Optimization over the Hypercube via Sums of Nonnegative Circuit Polynomials

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(Joint work with A. Kurpisz and T. de Wolff)

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Geometry of Real Polynomials, Convexity, and Optimization

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Key Problem / Motivation

Problem: Let $f, g_1, \ldots, g_s \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ and consider the CONSTRAINED POLYNOMIAL OPTIMIZATION PROBLEM (CPOP)

$$f_K^* \; := \; \inf_{\mathbf{x} \in K} f(\mathbf{x}),$$

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- Problem has countless applications, e.g., robotics, control theory, economics, theoretical computer science.
- Problem is decidable, but NP-hard in general.

Solving CPOPs:

Using Positivstellensätze and relaxations, such problems can be tackled via SEMIDEFINITE OPTIMIZATION PROBLEM (SDP).

Typically: Putinar's Positivstellensatz and Lasserre's relaxation:

$$f_{sos}^{(d)} = \sup \left\{ \gamma : f - \gamma = \sigma_0 + \sum_{i=1}^{s} \sigma_i g_i, \ \sigma_i \text{ is SOS and } \deg(\sigma_i g_i) \leq 2d \right\}$$

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For many applications, problems are too large or numerical issues are too severe to find a (proper) solution via SOS/SDP.

Idea:
Find *new* ways to certify nonnegativity *independent* of SOS.

Some Notation

- We define $\mathbb{R}[\mathbf{x}]_{n,2d}$ as the vector space of real polynomials in n variables of degree at most $2d$.

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$$P_{n,2d} \quad := \quad \{f \in \mathbb{R}[\mathbf{x}]_{n,2d} \ : \ f(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n\}.$$

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- We define the CONE OF SUMS OF SQUARES as

$$\Sigma_{n,2d} \quad := \quad \left\{ f \in P_{n,2d} \ : \ f = \sum_{j=1}^{r} s_j^2 \text{ with } s_1, \ldots, s_r \in \mathbb{R}[\mathbf{x}]_{n,d} \right\}.$$

Circuit Polynomials

Definition

Let $f \in \mathbb{R}[\mathbf{x}]$ be supported on $A \subset \mathbb{N}^n$. Then f is called a CIRCUIT POLYNOMIAL if it is of the form

$$f = \sum_{j=0}^{r} f_{\alpha(j)} \mathbf{x}^{\alpha(j)} + f_{\beta} \mathbf{x}^{\beta}$$

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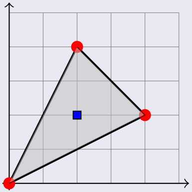

with the following conditions:

(C1) New(f) is a simplex with even vertices $\alpha(0), \ldots, \alpha(r)$.

(C2) $\beta = \sum_{j=0}^{r} \lambda_j \alpha(j)$ with $\lambda_j > 0$ and $\sum_{j=0}^{r} \lambda_j = 1$.

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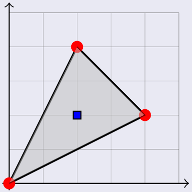

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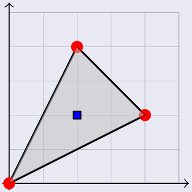

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Example: The Motzkin polynomial $1 + x^4y^2 + x^2y^4 - 3x^2y^2$ is a circuit polynomial.

Circuit Polynomials

For every circuit polynomial f, we define the corresponding CIRCUIT NUMBER as

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Facts:

- Nonnegativity of circuit polynomials can be checked easily via the condition: $|f_\beta| \leq \Theta_f$ or f is a sum of monomial squares.
- Writing a polynomial as a SUM OF NONNEGATIVE CIRCUIT POLYNOMIALS (SONC) is a certificate of nonnegativity.

Sums of Nonnegative Circuit Polynomials

Definition

Let the set of SUMS OF NONNEGATIVE CIRCUIT POLYNOMIALS (SONC) be

$$C_{n,2d} \quad := \quad \left\{ p \in \mathbb{R}[\mathbf{x}]_{n,2d} \; : \; \begin{array}{l} p = \sum_{i=1}^{k} \mu_i f_i, \ \forall\, i : \mu_i \geq 0, \\ f_i \text{ is NN circuit polyn. in } \mathbb{R}[\mathbf{x}]_{n,2d} \end{array} \right\}$$

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Let the set of sums of nonnegative circuit polynomials (SONC) be

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Theorem (Iliman, de Wolff, 2014 and D., 2018+)

$C_{n,2d}$ is a convex cone in $P_{n,2d}$ which satisfies:

- $C_{n,2d} \subseteq \Sigma_{n,2d}$ if and only if $(n, 2d) \in \{(1, 2d), (n, 2), (2, 4)\}$.
- $\Sigma_{n,2d} \not\subseteq C_{n,2d}$ for $2d \geq 4$.
- $\Sigma_{n,2} \not\subseteq C_{n,2}$ for $n \geq 2$.

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Theorem (D., Iliman, de Wolff, 2016)

For every $n, d \in \mathbb{N}^*$ the cone $C_{n,2d}$ is full-dimensional in $P_{n,2d}$.

Lemma (D., Iliman, de Wolff, 2016)

For every $n, d \in \mathbb{N}^*$ there exists $f, g \in C_{n,2d}$ such that $f \cdot g \notin C_{n,4d}$.

Sums of Nonnegative Circuit Polynomials

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For every $d \geq 2, n \in \mathbb{N}^*$ the SONC cone $C_{n,2d}$ is not closed under affine transformation of variables.

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Problem: How can one check efficiently, whether a polynomial has a SONC decomposition?

Consider the CPOP:

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Constrained Polynomial Optimization

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Key strength of $f_{sos}^{(d)}$: (Finite) convergence based on Putinar's Positivstellensatz.

A Positivstellensatz for SONC

- **Bad news:** Proof of Putinar's Positivstellensatz does not generalize from SOS to SONC, since $C_{n,2d}$ is not closed under multiplication.

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- **Bad news:** Proof of Putinar's Positivstellensatz does not generalize from SOS to SONC, since $C_{n,2d}$ is not closed under multiplication.
- **Good news:** We obtain straightforwardly:

Theorem (D., Iliman, de Wolff, 2016)

Schmüdgen-like Positivstellensatz holds for SONC.

A Positivstellensatz for SONC

Theorem (D., Iliman, de Wolff, 2016); rough version

Let $f(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ and K be a compact semi-algebraic set defined by $g_1(\mathbf{x}), \ldots, g_s(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$. If $f(\mathbf{x})$ is strictly positive for all $\mathbf{x} \in K$, then there exist $d, q \in \mathbb{N}^*$, such that

$$f(\mathbf{x}) \;=\; \sum_{\text{finite}} s(\mathbf{x}) H^{(q)}(\mathbf{x})$$

where

- every $s(\mathbf{x}) \in C_{n,2d}$,
- every $H^{(q)}(\mathbf{x})$ is a product of at most q of the $g_i(\mathbf{x})$: $H^{(q)}(\mathbf{x}) = \prod_{i=1}^{q} g_i(\mathbf{x})$.

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Note: An analog Positivstellensatz was given by Chandrasekaran and Shah for signomials via sums of arithmetic geometric exponentials (SAGE).

A Converging Hierarchy

We define:

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Clearly we have: $f_{\text{sonc}}^{(d)} \leq f_K^*$.

The SONC Positivstellensatz yields a degree dependent converging hierarchy:

Theorem (D., Iliman, de Wolff, 2016)

Let $f \in \mathbb{R}[\mathbf{x}]$, and K be a compact, semi-algebraic set. Then

$$f_{\text{sonc}}^{(d)} \uparrow f_K^* \ , \ \text{ for } \ d \to \infty.$$

Good news:

- The bounds $f_{\mathrm{sonc}}^{(d)}$ are given by a RELATIVE ENTROPY PROGRAM:

SONC Certificates via Relative Entropy Programming

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Definition

Let $\boldsymbol{\nu}, \boldsymbol{\zeta} \in \mathbb{R}^n_{\geq 0}$ and $\boldsymbol{\delta} \in \mathbb{R}^n$. A RELATIVE ENTROPY PROGRAM (REP) is of the form:

$$\begin{cases} \text{minimize} & p_0(\boldsymbol{\nu}, \boldsymbol{\zeta}, \boldsymbol{\delta}), \\ & (1) \quad p_i(\boldsymbol{\nu}, \boldsymbol{\zeta}, \boldsymbol{\delta}) \leq 1 \quad \text{for all} \quad i = 1, \ldots, m, \\ \text{subject to:} & (2) \quad \nu_j \log\left(\frac{\nu_j}{\zeta_j}\right) \leq \delta_j \quad \text{for all} \quad j = 1, \ldots, n, \end{cases}$$

where $p_0, \ldots, p_m$ are linear functionals and the constraints (2) are jointly convex functions in $\boldsymbol{\nu}, \boldsymbol{\zeta}$, and $\boldsymbol{\delta}$ defining the relative entropy cone.

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- Relative entropy programs are convex.
- Efficiently solvable with interior point methods.

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Theorem (D., Iliman, de Wolff, 2016)

Let $f \in \mathbb{R}[\mathbf{x}]$, and K be a compact, semi-algebraic set. Then for every d the bound $f_{\mathrm{sonc}}^{(d)}$ is computable via an explicit relative entropy program.

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Note: For a given support, searching through the space of degree d SONC certificates can be computed via a REP of size $n^{O(d)}$.

Let $f, g_1, \ldots, g_n, p_1, \ldots, p_m \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ with $g_j(\mathbf{x}) = (x_j - a_j)(x_j - b_j)$ for chosen $a_j, b_j \in \mathbb{R}$. Consider the CONSTRAINED HYPERCUBE OPTIMIZATION PROBLEM (CHOP)

$$\begin{aligned} & \min f(\mathbf{x}) \\ \text{s.t.} \quad & g_j(\mathbf{x}) = 0 \text{ for } j = 1, \ldots, n \\ & p_i(\mathbf{x}) \geq 0 \text{ for } i = 1, \ldots, m \\ & \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

Optimization over the Hypercube

Let $f, g_1, \ldots, g_n, p_1, \ldots, p_m \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ with $g_j(\mathbf{x}) = (x_j - a_j)(x_j - b_j)$ for chosen $a_j, b_j \in \mathbb{R}$. Consider the CONSTRAINED HYPERCUBE OPTIMIZATION PROBLEM (CHOP)

$$\begin{aligned} & \min f(\mathbf{x}) \\ \text{s.t.} \quad & g_j(\mathbf{x}) = 0 \text{ for } j = 1, \ldots, n \\ & p_i(\mathbf{x}) \geq 0 \text{ for } i = 1, \ldots, m \\ & \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

We denote $\mathcal{H}_{\mathcal{P}}$ as the feasible set: the *n-dimensional hypercube $\mathcal{H}$ constrained by polynomial inequalities given by $\mathcal{P}$*.

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Several key problems from theoretical computer science are equivalent to solving a CHOP. E.g., MAX CUT, Sparsest Cut, Knapsack, Maximum constraint satisfaction (CSP), Problem scheduling, etc.

Main goal: Find certificates with good complexity bounds in n and maximal total degree d.

Key Facts for SOS Certificates on the Boolean Hypercube

- For every feasible n-variate CHOP with constraints of degree at most d there exists a degree $2n + 2d$ SOS certificate.

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- For every feasible n-variate CHOP with constraints of degree at most d there exists a degree $2n + 2d$ SOS certificate.
- Finding a degree d SOS certificate for nonnegativity of a polynomial f on $\mathcal{H}_{\mathcal{P}}$ can be performed by solving an SDP of size $n^{O(d)}$.
  $\Rightarrow$ SOS certificate with at most $n^{O(d)}$ squared polynomials.

Our Main Results for SONC Certificates on $\mathcal{H}_{\mathcal{P}}$

Assumption: $|\mathcal{P}| = \text{poly}(n)$.

Theorem (D., Kurpisz, de Wolff, 2018)

For every polynomial f, nonnegative over the boolean hypercube, constrained with polynomial inequalities of degree at most d, there exists a degree $n + d$ SONC certificate.

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Theorem (D., Kurpisz, de Wolff, 2018)

Let f be an n-variate polynomial, nonnegative on the constrained hypercube $\mathcal{H}_{\mathcal{P}}$. If there exists a degree d SONC certificate for f, then there exists a degree d SONC certificate for f involving at most $n^{O(d)}$ many nonnegative circuit polynomials.

Proof strategy

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1. Develop a *Kronecker delta function* for SONC on $\mathcal{H}$:

$$\delta_{\mathbf{v}}(\mathbf{x}) := \prod_{j\in[n]:\ v_j=a_j} \left(\frac{-x_j+b_j}{b_j-a_j}\right) \cdot \prod_{j\in[n]:\ v_j=b_j} \left(\frac{x_j-a_j}{b_j-a_j}\right)$$

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   - For every $\mathbf{v} \in \mathcal{H}$ the Kronecker delta function can be written as

$$\delta_{\mathbf{v}} = \sum_{j=1}^{2^n} s_j H_j^{(n)} = \sum_{j=1}^{2^n} s_j \prod_{i=1}^{n} g_{i,j},$$

   for $s_1, \ldots, s_{2^n} \in \mathbb{R}_{\geq 0}$.

Proof strategy

2. Let $d \in \mathbb{N}$ and $f \in \mathbb{R}[\mathbf{x}]_{n,2d+2}$ such that f vanishes on $\mathcal{H}$. Then there exist $s_1, \ldots, s_{2n} \in C_{n,2d}$ such that

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- Confirm that the degrees did not increase.

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3. When restricted to the hypercube $\mathcal{H}$, a polynomial f can be represented as

$$f(\mathbf{x}) = \sum_{\mathbf{v}\in\mathcal{H}_{\mathcal{P}}} \delta_{\mathbf{v}}(\mathbf{x})f(\mathbf{v}) + \sum_{\mathbf{v}\in\mathcal{H}\setminus\mathcal{H}_{\mathcal{P}}} \delta_{\mathbf{v}}(\mathbf{x})f(\mathbf{v}).$$

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   This is a polynomial of degree at most $n + d$.

Proof strategy

4. Conclude a degree at most $n+d$ decomposition

$$
\begin{aligned}
f(\mathbf{x}) \; = \; & \sum_{j=1}^{n} s_j(\mathbf{x}) g_j(\mathbf{x}) + \sum_{j=1}^{n} s_{n+j}(\mathbf{x})(-g_j(\mathbf{x})) + \\
& \sum_{\mathbf{v}\in\mathcal{H}_{\mathcal{P}}} \delta_{\mathbf{v}}(\mathbf{x}) f(\mathbf{v}) + \sum_{\mathbf{v}\in\mathcal{H}\setminus\mathcal{H}_{\mathcal{P}}} \delta_{\mathbf{v}}(\mathbf{x}) p_{\mathbf{v}}(\mathbf{x}) \frac{f(\mathbf{v})}{p_{\mathbf{v}}(\mathbf{v})} \,,
\end{aligned}
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for some $s_1, \ldots, s_{2n} \in C_{n,n+d-2}$ and $p_{\mathbf{v}} \in \mathcal{P}$.

As a consequence of the decomposition of f in the previous theorem we can prove:

The function

$$f_a(\mathbf{x}) := (a-1)\prod_{i=1}^{n}\left(\frac{x_i+1}{2}\right)+1$$

has no Putinar-like SONC representation over $\mathcal{H} = \{\pm 1\}^n$ if $a > \frac{2n-1}{2^{n-2}-1}$.

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Corollary (D., Kurpisz, de Wolff, 2018)

There exists no equivalent of Putinar's Positivstellensatz for SONC.

We summarize

1. SONC polynomials provide a valid certificate for optimization over the n-variate constrained hypercube $\mathcal{H}_{\mathcal{P}}$.
2. For $f \geq 0$ on $\mathcal{H}_{\mathcal{P}}$, with $\deg(p_i) \leq d$, there exists a degree $n + d$ SONC certificate.
3. If f admits a degree d SONC certificate on $\mathcal{H}_{\mathcal{P}}$, then there exists a degree d SONC certificate for f involving at most $n^{O(d)}$ many nonnegative circuit polynomials.

Open Problems

1. We showed the existence of a ‘short’ SONC certificate containing at most $n^{O(d)}$ nonnegative circuit polynomials. But can the corresponding REP also be formulated in *time* $n^{O(d)}$?

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2. We also showed: SONC is not closed under affine transformation. What is its closure? How can we compute such extended SONC certificates efficiently?
3. How is the situation over other varieties?

Thank you for your attention!

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