Playing games with II_1 factors

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Introducing the game

- We fix a countably infinite set C of distinct symbols (*witnesses*) that are to represent generators of a separable tracial vNa that two players (traditionally named $\forall$ and $\exists$) are going to build together (albeit adversarially).
- The two players take turns playing finite sets of expressions of the form $|\|p(c)\|_2 - r| < \epsilon$, where c is a tuple of variables, $p(c)$ is a $*$-polynomial, and each player's move is required to extend the previous player's move. These sets are called (open) *conditions*.
- Moreover, these conditions are required to be *satisfiable*, meaning that there should be some vNa A and some tuple a from A such that $|\|p(a)\|_2 - r| < \epsilon$ for each such expression in the condition.

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Introducing the game (cont'd)

- We play this game for ω many steps.
- At the end of this game, we have enumerated some countable, satisfiable set of expressions.
- Provided that the players behave, they can ensure that the play is *definitive*, meaning that the final set of expressions yields complete information about all $*$-polynomials over the variables C (that is, for each $*$-polynomial $p(c)$, there should be a unique r such that the play of the game implies that $\|p(c)\| = r$) and that this data describes a countable, dense $*$-subalgebra of a unique vNa, which is often called the *compiled structure*.

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Enforceable properties

Definition

Given a property P of vNas, we say that P is an **enforceable** property is there a strategy for $\exists$ so that, regardless of player $\forall$'s moves, if $\exists$ follows the strategy, then the compiled structure will have that property.

Conjunction Lemma

If $(P_i : i \in \omega)$ are all enforceable properties, so is $\bigwedge_i P_i$.

It is natural to ask: are there any interesting enforceable properties of vNas?

Exercise

Being a locally universal II_1 factor is enforceable.

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An example of enforceability

Example

It is enforceable that the compiled vNa is a McDuff II_1 factor.

Proof.

- We use the fact that a separable II_1 factor A is McDuff if and only if, for every finite set $F \subseteq A$, there is a copy of $M_2(\mathbb{C})$ in A that almost commutes with F.
- Here's the strategy: suppose that $\forall$ played the open condition p that only mentions witnesses amongst $C_0 \subseteq C$ (finite).
- $\exists$ can respond by taking $(c_{ij}) \in C \setminus C_0$ and saying that (c_{ij}) are matrix units that almost commute with C_0.
- This is indeed a condition: if p were satisfied in A, then this new set of expressions is satisfiable in $A \otimes M_2(\mathbb{C})$.

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A crucial fact and a crucial definition

Definition

A vNa A is **enforceable** if the property of being isomorphic to A is enforceable.

- By the conjunction lemma, it thus follows that an enforceable algebra, should it exist, is necessarily a McDuff II_1 factor.
- We let $\mathcal{E}$ denote the enforceable II_1 factor, *should it exist*.
- If it exists, $\mathcal{E}$ is a *canonical* locally universal II_1 factor.

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Some examples

Example

The random graph is the enforceable graph.

Example

With respect to fields of some fixed characteristic p, the algebraic closure of the prime field is the enforceable structure.

Example

There is an enforceable Banach space, the *Gurarij Banach space*.

Non-example

There is no enforceable group. (Highly nontrivial!)

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CEP and enforceable models

Theorem

The following are equivalent:

1. *CEP has a positive solution.*
2. *Hyperfiniteness is an enforceable property.*
3. $\mathcal{R}$ *is the enforceable* II_1 *factor.*
4. $\mathcal{R}^{\mathcal{U}}$*-embeddability is enforceable.*

The dichotomy theorem

Theorem

Exactly one of the two possibilities occurs:

- *There is an enforceable vNa; or*
- *Chaos: for every enforceable property P of vNas, there are $2^{\aleph_0}$ many pairwise nonisomorphic vNas with property P.*

Intriguing Question

Suppose that we know that $\mathcal{E}$ exists. Must it be the case that $\mathcal{E} \cong \mathcal{R}$?

- If not, then $\mathcal{E}$ rivals $\mathcal{R}$ as the most canonical separable II_1 factor (and CEP is false).
- Evidence?
  - $\mathcal{E}$ embeds into all e.c. II_1 factors.
  - Every embedding of $\mathcal{E}$ into $\mathcal{E}^{\mathcal{U}}$ is elementary.

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Square roots and CEP

Definition

A vNa A is a **tensor square** or **has a tensor square root** if there is a vNa B such that $A \cong B\overline{\otimes}B$.

Clearly $\mathcal{R}$ is a tensor square.

Theorem (G.; G.-Sinclair; Connes)

CEP holds if and only if the property of being a tensor square is enforceable.

Key ingredient: Having every automorphism approximately inner is enforceable.

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Elementary equivalence

Evil Definition

II_1 factors M and N are **elementary equivalent**, denoted $M \equiv N$, if there is an ultrafilter $\mathcal{U}$ such that $M^{\mathcal{U}} \cong N^{\mathcal{U}}$.

Facts and Examples

1. (Farah-Hart-Sherman) Given any separable M, there are $2^{\aleph_0}$ many pairwise nonisomorphic separable N such that $M \equiv N$.
2. (Farah-Hart-Sherman) If M has Γ (resp. is McDuff) and N does not have Γ (resp. is not McDuff), then $M \not\equiv N$.
3. (Boutonnet-Chifan-Ioana) There are separable M_α ($\alpha \in 2^\omega$) such that $M_\alpha \not\equiv M_\beta$ for $\alpha \neq \beta$.

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First-order Dye's Theorem

Observation

$M \equiv N$ if and only if $U(M) \equiv U(N)$ (as metric groups).

Proof.

The following are equivalent:

1. $M \equiv N$
2. $(\exists \mathcal{U}) M^{\mathcal{U}} \cong N^{\mathcal{U}}$
3. $(\exists \mathcal{U}) U(M^{\mathcal{U}}) \cong U(N^{\mathcal{U}})$
4. $(\exists \mathcal{U}) U(M)^{\mathcal{U}} \cong U(N)^{\mathcal{U}}$
5. $U(M) \equiv U(N)$.

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The equivalence of (2) and (3) is by Dye's Theorem.

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The class $\mathcal{K}_{\mathrm{op}}$

Definition

We let $\mathcal{K}_{\mathrm{op}}$ denote the class of M such that $M \equiv M^{\mathrm{op}}$.

Remark

$\mathcal{K}_{\mathrm{op}}$ is an axiomatizable class.

Question

Does every II_1 factor belong to $\mathcal{K}_{\mathrm{op}}$?

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A strengthening

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Theorem (G.-Sinclair)

For $M, N \in \mathcal{K}_{op}$, we have $M \equiv N$ if and only if $U(M) \equiv U(N)$ as $\mathbb{Z}_4$-metric spaces.

- A $\mathbb{Z}_4$-metric space is a metric space together with an action of $\mathbb{Z}_4$ by isometries.
- We consider $U(M)$ as a $\mathbb{Z}_4$-metric space by letting the generator act by multiplication by i.

Question

Is it true that $M \equiv N$ if and only if $U(M) \equiv U(N)$ as metric spaces?

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Banach pairs

Definition

A **Banach pair** is a pair $(X, \mathcal{C})$, where X is a normed space and $\mathcal{C} \subseteq (X)_1$ are such that:

- $\mathcal{C}$ is complete;
- for all $x, y \in \mathcal{C}$ and $\lambda, \mu \in \mathbb{C}$ with $|\lambda| + |\mu| \leq 1$, we have $\lambda x + \mu y \in \mathcal{C}$;
- $X = \bigcup_n n \cdot \mathcal{C}$.

Main example

If M is a II_1 factor, then $(M, (M)_1)$ is a Banach pair, where M is considered a normed space in the 2-norm.

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A game for Banach pairs

Given $n \in \mathbb{N}$, $\epsilon > 0$, we describe the game $\mathcal{G}(n, \epsilon)$ played by two players with Banach pairs $(X, \mathcal{C})$ and $(Y, \mathcal{D})$:

- Player I chooses a one-dimensional subspace, either $E_1 \subset X$ or $F_1 \subset Y$. Player II then chooses a subspace, respectively $F_1 \subset Y$ or $E_1 \subset X$, and a linear bijection $T_1 : E_1 \to F_1$.
- At round i, Player I chooses an at most one-dimensional extension, either $E_i \supset E_{i-1}$ or $F_i \supset F_{i-1}$. Player II then chooses a subspace, respectively $F_i \subset Y$ or $E_i \subset X$, and a linear bijection $T_i : E_i \to F_i$ which extends T_{i-1}.
- The players make their choices for n rounds. Player II wins if $T_n : E_n \to F_n$ is an ϵ-almost isometry; otherwise, Player I wins.

Proposition

$(X, \mathcal{C}) \equiv (Y, \mathcal{D})$ if and only if player II has a winning strategy for $\mathcal{G}(n, \epsilon)$ for all n and ϵ.

A game for Banach pairs

Given $n \in \mathbb{N}$, $\epsilon > 0$, we describe the game $\mathcal{G}(n, \epsilon)$ played by two players with Banach pairs $(X, \mathcal{C})$ and $(Y, \mathcal{D})$:

- Player I chooses a one-dimensional subspace, either $E_1 \subset X$ or $F_1 \subset Y$. Player II then chooses a subspace, respectively $F_1 \subset Y$ or $E_1 \subset X$, and a linear bijection $T_1 : E_1 \to F_1$.
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First reduction

Fact (Kirchberg)

If M and N are II_1 factors and there is an isometry $T : L^2(M) \to L^2(N)$ that maps M onto N contractively, then either $M \cong N$ or $M \cong N^{\mathrm{op}}$.

Definition

We say that M and N are **locally equivalent**, denoted $M \equiv_{\mathrm{loc}} N$, if $(M, (M)_1) \equiv (N, (N)_1)$ as Banach pairs.

Corollary

$M \equiv_{\mathrm{loc}} N$ if and only if $M \equiv N$ or $M \equiv N^{\mathrm{op}}$. In particular, if $M, N \in \mathcal{K}_{\mathrm{op}}$, then $M \equiv_{\mathrm{loc}} N$ if and only if $M \equiv N$.

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We define the game $\mathcal{G}_{\mathrm{vN}}(n, \epsilon)$ played by two players with II_1-factors M and N as follows:

- At stage i, Player I chooses a unitary either $u_i \in U(M)$ or $v_i \in U(N)$. Player II then chooses a unitary, respectively $v_i \in U(N)$ or $u_i \in U(M)$ in the same manner.
- The players make their choices for n rounds. Player II wins if $|\langle u_i, u_j \rangle - \langle v_i, v_j \rangle| < \epsilon$ for all $1 \leq i, j \leq n$; otherwise, Player I wins.

Theorem

$M \equiv_{\mathrm{loc}} N$ if and only if Player II has a winning strategy for the game $\mathcal{G}_{\mathrm{vN}}(n, \epsilon)$ for all n and ϵ.

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Proof of the Main Theorem

- By first-order Dye, if $M \equiv N$, then $U(M) \equiv U(N)$ as metric groups, and thus as $\mathbb{Z}_4$-metric spaces.
- Now suppose that $U(M) \equiv U(N)$ as $\mathbb{Z}_4$-metric spaces.
- Since

$$\Re\langle u_i, u_j\rangle = 1 - \frac{1}{2}d(u_i, u_j)^2, \quad \Im\langle u_i, u_j\rangle = 1 - \frac{1}{2}d(u_i, i \cdot u_j)^2,$$

  we know that the ability to win the ordinary EF-game between $U(M)$ and $U(N)$ as $\mathbb{Z}_4$-metric spaces allows us to win the games $\mathcal{G}_{\mathrm{vN}}(n, \epsilon)$, whence $M \equiv_{\mathrm{loc}} N$.
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The infinite forcing game $\mathcal{G}(M, \varphi, r)$

Assume CEP.

- Suppose that M is a II_1 factor and φ is a $\forall_n$ sentence with parameters from M, that is, one of the form

$$\sup_{x_1} \inf_{x_2} \cdots Q_{x_n} \theta_\varphi(x_1, \ldots, x_n),$$

where θ_φ is an expression of the form

$$\max_{1 \leq i \leq m} \left| \|p_i(\vec{x})\|_2 - r_i \right|,$$

with each p_i a $*$-polynomial with coefficients from M.

- Further suppose that $r > 0$ is a positive number.
- We define a two player game $\mathcal{G}(M, \varphi, r)$ as follows: The players take turns playing pairs (M_i, a_i), with each M_i a II_1 factor, $M \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n$, and $a_i \in M_i$ for each i. (Note that if n is odd, then the game ends with a move by player I.).
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Definition

1. $M \Vdash \varphi \leq r$ if and only if player II has a winning strategy in $\mathcal{G}(M, \varphi, r)$.
2. $V^M(\varphi) = \inf\{r \ : \ M \Vdash \varphi \leq r\}$.

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Definition

M is called **infinitely generic** if $V^M(\varphi) = v^M(\varphi)$ for all φ.

Lemma

M is infinitely generic if and only if: for every φ, every $\bowtie \in \{\leq, \geq\}$, and every r,

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Fact

Infinitely generic II_1 factors exist. In fact, every II_1 factor is a subfactor of an infinitely generic factor of the same density character.

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Question

Is $\mathcal{R}$ infinitely generic?

- Any two infinitely generic II_1 factors are elementarily equivalent.
- It thus turns out that the above question is equivalent to knowing that $\mathcal{R}$ is elementarily equivalent to an infinitely generic II_1 factor.
- If the question has a negative answer, this would be interesting as then we would have our first example of two non-elementarily equivalent existentially closed II_1 factors.

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