On the Lee-Huang-Yang universal asymptotics for the ground state energy of a Bose gas in the dilute limit

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BIRS, Banff August 5, 2019
From Many Body Problems to Random Matrices

Based on works with Birger Brietzke and Søren Fournais

Outline of Talk

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The Bose gas

We consider N bosons moving in a box $\Omega = [0, L]^3$ (Dirichlet or periodic b.c.)

2-body potential: $v : \mathbb{R}^3 \to [0, \infty]$ measurable, spherically symmetric, compact support, e.g., **hard core potential:** $v(x) = \infty$ if $|x| < a$ and zero otherwise.

Hamiltonian:

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leq i < j \leq N} v(x_i - x_j)$$

Hilbert Space: $\bigvee^N L^2(\Lambda)$ (the symmetric tensor product)

Thermodynamic limit of ground state energy:

$$e(\rho) = \lim_{\substack{L \to \infty \\ N/L^3 \to \rho}} e_L(N), \qquad e_L(N) = L^{-3} \inf \mathrm{Spec}(H_N)$$

The scattering length and dilute limit

The Scattering Solution for the 2-body potential: $u : \mathbb{R}^3 \to \mathbb{R}$

$$-\Delta u + \frac{1}{2} v u = 0, \qquad \lim_{x\to\infty} u(x) = 1, \qquad u = 1 - \omega$$

$$0 \leq \omega \leq 1$$

Scattering length:

$$a = \lim_{x\to\infty} |x|\omega(x) = \frac{1}{8\pi}\int_{\mathbb{R}^3} v(1-\omega)$$

The dilute limit of the Bose gas:

$$\rho a^3 \to 0$$

The Lee-Huang-Yang formula and main results

Theorem (Brietzke-Fournais-Solovej 2019 LHY Order)

$$e(\rho) \geq 4\pi\rho^2 a(1 - C\sqrt{\rho a^3}),$$

$C > 0$ *depends on support and scattering length of* v.

Note it is enough to prove this for L^1 potentials.

Theorem (Lee-Huang-Yang (1957) Formula for dilute limit $\rho a^3 \to 0$, Fournais-Solovej 2019)

If we also have $v \in L^1(\mathbb{R}^3)$ *then*

$$e(\rho) \geq 4\pi\rho^2 a(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho a^3})).$$

Upper bound needs stronger assumptions on v *(Yau-Yin (2009)).*

Previous results

- **Lee-Huang-Yang (1957)** derived formula by summing selected terms in pertubation series and an uncontrolled pseudo potential approximation
- **Dyson (1957)** Got leading upper bound with error $(\rho a^3)^{1/3}$. His lower bound was not correct to leading order
- **Lieb (1963)** derived the formula under assumptions on the structure of the ground state
- **Lieb-Yngvason (1998)** established the leading order $4\pi\rho^2 a$ with error bound $(\rho a^3)^{1/17}$
- **Erdős-Schlein-Yau (2008)** had the LHY order as an upper bound under additional assumptions on v
- **Yau-Yin (2009)** established the LHY formula as an upper bound under additional assumptions on v
- **Giuliani-Seiringer (2009)** derived LHY for soft potential with radius of support $R \gg \rho^{-1/3}$, i.e., requirements on potential depend on density
- **Brietzke-Solovej (2018)** derived LHY for soft potentials with $a \ll R \ll \rho^{-1/3}$
- **Boccato-Brennecke-Cenatiempo-Schlein (2018)** derived the LHY formula in the confined case with additional assumptions on v

Sketch of the big-O proof

Sketch of the Big-O Proof

Localization of the kinetic energy

We localize to boxes

$$\Lambda_u = \ell_0(u + [-1/2, 1/2]^3), \quad u \in \mathbb{R}^3$$

of size $\ell_0 = K^{-1}(\rho a)^{-1/2}$ with K large. Consider the **projections**

$$P_u = |\Lambda_u|^{-1}|\mathbb{1}_{\Lambda_u}\rangle\langle\mathbb{1}_{\Lambda_u}|, \qquad Q_u = \mathbb{1}_{\Lambda_u} - P_u.$$

and the **localization function**

$$\chi_u(x) = \chi(x/\ell_0 - u), \quad \chi \in C^M(\mathbb{R}^3) \text{ support in } [-1/2, 1/2]^3.$$

$$\int \chi_u^2(x)dx = \ell_0^3, \quad \int \chi_u^2(x)du = 1.$$

We have the **kinetic energy localization** (Brietzke-Fournais-Solovej)

$$-\Delta \geq \int \left(Q_u\chi_u[-\Delta - (s\ell_0)^{-2}]_+\chi_u Q_u + b\ell_0^{-2}Q_u\right) du,$$

where $b, s > 0$ are constants.

Potential energy localization

Introduce the **localized potential**

$$w_u(x,y) = \chi_u(x)W(x-y)\chi_u(y), \qquad W(x) = \frac{v(x)}{\chi * \chi(x/\ell_0)}$$

Then

$$\sum_{i<j} v(x_i - x_j) = \int \sum_{i<j} w_u(x_i, x_j) du$$

In the localization we will use a **grand canonical formalism** and introduce a **chemical potential** term

$$\begin{aligned} -8\pi a\rho N &= -N\rho \int v(1-\omega) \\ &= -\int \left[\sum_{i=1}^{N} \rho \int w_u(x_i, y)(1-\omega(x_i - y))dy \right] du. \end{aligned}$$

Localized Hamiltonian

The **localized Hamiltonians** are translates of

$$H_0 = \sum_i (\tau_i - \mu_i) + \sum_{i<j} w_{ij}$$

where, with $P_{u=0} = P$, $Q_{u=0} = Q$,

$$\tau = b\ell_0^{-2} Q + Q\chi_0[-\Delta - (s\ell_0)^{-2}]_+ \chi_0 Q$$

$$\mu = \mu(x) = \rho \int w(x,y)(1 - \omega(x-y))dy$$

and $w = w_{u=0}(x,y)$. The first term in τ is a **Neumann gap**. We have to show that

$$\ell_0^{-3} H_0 \geq -4\pi\rho^2 a - C\rho^2 a\sqrt{\rho a^3}.$$

The potential decomposition

A key idea is to decompose the potential

$$
\begin{aligned}
w_{ij} &= (P_i + Q_i)(P_j + Q_j) w_{ij} (P_i + Q_i)(P_j + Q_j) \\
&= (Q_i Q_j + (P_i P_j + Q_i P_j + P_i Q_j)\omega) w_{ij} (Q_i Q_j + \omega(P_i P_j + \cdots)) \\
&\quad + \mathcal{Q}_3^{\mathrm{ren}} + \mathcal{Q}_2^{\mathrm{ren}} + \mathcal{Q}_1^{\mathrm{ren}} + \mathcal{Q}_0^{\mathrm{ren}}
\end{aligned}
$$

where $\mathcal{Q}_q^{\mathrm{ren}}$ denotes terms with $q = 0, \ldots, 3$ number of Q's.

- Notice the first term is positive.
- For a $O(\rho^2 a \sqrt{\rho a^3})$ the $\mathcal{Q}_3^{\mathrm{ren}}$ may be controlled by this positive term and a CS-inequality.
- The remaining $1Q$ terms can also be controlled by a CS inequality leading to the final estimate with only no-Q and $2Q$ terms.

The main estimate

With $n_0 = \sum_i P_i$, $n_+ = \sum_i Q_i$ being the number of **condensate** and **excited** particles:

$$-\sum_i \mu_i + \sum_{i<j} w_{ij} \geq \widetilde{\mathcal{Q}}_0^{\mathrm{ren}} + \widetilde{\mathcal{Q}}_2^{\mathrm{ren}} - Ca(\rho + n_0 \ell_0^{-3}) n_+$$

The error term can be absorbed in the Neumann gap and

$$\widetilde{\mathcal{Q}}_0^{\mathrm{ren}} = \frac{n_0(n_0-1)}{2\ell_0^6} \int v(1-\omega^2)(x)\, dx - 8\pi a \left(\rho \frac{n_0}{\ell_0^3} + \frac{1}{4}\left(\rho - \frac{n_0+1}{\ell_0^3}\right)^2 \right)$$

The term $\widetilde{\mathcal{Q}}_2^{\mathrm{ren}}$ has 2 Q's and with the kinetic energy will be treated by a **Bogolubov diagonalization**. This replaces $1-\omega^2$ above to $1-\omega$ and gives errors of order $\rho^2 a \sqrt{\rho a^3}$ and hence exactly what we want.

The LHY asymptotics

To get to the **LHY accuracy** the terms $\mathcal{Q}_3^{\rm ren}$ have to be treated more carefully. Here are the main steps sketched:

- **Double localization**

$$\ell_1 \gg (\rho a)^{-1/2}, \quad \ell_0 \ll (\rho a)^{-1/2}$$

  gives big-O apriori estimate on energy and hence condensation: restriction to subspace with upper bound on

$$n_+ = \sum_i Q_i$$

- Inspired by Yin-Yau **soft pair** calculation we show that the main contribution to 3-Q terms are from

$$PQw(1-\omega)QQ \to PQ_L w(1-\omega)Q_H Q_H,$$

$$Q_L = \text{low momenta}, \quad Q_H = \text{high momenta}$$

The LHY asymptotics II

- 2nd quantization and **c-number substitution**
- There are good and bad 1-Q and 2-Q terms. The good 2-Q terms give a **quadratic Hamiltonian** which after **Bogolubov diagonalization** has a ground state energy which leads to the LHY term.
- The excited part of the Bogolubov diagonalized Hamiltonian together with the main 3-Q terms can again be "Bogolubov diagonalized" to miraculously cancel all the bad 1-Q and 2-Q terms.

Happy Birthday Yau

and thanks for all the inspiration to this and lots of other works!