_{\#}\mu_T$ and (ε-K) holds

Existence of Radon representing measures

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)**

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and

$$J := \{S \subseteq A : S \text{ finitely generated subalgebra of } A, 1 \in S\}.$$

$$\left(\begin{array}{l} \forall S \in J, \exists\, X(S)-\text{representing} \\ \text{Radon measure for } L\restriction_S + (\varepsilon\text{-K}) \end{array} \right) \Longleftrightarrow \left(\begin{array}{l} \exists\, X(A)-\text{representing} \\ \text{Radon measure } \nu \text{ for } L \end{array} \right)$$

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Sketch of the proof

$\mathcal{P} := \{(X(S), \mathcal{B}_S), \pi_{S,T}, J\}$ projective system

$\Downarrow$ EXISTENCE HP+ (ε-K)

$\left.\begin{array}{l}\forall S \in J,\ \exists\ \mu_S \text{representing Radon measure for } L\restriction_S \\ \text{s.t. } \forall S \subseteq T,\ \mu_S = {\pi_{S,T}}_{\#}\mu_T \text{ and } (\varepsilon\text{-K}) \text{ holds}\end{array}\right\} \rightsquigarrow$ $\{\mu_S, S \in J\}$ exact projective sys of Radon probabilities fulfilling (ε-K)

Existence of Radon representing measures

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)**

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and

$$J := \{S \subseteq A : S \text{ finitely generated subalgebra of } A, 1 \in S\}.$$

$$\left(\begin{array}{l} \forall S \in J, \exists\ X(S)-\text{representing} \\ \text{Radon measure for } L\restriction_S + (\varepsilon\text{-K}) \end{array} \right) \Longleftrightarrow \left(\begin{array}{l} \exists\ X(A)-\text{representing} \\ \text{Radon measure } \nu \text{ for } L \end{array} \right)$$

Sketch of the proof

$\mathcal{P} := \{(X(S), \mathcal{B}_S), \pi_{S,T}, J\}$ projective system

$\Downarrow$ EXISTENCE HP+ (ε-K)

$$\left.\begin{array}{l} \forall S \in J,\ \exists\ \mu_S \text{representing Radon measure for } L\restriction_S \\ \text{s.t. } \forall S \subseteq T,\ \mu_S = \pi_{S,T\#}\mu_T \text{ and } (\varepsilon\text{-K}) \text{ holds} \end{array}\right\} \rightsquigarrow$$

$\{\mu_S, S \in J\}$
exact projective sys
of Radon probabilities
fulfilling (ε-K)

$\Downarrow$ THM 2 (Prokhorov)

$$\exists! \nu \text{ Radon measure on } (X(A), \mathcal{B}_J) \text{ s.t. } \pi_{S\#}\nu = \mu_S, \quad \forall\, S \in J$$

Existence of Radon representing measures

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)**

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and

$$J := \{S \subseteq A : S \text{ finitely generated subalgebra of } A, 1 \in S\}.$$

$$\left(\begin{array}{l} \forall S \in J, \exists\, X(S)-\text{representing} \\ \text{Radon measure for } L\restriction_S + (\varepsilon\text{-K}) \end{array} \right) \Longleftrightarrow \left(\begin{array}{l} \exists\, X(A)-\text{representing} \\ \text{Radon measure } \nu \text{ for } L \end{array} \right)$$

Sketch of the proof

$\mathcal{P} := \{(X(S), \mathcal{B}_S), \pi_{S,T}, J\}$ projective system

$\Downarrow$ EXISTENCE HP+ (ε-K)

$$\left. \begin{array}{l} \forall S \in J,\ \exists\ \mu_S \text{representing Radon measure for } L\restriction_S \\ \text{s.t. } \forall S \subseteq T,\ \mu_S = \pi_{S,T\#}\mu_T \text{ and } (\varepsilon\text{-K}) \text{ holds} \end{array} \right\} \rightsquigarrow$$

$\{\mu_S, S \in J\}$ exact projective sys of Radon probabilities fulfilling (ε-K)

$\Downarrow$ THM 2 (Prokhorov)

$$\exists! \nu \text{ Radon measure on } (X(A), \mathcal{B}_J) \text{ s.t. } \pi_{S\#}\nu = \mu_S, \quad \forall S \in J$$

Hence, for any $a \in A$ we have $a \in S$ for some $S \in J$ and so

$$L(a) = L\restriction_S(a) = \int_{X(S)} \hat{a}(\beta) d\mu_S(\beta) = \int_{X(A)} \hat{a}(\pi_S(\beta)) d\nu(\beta) = \int_{X(A)} \hat{a}(\alpha) d\nu(\alpha).$$

Localization of the support

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and

$$J := \{S \subseteq A : S \text{ finitely generated subalgebra of } A, 1 \in S\}.$$

$$(*) \left(\begin{array}{l} \forall S \in J, \exists\ X(S)\text{−representing} \\ \text{Radon measure for } L\restriction_S \end{array} \right) \Longrightarrow \left(\begin{array}{l} \exists\ X(A)\text{−representing} \\ \text{measure } \mu \text{ on } \Sigma_J \text{ for } L \end{array} \right)$$

$$(**) \left(\begin{array}{c} \forall S \in J, \exists\ X(S)\text{−representing} \\ \text{Radon measure for } L\restriction_S \\ +(\varepsilon\text{-K}) \end{array} \right) \Longrightarrow \left(\begin{array}{l} \exists\ X(A)\text{−representing} \\ \text{Radon measure } \nu \text{ for } L \end{array} \right)$$

Localization of the support

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and

$$J := \{S \subseteq A : S \text{ finitely generated subalgebra of } A, 1 \in S\}.$$

$\forall\ S \in J$, let $K^{(S)}$ be a closed subset of $X(S)$ s.t. $\pi_{S,T}\left(K^{(T)}\right) \subseteq K^{(S)}$ for $T \supseteq S$ in J.

$$(*) \left(\begin{array}{l} \forall S \in J, \exists\ K^{(S)}-\text{representing} \\ \text{Radon measure for } L\restriction_S \end{array} \right) \Longrightarrow \left(\begin{array}{l} \exists\ X(A)-\text{representing} \\ \text{measure } \mu \text{ on } \Sigma_J \text{ for } L \\ \mu\left(X(A) \setminus \pi_S^{-1}(K^{(S)})\right) = 0, \forall S \in J. \end{array} \right)$$

$$(**) \left(\begin{array}{c} \forall S \in J, \exists\ K^{(S)}-\text{representing} \\ \text{Radon measure for } L\restriction_S \\ +(\varepsilon\text{-K}) \end{array} \right) \Longrightarrow \left(\begin{array}{l} \exists \left(\bigcap_{S\in J} \pi_S^{-1}\left(K^{(S)}\right)\right)-\text{representing} \\ \text{Radon measure } \nu \text{ for } L \end{array} \right)$$

Existence of $K-$representing measures: generalized Riesz-Haviland

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$, $K \subseteq X(A)$ closed.

$$\left(L(\mathrm{Psd}_A(K)) \subseteq [0,\infty) \right) \Longrightarrow \left(\begin{array}{c} \exists\ X(A)-\text{representing measure } \mu \text{ on } \Sigma_J \text{ for } L \\ \mu\left(X(A) \setminus \pi_S^{-1}(\pi_S(K)) \right) = 0, \forall S \in J. \end{array} \right)$$

$$\left(\begin{array}{c} L(\mathrm{Psd}_A(K)) \subseteq [0,\infty) \\ +(\varepsilon\text{-K}) \end{array} \right) \Longrightarrow \left(\exists\ K\text{–representing Radon measure } \nu \text{ for } L. \right)$$

where $\mathrm{Psd}_A(K) := \{a \in A : \hat{a}(\alpha) \geq 0 \text{ for all } \alpha \in K\}$.

Theorem (Riesz, 1923; Haviland, 1936)

Let $L : \mathbb{R}[X_1, \dots, X_d] \to \mathbb{R}$ be linear s.t. $L(1) = 1$ and $K \subseteq \mathbb{R}^d$ closed.

$$\left(L(\mathrm{Psd}(K)) \subseteq [0,\infty) \right) \Longrightarrow \left(\exists\ K\text{–representing Radon measure } \nu \text{ for } L. \right)$$

where $\mathrm{Psd}(K) := \{p \in \mathbb{R}[X_1, \dots, X_d] : p(y) \geq 0 \text{ for all } y \in K\}$.

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where $\mathrm{Psd}_A(K) := \{a \in A : \hat{a}(\alpha) \geq 0 \text{ for all } \alpha \in K\}$.

Theorem (Ghasemi, Kuhlmann, Marshall, 2016)

Let $L : \mathbb{R}[X_i, i \in \Omega] \to \mathbb{R}$ be linear s.t. $L(1) = 1$ and $K \subseteq \mathbb{R}^d$ closed s.t. $K \in \Sigma_J$.

$$\left(L(\mathrm{Psd}(K)) \subseteq [0,\infty)\right) \Longrightarrow \left(\exists\ K\text{–representing constructibly Radon measure } \nu \text{ for } L. \right)$$

where $\mathrm{Psd}(K) := \{p \in \mathbb{R}[X_i, i \in \Omega] : p(y) \geq 0 \text{ for all } y \in K\}$.

Existence of $K-$representing measures: generalized Riesz-Haviland

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

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where $\mathrm{Psd}_A(K) := \{a \in A : \hat{a}(\alpha) \geq 0 \text{ for all } \alpha \in K\}$.

Theorem (Alpay, Jorgensen, Kimsey, 2015; Ghasemi, Kuhlmann, Marshall, 2016)

Let $L : \mathbb{R}[X_i, i \in \Omega] \to \mathbb{R}$ be linear s.t. $L(1) = 1$ and $K \subseteq \mathbb{R}^d$ closed s.t. $K \in \Sigma_J$.
If Ω is countable, then

$$\left(L(\mathrm{Psd}(K)) \subseteq [0,\infty)\right) \Longrightarrow \left(\exists\ K\text{–representing } \text{~~constructibly~~ Radon measure } \nu \text{ for } L.\right)$$

where $\mathrm{Psd}(K) := \{p \in \mathbb{R}[X_i, i \in \Omega] : p(y) \geq 0 \text{ for all } y \in K\}$.

Existence of $K-$representing measures: generalized Riesz-Haviland

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$, $K \subseteq X(A)$ closed.

$$\left(L(\mathrm{Psd}_A(K)) \subseteq [0, \infty)\right) \Longrightarrow \left(\begin{array}{l} \exists\ X(A)-\text{representing measure } \mu \text{ on } \Sigma_J \text{ for } L \\ \mu\left(X(A) \setminus \pi_S^{-1}(\pi_S(K))\right) = 0, \forall S \in J. \end{array} \right)$$

$$\left(\begin{array}{c} L(\mathrm{Psd}_A(K)) \subseteq [0, \infty) \\ +(\varepsilon\text{-K}) \end{array} \right) \Longrightarrow \left(\exists\ K-\text{representing Radon measure } \nu \text{ for } L. \right)$$

where $\mathrm{Psd}_A(K) := \{a \in A : \hat{a}(\alpha) \geq 0 \text{ for all } \alpha \in K\}$.

Theorem (Marshall, 2003)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$, $K \subseteq X(A)$ closed.
If $\exists\ a \in A$ s.t. $\hat{a} \geq 0$ on K and $\{\alpha \in X(A) : \hat{a}(\alpha) \leq n\}$ is compact $\forall n \in \mathbb{N}$, then

$$\left(L(\mathrm{Psd}_A(K)) \subseteq [0, \infty)\right) \Longrightarrow \left(\exists\ K-\text{representing Radon measure } \nu \text{ for } L. \right)$$

Existence of $K-$representing measures: Nussbaum's type results

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and if

(I.) $L(Q) \geq 0$ for some quadratic module Q of A,
i.e. $1 \in Q$, $Q + Q \subseteq Q$ and $a^2 Q \subseteq Q$ for each $a \in A$

(II.) For each $a \in A$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(a^{2n})}} = \infty$.

then $\exists!$ $X(A)-$representing measure ν on Σ_J for L s.t. $\mu\left(X(A) \setminus \pi_S^{-1}(K_{Q\cap S})\right) = 0$, $\forall S \in J$; $K_{Q\cap S} := \{\alpha \in X(A) : \alpha(q) \geq 0, \forall q \in Q \cap S\}$.

Theorem (Lasserre, 2013; Infusino-Kuna-Rota, 2014)

Let $L : \mathbb{R}[X_1, \ldots, X_d] \to \mathbb{R}$ be linear s.t. $L(1) = 1$. If

(i) $L(Q) \geq 0$ for some quadratic module Q of $\mathbb{R}[X_1, \ldots, X_d]$,

(ii) $\forall i = 1, \ldots, d : \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(X_i^{2n})}} = \infty$ Carleman Condition

then $\exists!$ K_Q-representing Radon measure for L; $K_Q := \{x \in \mathbb{R}^d : q(x) \geq 0, \ \forall\ q \in Q\}$.

Existence of $K-$representing measures: Nussbaum's type results

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and if

(I.) $L(Q) \geq 0$ for some quadratic module Q of A,
i.e. $1 \in Q$, $Q + Q \subseteq Q$ and $a^2 Q \subseteq Q$ for each $a \in A$

(II.) For each $a \in A$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(a^{2n})}} = \infty$.

(III.) (ε-K) holds

then $\exists!$Radon K_Q–representing measure for L; $K_Q := \{\alpha \in X(A) : \alpha(q) \geq 0, \forall q \in Q\}$.

Theorem (Lasserre, 2013; Infusino-Kuna-Rota, 2014)

Let $L : \mathbb{R}[X_1, \dots, X_d] \to \mathbb{R}$ be linear s.t. $L(1) = 1$. If

(i) $L(Q) \geq 0$ for some quadratic module Q of $\mathbb{R}[X_1, \dots, X_d]$,

(ii) $\forall i = 1, \dots, d : \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(X_i^{2n})}} = \infty$ Carleman Condition

then $\exists!$ K_Q-representing Radon measure for L; $K_Q := \{x \in \mathbb{R}^d : q(x) \geq 0, \ \forall \ q \in Q\}$.

Existence of $K-$representing measures: the Archimedean property

Theorem (Jacobi-Prestel Positivstellensatz (2001))

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$.

$$\left(\begin{array}{l} L(Q) \subseteq [0, +\infty) \text{ for some} \\ \text{Archimedean quadratic module } Q \text{ of } A, \\ \text{i.e. } \forall\, a \in A,\ \exists\, N \in \mathbb{N}:\ N \pm a \in Q \end{array} \right) \Longrightarrow \left(\begin{array}{l} \exists!\ K_Q-\text{representing} \\ \text{Radon measure for } L \end{array} \right)$$

where $K_Q := \{\alpha \in X(A) : \hat{q}(\alpha) \geq 0, \forall q \in Q\}$.

Proof. For each $S \in J$, $Q \cap S$ is Archimedean. Then use the following to get a $K_{Q\cap S}$-representing Radon measure for $L \restriction_S$ and to prove that $\Rightarrow$ (ε-K) holds.

Theorem (Putinar, 1993)

Let $L : \mathbb{R}[X_1, \ldots, X_d] \to \mathbb{R}$ be linear s.t. $L(1) = 1$.

$$\left(\begin{array}{l} L(Q) \subseteq [0, +\infty) \text{ for some} \\ \text{Archimedean quadratic module } Q \text{ of } A \end{array} \right) \Longrightarrow \left(\begin{array}{l} \exists!\ K_Q-\text{representing} \\ \text{Radon measure for } L \end{array} \right)$$

where $K_Q := \{y \in \mathbb{R}^d : q(y) \geq 0, \forall q \in Q\}$ i.e. basic closed semi-algebraic set.

Existence of $K-$representing measures: the Archimedean property

Theorem (Jacobi-Prestel Positivstellensatz (2001))

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$.

$$\left(\begin{array}{l} L(Q) \subseteq [0, +\infty) \text{ for some} \\ \text{Archimedean quadratic module } Q \text{ of } A, \\ \text{i.e. } \forall\, a \in A,\ \exists\, N \in \mathbb{N}\text{: } N \pm a \in Q \end{array} \right) \Longrightarrow \left(\begin{array}{l} \exists!\ K_Q-\text{representing} \\ \text{Radon measure for } L \end{array} \right)$$

where $K_Q := \{\alpha \in X(A) : \hat{q}(\alpha) \geq 0, \forall q \in Q\}$.

Theorem (Alpay, Jorgensen, Kimsey, 2015)

Let $L : \mathbb{R}[X_i, i \in \mathbb{N}] \to \mathbb{R}$ be linear s.t. $L(1) = 1$.

$$\left(\begin{array}{l} L(P) \subseteq [0, +\infty) \text{ for some preordering } P \\ \text{s.t. } P \cap S \text{ is finitely generated and} \\ K_{P \cap \mathbb{R}[X_1, \ldots, X_d]} \text{ is compact } \forall d \in \mathbb{N}. \end{array} \right) \Longrightarrow \left(\begin{array}{l} \exists!\ K_P-\text{representing} \\ \text{Radon measure for } L \end{array} \right)$$

Note: $K_{P \cap \mathbb{R}[X_1, \ldots, X_d]}$ is compact $\forall d \in \mathbb{N} \Rightarrow P$ Archimedean in $\mathbb{R}[X_i, i \in \mathbb{N}]$.

Existence of $K-$representing measures: the Archimedean property

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, Q a quadratic module in A and $L: A \to \mathbb{R}$ s.t. $L(1) = 1$. If $\exists\, B_{\mathrm{a}}$, B_{c} subalgebras of A such that $B_{\mathrm{a}} \cup B_{\mathrm{c}}$ generates A as a real algebra with B_{c} countably generated and

(i) $Q \cap B_{\mathrm{a}}$ is Archimedean in B_{a}

(ii) For each $a \in B_{\mathrm{c}}$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(a^{2n})}} = \infty$

(iii) $L(Q) \subseteq [0, +\infty)$

then $\exists!$ K_Q-representing Radon measure with $K_Q := \{\alpha \in X(A) : \alpha(q) \geq 0, \forall q \in Q\}$.

Theorem (Ghasemi, Kuhlmann, Marshall, 2016)

Let Q be a quadratic module in $\mathbb{R}[X_i, i \in \Omega]$ and $L : \mathbb{R}[X_i, i \in \Omega] \to \mathbb{R}$ be linear s.t. $L(1) = 1$. and . If $\exists\, \Lambda \subseteq \Omega$ countable such that

(i) $Q \cap \mathbb{R}[X_i]$ is Archimedian for all $i \in \Omega \setminus \Lambda$.

(ii) For each $i \in \Lambda$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(X_i^{2n})}} = \infty$

(iii) $L(Q) \subseteq [0, +\infty)$

then $\exists!$ K_Q-representing Radon measure with $K_Q := \{y \in \mathbb{R}^{\Omega} : q(y) \geq 0, \forall q \in Q\}$.

Final remarks and open questions

Open questions

- The partial archimedeanity of Q implies (ε-K). Does the converse hold?
- Does this approach applies to topological algebras?
- Does this approach apply to the truncated case?

Advantages & Potential of the projective limit approach

- it is powerful technique to exploit the finite dimensional moment theory to get new advances in the infinite dimensional one.
- it provides a direct bridge from the KMP to a rich spectrum of tools coming from the theory of projective limits.
- it offers a unified setting in which compare the results known so far about the infinite dimensional KMP.

Thank you for your attention

M. Infusino, S. Kuhlmann, T. Kuna, P. Michalski, *Projective limits techniques for the infinite dimensional moment problem*, arXiv:1906.01691