The Monge problem in Brownian stopping optimal transport

Young-Heon Kim

University of British Columbia

Based on joint work with

Nassif Ghoussoub and **Aaron Zeff Palmer** (UBC).

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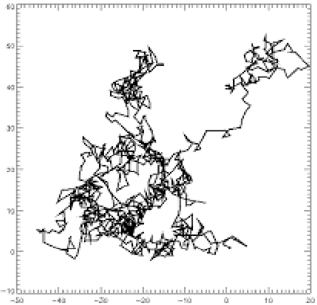
Brownian motion and stopping time

- **Brownian motion**:



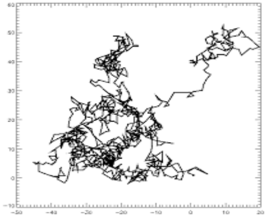
from CRM-physmath

- A **stopping time** τ of Brownian motion is, roughly speaking, a random time, prescribed to satisfy a certain probabilistic condition, at which one stops a particle following the Brownian motion.

Brownian motion and stopping time

[Skorokhod problem in R^n]
For given probability measures μ, ν, does there exist a **stopping time** τ of the Brownian motion such that

$$B_0 \sim \mu \quad \& \quad B_\tau \sim \nu?$$



from CRM-physmath

Remark:

- For such a stopping time τ to exist (with $\mathbb{E}[\tau] < \infty$), we need
  - μ and ν are in **subharmonic** order, $\mu \prec_{SH} \nu$, i.e. $\int \xi d\mu \leq \int \xi d\nu$, $\forall$ subharmonic $\xi : \mathbb{R}^n \to \mathbb{R}$ ($\Delta \xi \geq 0$).

Skorokod problem

[Skorokhod problem in R^n]
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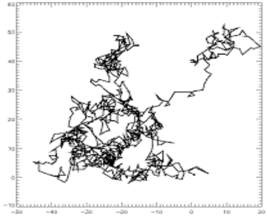


from CRM-physmath

- [Skorokhod] [Root] [Rost] [Azéma&Yor] [Vallois] [Perkins] [Jacka] ...[Obloj]...
- [Hobson] .. ....
- [Beiglеböck, Cox, & Huesmann '13].
  - **Optimal transport** unifies the previous results on Skorokhod problem.
- And many many more people.

Optimal Skorokhod problem

transportation cost $c(x,y)$.

- e.g. $c(x,y) = |x-y|$.
- Can also consider cost $\mathbb{E}\left[\int_0^{\tau} L(t,B_t)dt\right]$, etc.

Question: What can we say
about an **optimal** stopping time τ for

$$P_c(\mu,\nu) := \inf_{\tau}\{ \mathbb{E}\,[c(B_0,B_\tau)] \quad | \quad B_0 \sim \mu \quad \& \quad B_\tau \sim \nu\}?$$

- Existence?
- Uniquenss?
- Any extremal structure?
  - Does τ drop mass only in lower dimensional sets (called barrier) ?

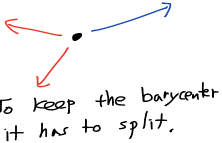
Martingale optimal transport:

Optimal Skorokhod problem is a special case of martingale optimal transport:

- The joint distribution $\pi \sim (B_0, B_\tau)$ is martingale: The distribution $\pi_x \sim B^x_\tau$ satisfies martingale constraint for $x \mapsto \pi_x \in P(\mathbb{R}^n)$:

$$\int y \, d\pi_x(y) = x.$$

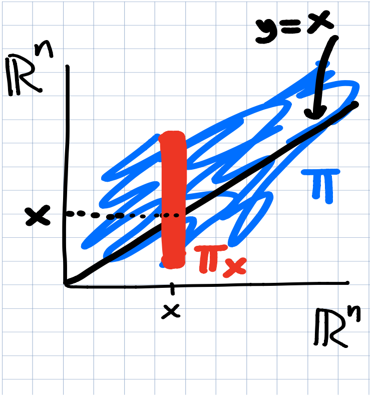
(Branches out while keeping the barycentre.)



Martingale optimal transport

- $MT(\mu, \nu)$:
  probability measures π on $\mathbb{R}^n \times \mathbb{R}^n$
  with the marginals μ, ν,
  such that its disintegration $(\pi_x)_{x \in \mathbb{R}^n}$ has barycenter at x
  (martingale constraint):
  $\int y d\pi_x(y) = x$.

$$\inf_{\pi \in MT(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x,y) d\pi(x,y).$$



Remark: [Strassen]

- $MT(\mu, \nu) \neq \emptyset$

$\Leftrightarrow$ μ and ν are in convex order;
$\mu \prec_C \nu$, i.e. $\int \xi d\mu \leq \int \xi d\nu$, $\forall$ convex $\xi : \mathbb{R}^n \to \mathbb{R}$.

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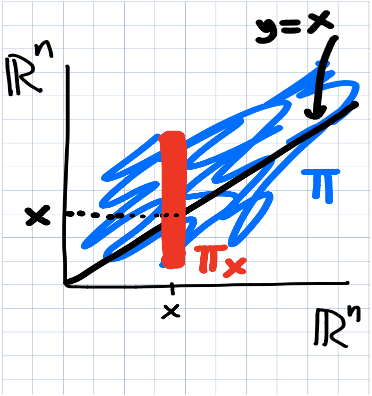

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Optimal transport

Martingale optimal transport is optimal transport with the additional martingale constraint.

- $T(\mu, \nu)$:
  probability measures π on $\mathbb{R}^n \times \mathbb{R}^n$
  with the marginals μ, ν.

Monge-Kantorovich problem:

$$\inf_{\pi \in T(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y) d\pi(x, y).$$

Many people contributed to this theory and related problems in PDE, geometry, probability, ..., machine learning, etc:
[Monge][Kantorovich][Brenier][McCann][Delanoë][Urbas] [Caffarelli] [Evans-Gangbo][Gangbo-McCann] [Trudinger-Wang] [Ambrosio][Otto][Villani] ......[Figalli] ......

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Some motivating comments

Many breakthroughs I know in optimal transport (OT) came when it meets with other areas:

- economics: matching theory
- fluids: Brenier theory
- physics of gas and crystals: McCann's displacement convexity
- diffusion phenomena: Otto calculus.
- Ricci curvature: Lott-Villani-Sturm theory.
- machine learning: Wasserstein GAN
- density functional theory ..
- general relativity ..
- stem cell research ..

Q. What if OT meets convex integration?

Let us get back to the discussion of martingale optimal transport and Skorokhod problem.

Martingale optimal transport:

- Backhoff, Bayraktar, Beiglböck, Bouchard, Claisse, Cox, Davis, Dolinsky, De March, Galichon, Ghoussoub, Griessler, Guo, Henry-Labordère, Hobson, Hu, Huesmann, Juillet, Kallblad, K., Klimmek, Lim, Neuberger, Nutz, Oblój, Palmer, Penkner, Perkowski, Proemel, Schachermayer, Siorpaes, Soner, Spoida, Stebegg, Tan, Touzi, Zaev, and many more people$\cdots\cdots$.

Martingale optimal transport vs. optimal Skorokhod problem

- π is martingale
  $\iff$

  $$\int \psi(y) d\pi_x(y) \geq \psi(y) \text{ for any convex function } \psi.$$

- [Ghoussoub, K., & Lim '17]
  $\pi \sim (B_0, B_\tau)$ for a (randomized) stopping time τ
  $\iff$ π is subharmonic martingale:

  $$\int \psi(y) d\pi_x(y) \geq \psi(y) \text{ for any subharmonic function } \psi.$$

They are the same in $1D$:

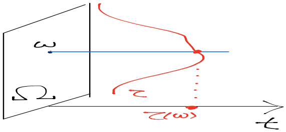
- 1D $\Rightarrow$ subharmonic = convex.

Different in general dimensions.

Randomized stopping time

Let $\Omega := C(\mathbb{R}_{\geq 0}; \mathbb{R}^n)$.

Stopping time
is a measurable **function** τ on the probability space $(\Omega, \mathbb{P}^\mu)$.
($\mathbb{P}^\mu$= the Wiener measure with $B_0 \sim \mu$).



Randomized stopping time
is a probability **measure** τ on the space $\mathbb{R}_{\geq 0} \times \Omega$,
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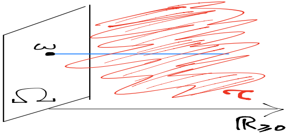

A (nonradomized) stopping time gives Dirac mass along each path.

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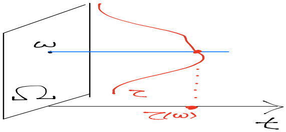

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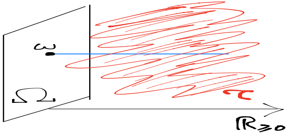

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Optimal Skorokhod problem: Kantorovich solution (a measure-valued solution)

- **[Beiglböck, Cox & Huesmann '13]**
  Randomized stopping times give
  Kantorovich relaxation to optimal Skorokhod problem.

  - **The set of randomized stopping times** from μ to ν is nonempty if $\mu \prec_{SH} \nu$.
  - Space of randomized stopping times is compact: weak* -**compactness** of the space of probability measures.
  - Optimal randomized stopping time exists through lower semi-continuity of the functional $\tau \to \mathbb{E}[c(B_0, B_\tau)]$ over **randomized stopping times.**

Optimal Skorokhod problem: Monge solution?

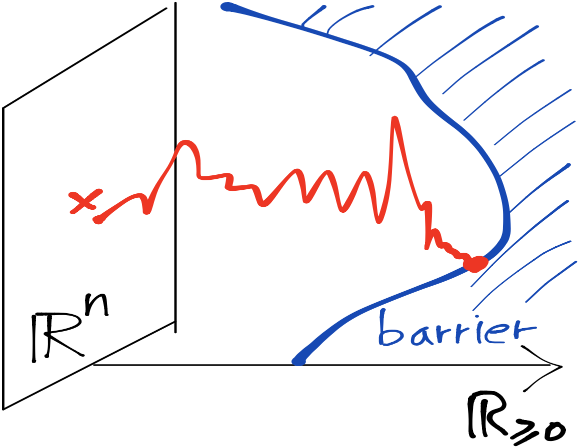
- **Question:**
  - When is the optimal Kantorovich solution a Monge solution?
    - In what case, does the optimal randomized stopping time become pure, that is, non-randomized stopping time?
  - Any associated structure?

Optimal Skorokhod problem: Monge solutions (non-randomized stopping)

- [Beigleböck, Cox, & Huesmann '13].
  - Some **variational** tools, called monotonicity principle, comparing different paths.
  - geometric structures for the cost $\mathbb{E}\left[\int_0^\tau L(t)dt\right]$.
    - Stopping time is given by hitting a certain **barrier**.
- [Ghoussoub, K. & Palmer '18]. For the cost $\mathbb{E}\left[\int_0^\tau L(t, B_t)dt\right]$.
  - Some **analytical** tools based on dual formulation.
    - **dual attainment**
  - geometric structures
    - Stopping time is determined by hitting a certain **barrier given by the optimal dual function**.

Hitting time to a barrier in $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$

Barrier looks like the graph of a function on $\mathbb{R}^n$.



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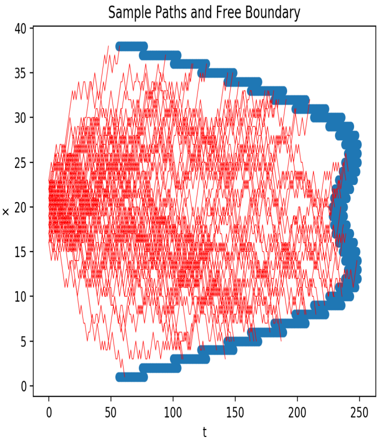
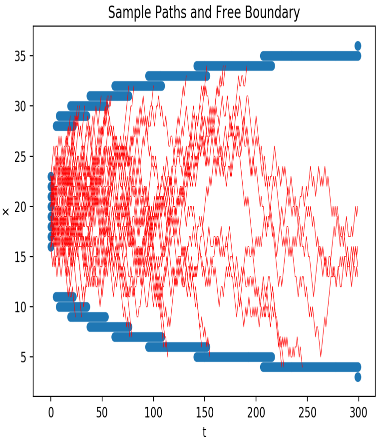




hitting from below

hitting from above

$c(x, y)$

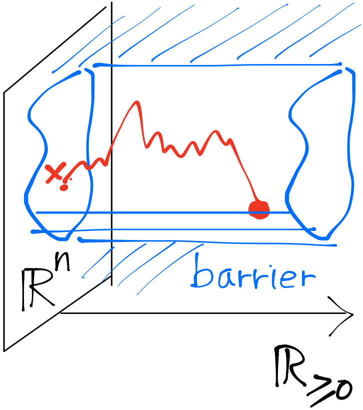
- [Ghoussoub, K. & Lim '17]
  - For $c(x, y) = |x - y|^p, p > 1, p \neq 2$: geometric structures when μ, ν are **radially symmetric** in $\mathbb{R}^n$.
    - Stopping time is given by hitting a certain barrier.
- [Ghoussoub, K. & Palmer '19]
  - Some **analytical** tools based on dual formulation.
    - **dual attainment**
  - For $c(x, y) = |x - y|$, geometric structures for **general** cases in $\mathbb{R}^n$.
    - Stopping time is given by hitting a certain **barrier determined by the optimal dual function**.

Hitting time to a barrier in $\mathbb{R}^n \times \mathbb{R}^n$

The barrier depends on the starting point $x \in R^n$.

Hitting time to a space-time barrier

The barrier (depending on the starting point *x*) looks like a vertical wall in the space-time.



Fundamental tool:

Duality and dual attainment

We will focus on the case:

- $dim \geq 2$.
- $c(x, y) = |x - y|$.

Assume:

- O bounded open convex set in $\mathbb{R}^n$.
- $\operatorname{supp} \mu, \operatorname{supp} \nu \subset O$
- $c \in C(\overline{O} \times \overline{O})$

Duality for OT with probabilistic constraints

Theorem

Weak duality: $P_c(\mu,\nu) = D_c(\mu,\nu)$.

- $P_c(\mu,\nu) := \inf\{\, \mathbb{E}\,[c(B_0, B_\tau)] \mid B_0 \sim \mu \quad \& \quad B_\tau \sim \nu\}$
- $D_c(\mu,\nu) := \sup\limits_{\psi,\phi,p} \left\{ \int \psi(y)d\nu(y) - \int \phi(x)d\mu(x) \right\}.$

  while
  - $\psi(y) - \phi(x) + p(x,y) \le c(x,y) \forall x, y$ and
  - $y \mapsto p(x,y)$ **subharmonic** and $p(x,x) = 0$.

Question: Dual attainment? (Does the dual optimizer (ψ, ϕ, p) exist?)

Dual attainment?

$$\sup_{\psi(y)-\phi(x)+p(x,y)\leq c(x,y)\forall x,y} \left\{ \int \psi(y)d\nu(y) - \int \phi(x)d\mu(x) \right\}.$$

$y \mapsto p(x,y)$ **subharmonic** and $p(x,x) = 0$.
This additional term $p(x,y)$ adds **non-compactness** of the problem for the dual attainment.

Remark

dim$=1$: *Dual attainment is shown* **[Beiglböck, Nutz, & Touzi] [Beiglböck, Lim & Obloj]**.

dim≥ 2: *We show dual attainment* **(for the Skorokhod problem)** *for a certain class of c; e.g.* $|x-y|$.

dim≥ 2: **For martingale transport** *($y \mapsto p(x,y)$ convex), dual attainment is* **open in general.**

Dual attainment?

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‘Brownian’ optimal tranport dual attainment

For attainment of the dual problem $D_c(\mu, \nu)$, we want to **reduce it to a compact set of functions.**

- Will use **dynamic programming** for the duality.
- Will find a **normalization** for the functions ψ.

Duality via dynamic programming

Theorem:

$$\inf\{\,\mathbb{E}\,[c(B_0,B_\tau)] \quad | \quad B_0 \sim \mu \quad \& \quad B_\tau \sim \nu\}$$
$$= \sup_{\psi \in LSC(\overline{O})} \left\{ \int \psi(y)d\nu(y) - \int J_\psi(x,x)d\mu(x) \right\}.$$

The value function:

$$J_\psi(x,y) = \sup_{\tau \le \tau_O} \mathbb{E}\,[\psi(B_\tau^y) - c(x,B_\tau^y)]$$

Notation:

- τ: (randomized) stopping time.
- τ_O the exit time of O:

$$\tau_O = \inf\{t \mid B_t \notin O\}.$$

The value function:

$$J_\psi(x, y) = \sup_{\tau \le \tau_O} \mathbb{E}\left[\psi(B^y_\tau) - c(x, B^y_\tau)\right].$$

Remark: Compare

- the usual value function in dynamic programming:
  $$J_\psi(t, y) = \sup_{\tau \ge t \,\&\, B_t = y} \mathbb{E}\left[\psi(B_\tau) - \int_t^\tau L(s, B_s)ds\right].$$
- $J_\psi(x, x)$ with the c-Legendre transform
  $\psi^c(x) = \sup_y [\psi(y) - c(x, y)]$.

Dynamic programming principle

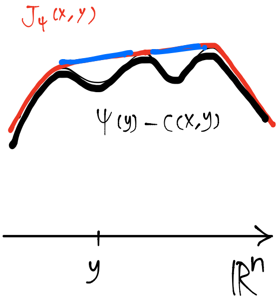
For the value function:

$$J_\psi(x,y) = \sup_{\tau \le \tau_O} \mathbb{E}\left[\psi(B^y_\tau) - c(x, B^y_\tau)\right]$$

We have

Dynamic programming principle:

- $y \mapsto J(x,y)$ is the smallest superharmonic function over $y \mapsto \psi(y) - c(x,y)$.



We will reduce the dual maximization problem for ψ,

$$D_c(\mu,\nu) = \sup_{\psi \in LSC(\overline{O})} \left\{ \int \psi(y) d\nu(y) - \int J_\psi(x,x) d\mu(x) \right\}.$$

to a compact function space, say $\mathcal{B}_D$:

$$D_c(\mu,\nu) = \sup_{\psi \in \mathcal{B}_D} \left\{ \int \psi(y) d\nu(y) - \int J_\psi(x,x) d\mu(x) \right\}.$$

Assumptions for dual attainment

Assume further

- $0 \leq \Delta_y c(x, y) \leq D$ (in the sense of viscosity)
- $\mu \prec_{SH} \nu$
- $\mu \in H^{-1}(O)$

Remark

- *OK with $\Delta_y c(x, y) \geq -M$.*
  Let $\tilde{c}(x, y) = c(x, y) + h(y)$, with h solving $\Delta h = M$, then, an optimizer $\tilde{\psi}$ for $\tilde{c}$ $\Leftrightarrow$ an optimizer $\tilde{\psi} - h$ for c.
- *$\Delta_y |x - y| = \infty$ along $x = y$. But, still can handle this by reducing to the case $\operatorname{supp} \mu \cap \operatorname{supp} \nu = \emptyset$.*
- *counterexample to dual attainment [Beiglböck/Juillet] if $\Delta_y c(x, y) = -\infty$ (e.g. $c(x, y) = -|x - y|$).*

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- $\Delta_y |x - y| = \infty$ *along* $x = y$. *But, still can handle this by reducing to the case* $\operatorname{supp} \mu \cap \operatorname{supp} \nu = \emptyset$.
- *counterexample to dual attainment [Beiglböck/Juillet] if* $\Delta_y c(x, y) = -\infty$ *(e.g.* $c(x, y) = -|x - y|$*).*

A key compact function space

Definition

$$\psi \in \mathcal{B}_D \Longleftrightarrow \begin{cases} \psi \in H_0^1(O), \\ \psi \leq 0, \textit{ and} \\ \Delta\psi(y) \leq D \text{ (weakly).} \end{cases}$$

Remark

The class $\mathcal{B}_D$ is weakly **compact**
as $\|\psi\|_{H_0^1(O)} \leq M \quad$ for all $\psi \in \mathcal{B}_D$.

Want to normalized ψ to a function in $\mathcal{B}_D$.

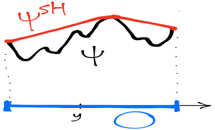
A key lemma for dual attainment

- Value function:

$$J_\psi(x,y) = \sup_{\tau \le \tau_O} \mathbb{E}\left[\psi(B^y_\tau) - c(x, B^y_\tau)\right]$$

Lemma

Assume $\Delta_y c(x,y) \ge 0$. ($\leftarrow$ essential!)
the superharmonic envelope
$\psi^{SH}(y) = \sup_{\tau \le \tau_O} \mathbb{E}[\psi(B^y_\tau)]$



Then,

$$J_{\psi - \psi^{SH}}(x,y) \le J_\psi(x,y) - \psi^{SH}(y) \quad \forall x, y \in O \times O.$$

(In fact, $=$.)

Normalization for dual attainment

- $\psi \longrightarrow \tilde{\psi} := \psi - \psi^{SH}$. Then, $\tilde{\psi} \leq 0$ in O and $\tilde{\psi} = 0$ on ∂O.
- $$\int (\psi - \psi^{SH}) d\nu - \int \underbrace{J_{\psi - \psi^{SH}}}_{\leq J_\psi - \psi^{SH}} d\mu \geq \int \psi d\nu - \int J_\psi d\mu.$$
  (Used $\Delta_y c(x, y) \geq 0$ as well as $\mu \prec_{SH} \nu$ here.)
  Dual value increases.
- $\bar{\psi}(y) := \inf_x [J_{\tilde{\psi}}(x, y) + c(x, y)]$.
  - Dual value increases ($\bar{\psi} \geq \tilde{\psi}$ & $J_{\bar{\psi}} = J_{\tilde{\psi}}$.)
- $\bar{\psi} \in \mathcal{B}_D$:
  - $\bar{\psi} \leq 0$ in O & $\bar{\psi} = 0$ on ∂O.
  - $\Delta_y c(x, y) \leq D \Longrightarrow \Delta_y \bar{\psi}(y) \leq D$.

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  (Used $\Delta_y c(x, y) \ge 0$ as well as $\mu \prec_{SH} \nu$ here.)
  Dual value increases.
- $\bar{\psi}(y) := \inf_x [J_{\tilde{\psi}}(x, y) + c(x, y)]$.
  - Dual value increases ($\bar{\psi} \ge \tilde{\psi}$ & $J_{\bar{\psi}} = J_{\tilde{\psi}}$.)
- $\bar{\psi} \in \mathcal{B}_D$:
  - $\bar{\psi} \le 0$ in O & $\bar{\psi} = 0$ on ∂O.
  - $\Delta_y c(x, y) \le D \Longrightarrow \Delta_y \bar{\psi}(y) \le D$.

Dual attainment

Theorem

There exists $\psi^* \in \mathcal{B}_D$ that attains the maximum value of the dual problem, i.e.,

$$\mathcal{D}_c(\mu,\nu) = \int_O \psi^*(y)\nu(dy) - \int_O J_{\psi^*}(x,x)\mu(dx).$$

Optimal paths stop in the contact set

- τ^* optimal (randomized) stopping time.
- $\pi^* \sim (B_0, B_{\tau^*})$. the optimal plan.

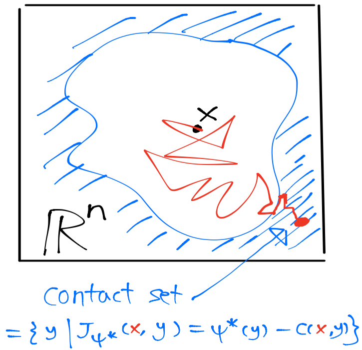
Theorem

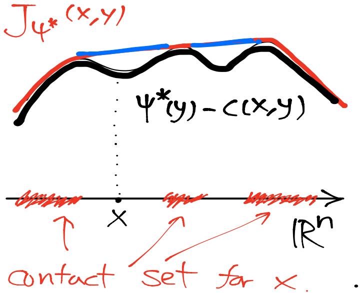
The optimal Brownian path stops at the contact set, namely, for π^*-a.e. $(x, y) \in O \times O$,

$$J_{\psi^*}(x, y) = \psi^*(y) - c(x, y).$$

In particular
$\tau^* \geq \eta := \inf\{t \;;\; J_{\psi^*}(B_0, B_t) = \psi^*(B_t) - c(B_0, B_t)\}$.

Optimal paths stop in the contact set

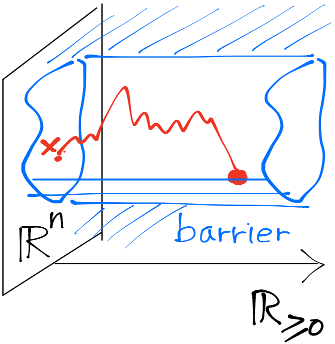




An optimal path stops at the contact set, but, may enter inside, not necessarily stopping when it hits the boundary.

Hitting a barrier and stopping immediately?

Monge solution:
characterization as the hitting time to a barrier.



We need a key condition called,
the **stochastic twist condition**.

Stochastic twist condition

Definition

the **stochastic twist (ST) condition** at (x, y):
$\forall$ stopping time ξ

$$\mathbb{E}\left[\nabla_x c(x, B_\xi^y)\right] = \nabla_x c(x, y) \quad \Longrightarrow \quad \xi = 0.$$

Remark

- *Compare with the usual twist condition in optimal transport:* $\nabla_x c(x, y_1) = \nabla_x c(x, y_2) \Longrightarrow y_1 = y_2$.
- **[Henry-Labordere & Touzi '16] the martingale counterpart of the Spence-Mirrlees condition***:*

$$c_{yyx}(x, y) > 0, \quad x, y \in \mathbb{R}^1.$$

Stochastic twist condition: Examples

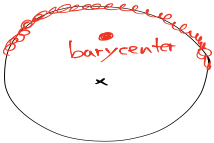
The quadratic cost $c(x, y) = |x - y|^2$ does not satisfy **ST**, because $\nabla_x |x - y|^2 = 2(x - y)$,

Example

- **[Lim]** $c(x, y) = |x - y|$ because $\nabla_x c(x, y) = \frac{x-y}{|x-y|} \in S^{n-1}$.
- Riemannian distance $c(x, y) = d(x, y)$ (as long as it is differentiable).
- Separable costs

$$c(x, y) = g(x)h(y)$$

with $\nabla g(x) \neq 0$ and $y \mapsto h(y)$ is either strictly superharmonic or strictly subharmonic.



Monge solution: the hitting time to a barrier

Theorem

Suppose **additionally**

- *c satisfies the stochastic twist condition (***ST***) for all* $(x, y) \in O \times O$.
- $\mu \ll$ *Leb*, $\mu(\partial \mathrm{supp}\, \mu) = 0$, *and* $\mu \wedge \nu = 0$.

Then,

$\exists$ *unique optimal stopping time* τ^**:*

$$\tau^* = \eta := \inf\{t;\ J_{\psi^*}(B_0, B_t) = \psi^*(B_t) - c(B_0, B_t)\}.$$

Monge solution: the hitting time to a barrier

Theorem

Assume

- $c(x, y) = |x - y|$, *dim* ≥ 2,
- $\operatorname{supp} \mu \cap \operatorname{supp} \nu = \emptyset$.

Then,

- $\exists$ *a constant D and* $\psi^* \in \mathcal{B}_D$ *such that* (ψ^*, J_{ψ^*}) *maximize the dual problem.*
- $\exists$ **unique** *optimal stopping time* τ^**:*

$$\tau^* = \eta = \inf\{t; J_{\psi^*}(B_0, B_t) = \psi^*(B_t) - |B_0 - B_t|\}.$$

Remark

May allow $\operatorname{supp} \mu \cap \operatorname{supp} \nu \neq \emptyset$*, but, the barrier will not be determined by a single dual optimizer* ψ^**.*

Monge solution: the hitting time to a barrier

Theorem

Assume

- $c(x, y) = |y - x|$ *and* $d \geq 2$
- $\mu \prec_{SH} \nu$, *and* μ *and* ν *have densities* $f \in C(\overline{O})$ *and* $g \in C(\overline{O})$,

Then

- $\exists!$ *optimal stopping time* τ^* *that is randomized* **only at time** 0.
- $\tau^* = 0$ *with density* $g \wedge f$ *and otherwise* τ^* *is the hitting time* η,

$$\eta = \inf\{t > 0;\ (B_0, B_t) \in R\}$$

for some $R \subset \overline{O} \times \overline{O}$ *measurable.*

Key steps in the proof:

Let

- ψ^* dual optimal solution
- τ^* optimal (randomized) stopping time
- π^* optimal plan
- the hitting time to the barrier
  $\eta = \inf\{t; J_{\psi^*}(B_0, B_t) = \psi^*(B_t) - |B_0 - B_t|\}$.

We can show

- $\frac{d}{dh}\Big|_{h=0} J_{\psi^*}(x+h, x)$ **exists** μ**-a.e.** x**.**

From this we derive for $\xi = \tau^* - \eta$**,**

- $$\mathbb{E}\left[\nabla_x c(x, B^y_\xi)\right] - \nabla_x c(x, y) = 0 \text{ for } \pi^*\text{-a.e. } (x, y).$$

ST implies $\xi = 0$ **so** $\tau^* = \eta$**.**

(a.e.) differentiability of optimal J_{ψ^*}

$$J_\psi(x, y) = \sup_\tau \mathbb{E}\left[\psi(B^y_\tau) - c(x, B^y_\tau)\right]$$

Lemma

$\|x \mapsto c(x, y)\|_{Lip} \leq K \Longrightarrow \|x \mapsto J_\psi(x, y)\|_{Lip} \leq K.$

Lemma

$x \notin \operatorname{supp} \nu \Longrightarrow y \mapsto J_{\psi^*}(x, y)$ *is harmonic near* x.

(a.e.) differentiability of optimal J_{ψ^*}

Lemma

Let

- τ^* *an optimal stopping time*
- ζ *be any stopping time,* $\zeta \leq \tau^*$ *satisfying*

$$\mathbb{E}\left[J_{\psi^*}(x, B^x_\zeta)\right] = \mathbb{E}\left[\psi^*(B^x_\zeta) - c(x, B^x_\zeta)\right] \quad \textit{for } \mu\textit{-a.e. } x.$$

Then, *for* μ*-a.e.* x

- $h \mapsto J_{\psi^*}(x+h, x)$, $h \mapsto \mathbb{E}\left[J_{\psi^*}(x+h, B^x_\zeta)\right]$, *and* $h \mapsto \mathbb{E}\left[J_{\psi^*}(x+h, B^x_{\tau^*})\right]$ *are differentiable at* $h = 0$
- $$\frac{d}{dh}\Big|_{h=0} J_{\psi^*}(x+h, x) = \frac{d}{dh}\Big|_{h=0} \mathbb{E}\left[J_{\psi^*}(x+h, B^x_\zeta)\right] = \mathbb{E}\left[-\nabla_x c(x, B^x_\zeta)\right] = \frac{d}{dh}\Big|_{h=0} \mathbb{E}\left[J_{\psi^*}(x+h, B^x_{\tau^*})\right] = \mathbb{E}\left[-\nabla_x c(x, B^x_{\tau^*})\right].$$

Remark

The results (the dual attainment and the hitting time property) hold for Brownian motion valued in Riemannian manifold, if

- $c(x, y) = d(x, y)$, *the Riemannian distance (as long as it is differentiable).* *($\Longrightarrow$ **ST**.)*

Some future work

- With control. $dX_t = Adt + dB_t$.
- More general cost.
- Multi-marginals / multiple stopping.
- Regularity of the ψ^*, J_{ψ^*} and the corresponding barriers (free boundaries).

References

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Thank you very much!