Universality for Lozenge Tiling Local Statistics

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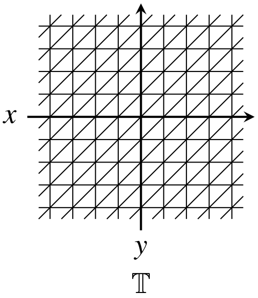
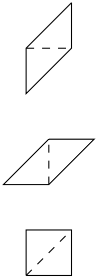
Lozenge Tilings

- Triangular lattice $\mathbb{T}$

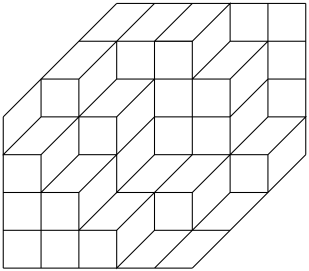




- Consider tilings of subdomains of $\mathbb{T}$ using three types of *lozenges*.

Lozenge Tiling of a Hexagon

- Tiling of a hexagon



- How does a uniformly random tiling “look” when the domain is large?

Tiling of a Small Hexagon

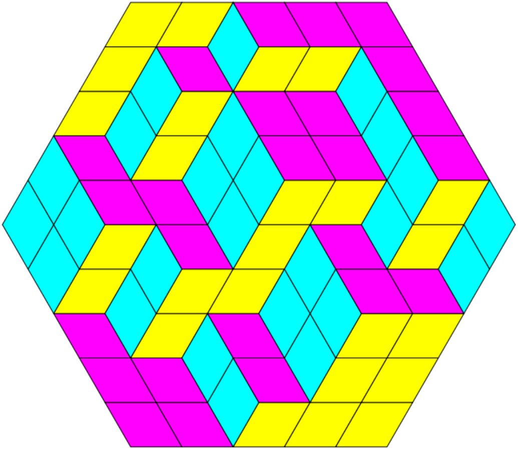


Figure 2 of "Lectures on Dimers," by R. Kenyon.

Tiling of a Large Hexagon

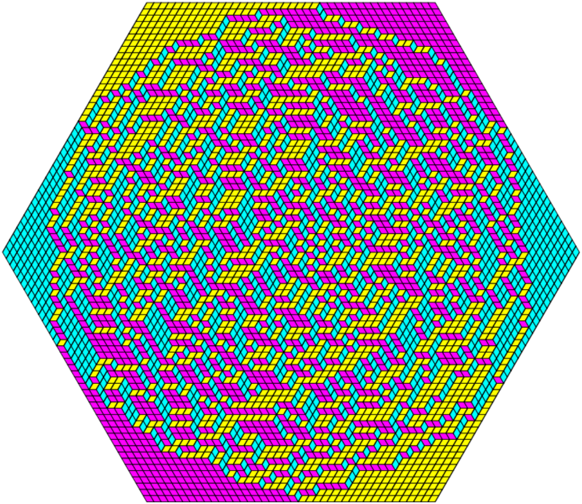


Figure 4 of “Lectures on Dimers,” by R. Kenyon.

Tiling of a Larger Hexagon

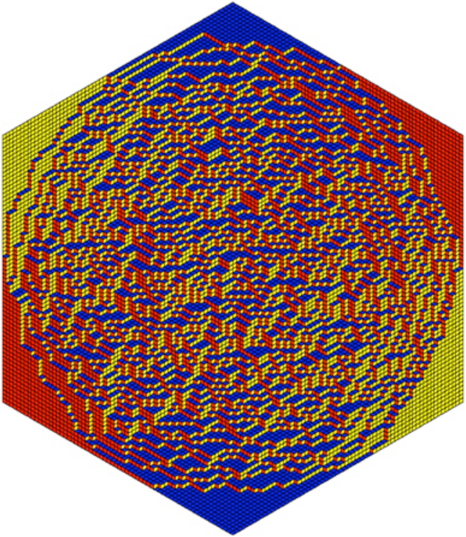


Figure from https://www.math.colostate.edu/~kenmcl/RMT.html

Tilings of Other Shapes

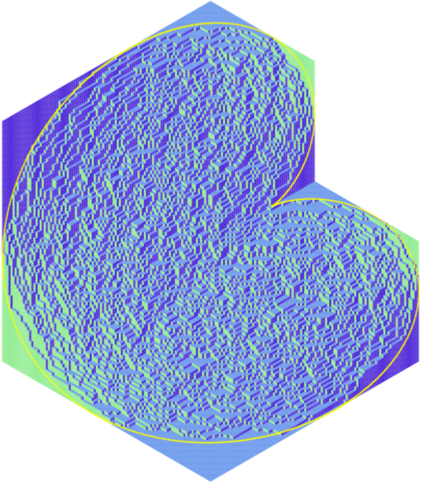


Figure 1 of “Limit shapes and the complex Burgers equation,” by R. Kenyon and A. Okounkov.

Tilings of Other Shapes

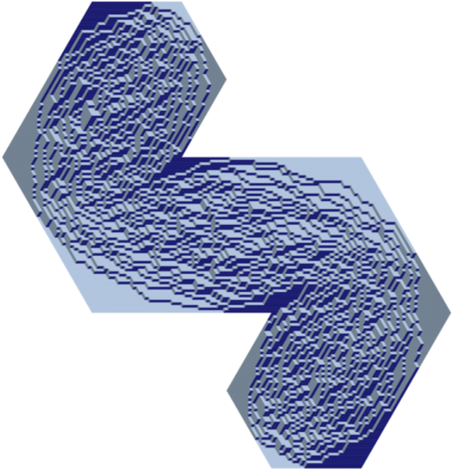
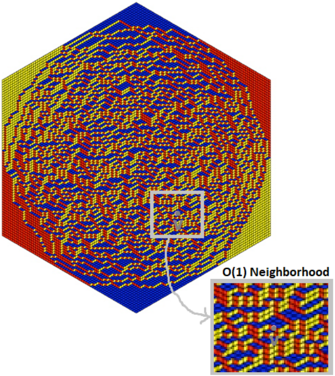


Figure 15 of “Random Tilings with the GPU,” by D. Keating and A. Sridhar.

Local Statistics of Lozenge Tilings

- Consider a uniformly random tiling of a domain $R \subset \mathbb{T}$.
- Fix a vertex $v \in R$ and consider an $O(1)$-neighborhood of v.



- How does the tiling look in this neighborhood? Equivalently, what are the correlation functions for nearly neighboring tiles?

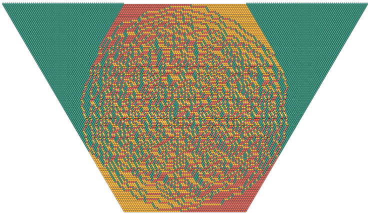
Boundary Conditions

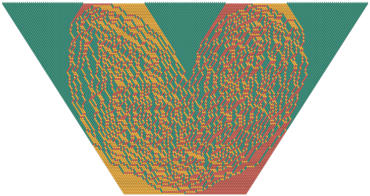
- Kasteleyn (1961): How do the local statistics around v depend on R?

Theorem (A., 2019; Informal Version)

Let R be a large tileable domain. Then the local statistics of a uniformly random tiling of R around a vertex $v \in R$ are asympotically determined by the local densities of the three types of tiles around v.

Different Behaviors for Similar Domains

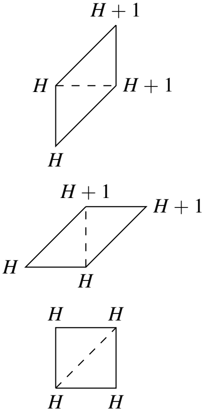


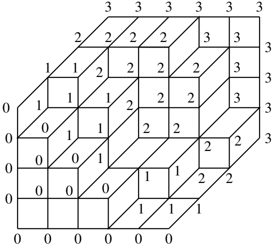


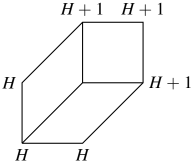
Figures due to L. Petrov.

Height Functions

- A **height function** $H : R \to \mathbb{Z}$ is one that satisfies $f(v) - f(u) \in \{0, 1\}$ whenever $u = (x, y)$ and $v \in \big\{(x+1, y), (x, y+1), (x+1, y+1)\big\}$.
- If R is simply-connected, then associated with any tiling of R is a height function (unique up to shifts).

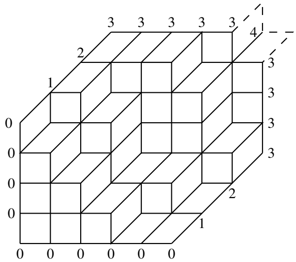






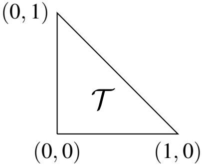
Boundary Height Functions

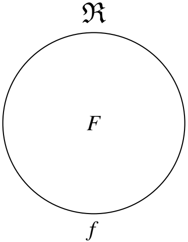
- Up to global shifts, the restriction of this height function to ∂R is independent of the tiling.
- Any height function with this restriction to ∂R gives rise to a tiling on R.
- Any height function on R gives rise to a *free tiling* on R, whose tiles are permitted to extend past ∂R and include a face of $\mathbb{T} \setminus R$.



Admissible Functions

- Define $\mathcal{T} = \{(s,t) \in \mathbb{R}^2_{>0} : s + t < 1\}$.
- Fix a bounded, open, nonempty set $\mathfrak{R} \subset \mathbb{R}$ with boundary $\partial\mathfrak{R}$.
- Let $\mathrm{Adm}(\mathfrak{R})$ denote the set of Lipschitz functions $F : \overline{\mathfrak{R}} \to \mathbb{R}$ such that $\nabla F(z) \in \overline{\mathcal{T}}$ for almost every $z \in \mathfrak{R}$.





- For any $f : \partial\mathfrak{R} \to \mathbb{R}$, set $\mathrm{Adm}(\mathfrak{R}; f) = \{F \in \mathrm{Adm}(\mathfrak{R}) : F|_{\partial\mathfrak{R}} = f\}$.
- If $\mathrm{Adm}(\mathfrak{R}; f)$ is not empty, then f *admits an admissible extension to* $\mathfrak{R}$.
- We view $\mathrm{Adm}(\mathfrak{R})$ as possible scaling limits for a height function.

Entropy Functional and Maximizers

- Define the *Lobachevsky function* $L : \mathbb{R}_{>0} \to \mathbb{R}$ by setting

$$L(x) = -\int_0^x \log|2\sin z| dz.$$

- Define the *surface tension* $\sigma : \overline{\mathcal{T}} \to \mathbb{R}$ by, for any $(s,t) \in \overline{\mathcal{T}}$, setting

$$\sigma(s,t) = \frac{1}{\pi}\Big(L(\pi s) + L(\pi t) + L\big(\pi(1-s-t)\big)\Big).$$

- For any $F \in \operatorname{Adm}(\mathfrak{R})$, define the (weakly concave) *entropy functional*

$$\mathcal{E}(F) = \int_{\mathfrak{R}} \sigma\big(\nabla F(z)\big) dz.$$

- If $\mathfrak{h} : \partial\mathfrak{R} \to \mathbb{R}$ admits an admissible extension to $\mathfrak{R}$, then $\mathcal{H} \in \operatorname{Adm}(\mathfrak{R};\mathfrak{h})$ is a *maximizer of $\mathcal{E}$ on $\mathfrak{R}$ with boundary data* $\mathfrak{h}$ if $\mathcal{E}(\mathcal{H}) \geq \mathcal{E}(\mathcal{G})$ for any $\mathcal{G} \in \operatorname{Adm}(\mathfrak{R};\mathfrak{h})$.

Variational Principle

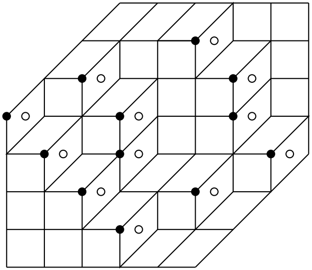
- Let $\mathfrak{R} \subset \mathbb{R}^2$ denote a simply-connected, bounded domain with piecewise smooth, simply boundary.
- Let $\mathfrak{h} : \partial\mathfrak{R} \rightarrow \mathbb{R}$ admit an admissible extension to $\mathfrak{R}$.
- Let $\mathcal{H} \in \mathrm{Adm}(\mathfrak{R}; \mathfrak{h})$ be the maximizer of $\mathcal{E}$ on $\mathfrak{R}$ with boundary data $\mathfrak{h}$.
- Let $R_1, R_2, \ldots \subset \mathbb{T}$ denote simply-connected, tileable domains with boundary height functions $h_1, h_2, \ldots$, respectively.
- Suppose that $\lim_{N\rightarrow\infty} N^{-1} R_N = \mathfrak{R}$.
- Define $\mathfrak{h}_N : \partial(N^{-1}R_N) \rightarrow \mathbb{R}$ by setting $\mathfrak{h}_N(N^{-1}u) = N^{-1}h_N(u)$ for each $u \in \partial R_N$.
- Suppose that $\lim_{N\rightarrow\infty} \mathfrak{h}_N = \mathfrak{h}$.

Cohn–Kenyon–Propp (2001): Let H_N denote the height function associated with a uniformly random lozenge tiling of R_N, with boundary height function h_N. Then, for any $\delta > 0$,

$$\lim_{N\rightarrow\infty} \mathbb{P}\left[\max_{v\in R_N} \left|N^{-1}H_N(v) - \mathcal{H}(N^{-1}v)\right| > \delta\right] = 0.$$

The Set $\mathcal{X}$

- If $\mathcal{M}$ is a tiling, then let $\mathcal{X} = \mathcal{X}(\mathcal{M})$ denote the set of all $(x, y) \in \mathbb{Z}^2$ such that $\left(x + \frac{1}{2}, y\right)$ is the center of some vertical lozenge in $\mathcal{M}$.
- The set $\mathcal{X}(\mathcal{M})$ determines $\mathcal{M}$.



- For any $\xi \in \mathbb{H}$ and $x_1, x_2, y_1, y_2 \in \mathbb{Z}$, the *extended discrete sine kernel* is

$$\mathcal{K}_\xi(x_1, y_1; x_2, y_2) = \frac{1}{2\pi\mathbf{i}} \int_{\overline{\xi}}^{\xi} (1 - z)^{y_1 - y_2} z^{x_2 - x_1 - 1} dz.$$

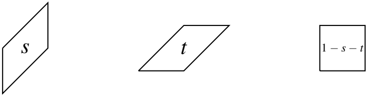
Infinite Volume Measures

- Okounkov–Reshetikhin (2003): For any $\xi \in \mathbb{H}$, there exists a probability measure μ_ξ on the set of tilings of $\mathbb{T}$ such that

$$\mathbb{P}\left[\bigcap_{k=1}^{m} \big\{(x_k, y_k) \in \mathfrak{X}(\mathcal{M})\big\}\right] = \det\big[\mathcal{K}_\xi(x_i, y_i; x_j, y_j)\big]_{1 \le i,j \le m},$$

  where $\mathcal{M} \in \mathfrak{E}(\mathbb{T})$ is sampled under μ_ξ.

- If $(s, t) \in \mathcal{T}$ and $\xi = e^{\pi \mathbf{i} s} \frac{\sin(\pi t)}{\sin(\pi - \pi s - \pi t)}$, then the proportion of the tiles below are s, t, and $1 - s - t$.



- The measure $\mu_{s,t}$ is an **translation-invariant, extremal Gibbs measure**.
- Its height function satisfies $\mathbb{E}\big[H(1,0) - H(0,0)\big] = s$ and $\mathbb{E}\big[H(0,1) - H(0,0)\big] = t$, and so its **slope** is (s, t).

Local Statistics Results

- Adopt the notation and assumptions in the variational principle.
- Let $\mathcal{M}_N$ denote a uniformly random lozenge tiling of R_N.
- Fix $\mathfrak{v} \in \mathfrak{R}$ such that $\nabla \mathcal{H}(\mathfrak{v}) \in \mathcal{T}$.
- Let $v_N \in R_N$ be such that $\lim_{N \to \infty} N^{-1} v_N = \mathfrak{v}$.
- Set $(s, t) = \nabla \mathcal{H}(\mathfrak{v})$.

Theorem (A., 2019)

The local statistics of $\mathcal{M}_N$ around v_N are given by $\mu_{s,t}$.

- Predicted by Cohn–Kenyon–Propp in 2001
- Universality in the domain, in that the limiting local statistics around v_N only depend on $\nabla \mathcal{H}(\mathfrak{v})$

Previous Results

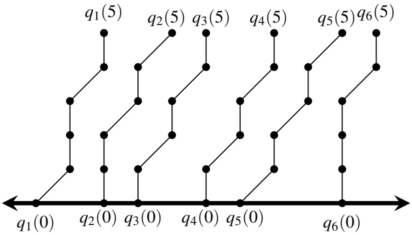
- Domains
  - Baik–Kreicherbauer–McLaughlin–Miller (2007), Gorin (2008): Hexagons
  - Petrov (2014): Trapezoids
  - Gorin (2017): Domains “covered” by trapezoids
  - Laslier (2017): Bounded perturbations of the above
- Many of these results are based on analysis of an *Kasteleyn matrix* $\mathbf{K} = \mathbf{K}_R$, which satisfies

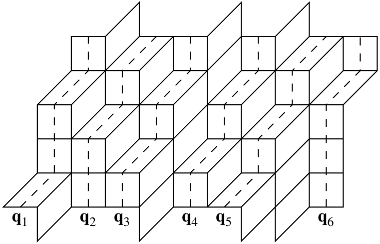
$$\mathbb{P}\left[\bigcap_{k=1}^{m} \{v_k \in \mathcal{X}(\mathcal{M})\}\right] = \det \left[\mathbf{K}^{-1}(v_i, v_j)\right]_{1 \le i,j \le m}.$$

- Issue: Inverse Kasteleyn matrix entries unstable under perturbations of R

Non-Intersecting Paths

- A *path* is a integer sequence $\mathbf{q} = \big(q(0), q(1), \ldots, q(t)\big)$ such that $q(i+1) - q(i) \in \{0, 1\}$ for each i.
- An ensemble $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n)$of paths is *non-inersecting* if $q_1(s) < q_2(s) < \cdots < q_n(s)$ for each s.





- Bijection between non-intersecting path ensembles and lozenge tilings

Random Non-Intersecting Path Ensembles

- Fix initial data $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ and $\beta \in (0, 1)$.
- Let $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n)$ be an ensemble of n Bernoulli random walks, with jump probability β, starting at $a_1, a_2, \ldots, a_n$ and conditioned to never intersect.
- Its probability distribution is given by

$$\mathbb{P}_{\beta;\mathbf{a}}[\mathbf{Q}] = \beta^{|\mathbf{q}(t)|-|\mathbf{a}|}(1-\beta)^{(m+n+1)t-|\mathbf{q}(t)|+|\mathbf{a}|} \prod_{-m\leq j<k\leq n} \frac{q_k(t)-q_j(t)}{a_k-a_j},$$

  if $\mathbf{Q}$ is non-intersecting and 0 otherwise, where $\mathbf{q}(t) = \big(q_1(t), q_2(t), \ldots, q_n(t)\big)$ and $|\mathbf{p}| = \sum_{p\in\mathbf{p}} p$.
- Conditional on the final data $\mathbf{q}(t)$, $\mathbf{Q}$ is uniform (for any β).

Universality Results for Non-Intersecting Random Walks

- The model $\mathbb{P}_{\beta;\mathbf{a}}$ is the discrete analog of $\beta = 2$ Dyson Brownian motion.
- Gorin–Petrov (2016): It is a determinantal point process with explicit kernel (discrete analog of Brézin–Hikami identity).
- Gorin–Petrov (2016): Suppose $1 \ll U \ll T \ll V \ll N$ are scales and $\mathbf{a} = (a_1, a_2, \ldots, a_N)$ is an initial data sequence that is approximately uniform on any length U subinterval of $[x_0 - V, x_0 + V]$. Then the local statistics of the non-intersecting random walk model $\mathbb{P}_{\beta;\mathbf{a}}$, run for time T, converge around site x_0 to a measure $\mu_{s,t}$.
- Discrete analog of results for Dyson Brownian motion by Erdős–Schnelli (2017), Landon–Yau (2017), and Landon–Sosoe–Yau (2019)

Outline

- Tileable $R = R_N \approx N\mathfrak{R} \subset \mathbb{T}$
- Uniformly random tiling $\mathcal{M} = \mathcal{M}_N$
- Associated height function $H : R \to \mathbb{Z}$
- Vertex $v = v_N \approx N\mathfrak{v}$ of R

We will prove universality by "locally comparing" $\mathcal{M}$ around v with a random non-intersecting path ensemble.

1. *Local Law*: Establish a local law for $\mathcal{M}$, that is, H is approximately linear with slope $\nabla\mathcal{H}(\mathfrak{v})$ on any mesoscopic scale.
2. *Comparison*: Exhibit a coupling between $\mathcal{M}$ and a non-intersecting random path ensemble $\mathbf{P}$ sampled under some $\mathbb{P}_{\beta;\mathbf{a}}$, such that the two models coincide around v with high probability.
3. *Universality*: Use results of Gorin–Petrov (and the local law) to show that the local statistics of $\mathbf{P}$ around v are universal, and conclude that the same holds for $\mathcal{M}$.

Analogous to "three-step strategy" in random matrix theory

The Local Law

Assume $\mathfrak{R} = \mathcal{B}$ and $\mathcal{B}_{N-2} \subset R \subset \mathcal{B}_N$ (but no assumptions on the boundary height function).

Proposition (A., 2019)

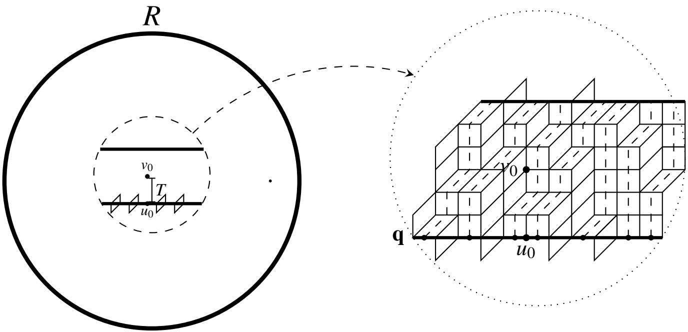
For $c = \frac{1}{20000}$ *and any* $1 \le M \le N$,

$$\mathbb{P}\left[\max_{|u-v|<M} \left| M^{-1}\big(H(u) - H(v)\big) - M^{-1}(u-v) \cdot \nabla \mathcal{H}(\mathfrak{v}) \right| > (\log M)^{-c} \right] < CM^{-D}.$$

- Proof is based on a combination of a multi-scale analysis with estimates obtained using the integrability of the tiling model.

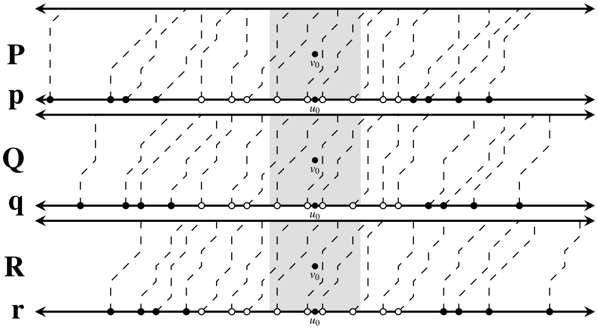
Outline of the Comparison

- Let $v_0 = (x_0, y_0) \in R$.
- Fix an integer $1 \ll T \ll N \sim \mathrm{diam}(R)$.
- Define the vertex $u_0 = v - (0, T) = (x_0, y_0 - T) \in R$.
- Interpret $\mathcal{M}$ as an ensemble $\mathbf{Q}$ of non-intersecting paths, and let $\mathbf{q}$ denote the locations where these paths intersect the horizontal line $\{y = y_0 - T\}$.



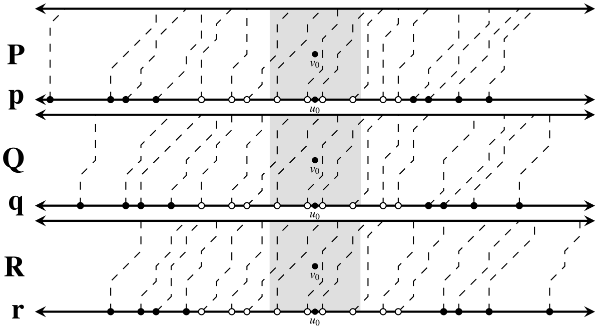
Outline of the Comparison

- Introduce particle configurations $\mathbf{p}$ and $\mathbf{r}$ that coincide with $\mathbf{q}$ near u_0, but are to the left and right of $\mathbf{q}$, respectively, away from u_0.
- Define two random path ensembles $\mathbf{P} \sim \mathbb{P}_{\beta_1;\mathbf{p}}$ and $\mathbf{R} \sim \mathbb{P}_{\beta_2;\mathbf{r}}$ with $\beta_1 \approx \beta_2$, and show that there exists a coupling between $(\mathbf{P}, \mathbf{Q}, \mathbf{R})$ such that $\mathbf{Q}$ is likely bounded between $\mathbf{P}$ and $\mathbf{R}$.



Outline of the Comparison

- Use identities from Gorin–Petrov to prove that the expected difference between the height functions associated with **P** and **R** tends to 0 in a large neighborhood of u_0 (containing v_0).
- Using the ordering between $(\mathbf{P}, \mathbf{Q}, \mathbf{R})$ and a Markov bound, conclude that one can couple them to coincide near v_0 with high probability.




Happy Birthday
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