Measure-valued - strong uniqueness for general conservation laws and some convex integration for nonlocal Euler system

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A couple of obvious things about measure-valued solutions

- The basic concept behind the approach of measure-valued solutions is to embed the problem into a wider space.
- The benefit of this idea is passing from a nonlinear problem to a linear one. The essence of the proof of existence of such solutions becomes a matter of appropriate estimates rather than subtle weak sequential stability arguments.
- There is of course a cost to be paid – the result of a limit is only a weak object represented by a *Young measure*, namely by a parametrized family of measures.
- For general systems of conservation laws there is no hope to obtain entropy weak solutions and therefore there seems to be no alternative to the use of measure-valued solutions or related concepts.

Relative entropy method

- The foundation of the mv-strong uniqueness principle is a relative entropy method, whose origins can be traced back to physics.
- It is a useful tool in obtaining a variety of interesting analytical results: uniqueness of solutions to a conservation law in the scalar case, while for many systems of equations it provides the so-called weak-strong uniqueness property.
- Other areas where relative entropy method is found useful include stability studies, singular limits, such as hyperbolic and diffusive limits, dimension reduction problems, the method is also applied to problems arising from biology, known in this context as *general relative entropy method*. It is essentially used for showing asymptotic convergence of solutions to steady-state solutions.

Can the Young measures describe a concentration effect?

Definition

A bounded sequence $\{z^j\}$ in $L^1(\Omega)$ converges in biting sense to a function $z \in L^1(\Omega)$, written $z^j \overset{b}{\to} z$ in Ω, provided there exists a sequence $\{E_k\}$ of measurable subsets of Ω, satisfying $\lim_{k\to\infty} |E_k| = 0$, such that for each k

$$z^j \rightharpoonup z \quad \text{in} \quad L^1(\Omega \setminus E_k).$$

Remarks

Biting limit can be also express as $\lim\limits_{n\to\infty} \lim\limits_{j\to\infty} T^n(z^j)$, where by $T^n(\cdot)$ we denote standard truncation operator.

Lemma

Let u^j be a sequence of measurable functions and ν_x a Young measure associated to a subsequence u_{j_k}. Then $f(\cdot, u^{j_k}) \overset{b}{\rightarrow} \langle \nu_x, f \rangle$ for every Carathéodory function $f(\cdot,\cdot)$ s.t. $f(\cdot, u^{j_k})$ is a bounded sequence in $L^1(\Omega)$. Here $\langle \nu_x, f \rangle = \int_{\mathbb{R}^d} f \, d\nu_x$.

Remarks

In view of the above facts the classical Young measures prescribe only the oscillation effect, not the concentration one. The attempt to prescribe also concentration effect by some generalizations of the Young measures was initiated by DiPerna and Majda

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Remarks

In view of the above facts the classical Young measures prescribe only the oscillation effect, not the concentration one. The attempt to prescribe also concentration effect by some generalizations of the Young measures was initiated by DiPerna and Majda

Incompressible Euler equations

$$\begin{aligned} v_t + \operatorname{div}(v \otimes v) + \nabla p &= 0, \\ \operatorname{div} v &= 0. \end{aligned}$$

- R. J. DiPerna and A. J. Majda, Oscillations and concentrations in weak solutions of the incompressible fluid equations, Comm. Math. Phys. 1987.
- J. J. Alibert and G. Bouchitté, Non-uniform integrability and generalized Young measures, J. Convex Anal. 1997.

Theorem (DiPerna-Majda, Alibert-Bouchitté)

Let $\{u_k\}$ be a bounded sequence in $L^p(\Omega;\mathbb{R}^n)$. There exists a subsequence $\{u_{k_j}\}$, a nonnegative Radon measure λ and parametrized families of probability measures $\nu \in \mathcal{P}(\Omega;\mathbb{R}^n)$, $\nu^\infty \in \mathcal{P}(\lambda;\mathbb{S}^{n-1})$ such that:

$$g(u_{k_j}) \stackrel{*}{\rightharpoonup} \langle \nu, g \rangle + \langle \nu^\infty, g^\infty \rangle \lambda$$

in the sense of measures, for every $g \in \mathcal{F}$.

Measure-valued solutions to incompressible Euler system

We say that (ν, m, ν^∞) is a measure-valued solution of incompressible Euler system with initial data u_0 if for every $\phi \in C^1_{c,\mathrm{div}}([0,T) \times \mathbb{T}^n; \mathbb{R}^n)$ it holds that

$$\int_0^T \int_{\mathbb{T}^n} \partial_t \phi \cdot \overline{u} + \nabla \phi : \overline{u \otimes u} dxdt + \int_{\mathbb{T}^n} \phi(\cdot, 0) \cdot u_0 dx \quad = 0.$$

Where

$$\overline{u} = \langle \lambda, \nu \rangle$$
$$\overline{u \otimes u} = \langle \lambda \otimes \lambda, \nu \rangle + \langle \beta \otimes \beta, \nu^\infty \rangle m$$

If the solution is generated by some approximation sequences, then the black terms on right-hand side correspond to the biting limit of sequences whereas the blue ones corespond to concentration measure.

Admissibility of measure-valued solutions

Let us set

$$E_{mvs}(t) := \int_{\mathbb{T}^n} \frac{1}{2}\overline{|u|^2}(t,x)dx$$

for almost every t, where

$$\overline{|u|^2} = \langle |\lambda|^2, \nu\rangle + \langle |\beta|^2, \nu^\infty\rangle m$$

and

$$E_0 := \int_{\mathbb{T}^n} \frac{1}{2}|u_0|^2(x)dx.$$

We then say that a measure-valued solution is admissible if

$$E_{mvs}(t) \leq E_0$$

in the sense of distributions.

Mv - strong uniqueness for concrete systems

- Y. Brenier, C. De Lellis, L. Székelyhidi, Jr., Weak-strong uniqueness for measure-valued solutions. Comm. Math. Phys. 2011,

  Incompressible Euler -oscillation and concentration measure,
- S. Demoulini, D. M. A. Stuart, A. E. Tzavaras, Weak-strong uniqueness of dissipative measure-valued solutions for polyconvex elastodynamics. Arch. Ration. Mech. Anal. 2012

  In weak formulation only oscillation measure, in entropy inequality there appears non-negative concentration measure

Mv - strong uniqueness for concrete systems

- P. Gwiazda, A. Ś.-G., E. Wiedemann, Weak-Strong Uniqueness for Measure-Valued Solutions of Some Compressible Fluid Models, Nonlinearity, 2015
  Oscillatory and vector-valued concentration measure both in weak formulation and entropy inequality
- E. Feireisl, P. Gwiazda, A. Ś.-G., E. Wiedemann, Dissipative measure-valued solutions to the compressible Navier–Stokes system, Calculus of Variations and Partial Differential Equations, 2016
  Instead of vector-valued concentration measure the dissipation defect is introduced
- J. Březina, E. Feireisl, Measure-valued solutions to the complete Euler system, J. Math. Soc. Japan, 2018
  J. Březina, E. Feireisl, Measure-valued solutions to the complete Euler system revisted, ZAMP 2018

Mv - Strong Uniqueness

Theorem (Y. Brenier, C. De Lellis, L. Székelyhidi, Jr., 2011)

Let $U \in C^1([0,T] \times \mathbb{T}^n)$ *be a solution of IE . If* (ν, m, ν^∞) *is an admissible measure-valued solution with the same initial data, then*

$$\nu_{t,x} = \delta_{U(t,x)} \text{ for a.e. } t, x, \text{ and } m = 0.$$

Remark:

Some generalization of this result: Emil Wiedemann, *Weak-strong uniqueness in fluid dynamics*, to appear in Partial Differential Equations in Fluid Mechanics, Cambridge University Press.

General hyperbolic conservation law

We consider the hyperbolic system of conservation laws in the form

$$\partial_t A(u) + \partial_\alpha F_\alpha(u) = 0$$

with the initial condition $u(0) = u_0$. Here $u : [0, T] \times \mathbb{T}^d \to \mathbb{R}^n$.

General hyperbolic conservation law

- Y. Brenier, C. De Lellis, L. Székelyhidi, Jr., Weak-strong uniqueness for measure-valued solutions. Comm. Math. Phys. 2011,
  General hyperbolic systems - only oscillation measure, both in weak formulation and entropy inequality
- S. Demoulini, D. M. A. Stuart, A. E. Tzavaras, Weak-strong uniqueness of dissipative measure-valued solutions for polyconvex elastodynamics. Arch. Ration. Mech. Anal. 2012
  In weak formulation only oscillation measure, in entropy inequality there appears non-negative concentration measure

General hyperbolic conservation law

- C. Christoforou, A. Tzavaras, Relative entropy for hyperbolic-parabolic systems and application to the constitutive theory of thermoviscoelasticity, Arch. Ration. Mech. Anal. (2018)
  An analogue result for more general form of a system, hyperbolic-parabolic case, also only with a non-negative concentration measure in entropy inequality
- P. Gwiazda, O. Kreml, A. Ś.-G. Dissipative measure valued solutions for general hyperbolic conservation laws
  Concentration measure both in the weak formulation and the entropy inequality

General hyperbolic conservation law

We consider the hyperbolic system of conservation laws in the form

$$\partial_t A(u) + \partial_\alpha F_\alpha(u) = 0 \tag{1}$$

with the initial condition $u(0) = u_0$. Here $u : [0, T] \times \mathbb{T}^d \to \mathbb{R}^n$. There exists an open convex set $X \subset \mathbb{R}^n$ such that the mappings $A : \overline{X} \to \mathbb{R}^n$, $F_\alpha : X \to \mathbb{R}^n$ are C^2 maps on X, A is continuous on $\bar{X}$ and $\nabla A(u)$ is nonsingular for all $u \in X$.

Examples

Compressible Euler system

$$\partial_t \rho + \mathrm{div}(\rho v) = 0,$$
$$\partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) + \nabla p(\rho) = 0.$$

The associated entropy is given by

$$\eta(\rho, v) = \frac{1}{2}\rho|v|^2 + P(\rho),$$

here the pressure potential $P(\rho)$ is related to the original pressure $p(\rho)$ through

$$P(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2} dr.$$

Examples

$$A(u) = \begin{pmatrix} \rho \\ \rho v \end{pmatrix}, \quad F(u) = \begin{pmatrix} \rho v \\ v \otimes v + p(\rho) \end{pmatrix}.$$

$$\eta(\rho, v) = \frac{1}{2}\rho|v|^2 + P(\rho),$$

We show that

$$\frac{|A(u)|}{\eta(u)} \to 0$$

as $|u| \to \infty$ and

$$\frac{|F(u)|}{\eta(u)} \leq C.$$

Examples

Shallow water magnetohydrodynamics

$$
\begin{aligned}
\partial_t h + \mathrm{div}_x(hv) &= 0,\\
\partial_t(hv) + \mathrm{div}_x\left(hv\otimes v - hb\otimes b\right) + \nabla_x(gh^2/2) &= 0,\\
\partial_t(hb) + \mathrm{div}_x(hb\otimes v - hv\otimes b) + v\,\mathrm{div}_x(hb) &= 0,
\end{aligned}
$$

where $g > 0$ is the gravity constant, $h\colon Q \to \mathbb{R}_+$ is the thickness of the fluid, $v\colon Q \to \mathbb{R}^2$ is the velocity, $b\colon Q \to \mathbb{R}^2$ is the magnetic field.

Polyconvex elasticity

Consider the evolution equations of nonlinear elasticity

$$\left.\begin{aligned} \partial_t F &= \nabla_x \mathbf{v} \\ \partial_t \mathbf{v} &= \mathrm{div}_x \left(D_F W(F)\right) \end{aligned}\right. \quad \text{in } \mathcal{X},$$

for an unknown matrix field $F\colon \mathcal{X} \to \mathbb{M}^{k\times k}$, and an unknown vector field $\mathbf{v}\colon \mathcal{X} \to \mathbb{R}^k$. Function $W\colon \mathcal{U} \to \mathbb{R}$ is given.

Hypotheses

(H1) There exists an entropy-entropy flux pair (η, q_α), $\eta(u) \geq 0$ and $\lim_{|u|\to\infty} \eta(u) = \infty$
This yields the existence of a C^1 function $G : \overline{X} \to \mathbb{R}^n$ such that

$$\nabla\eta = G \cdot \nabla A, \quad \nabla q_\alpha = G \cdot \nabla F_\alpha, \qquad \alpha = 1, ..., d.$$

(H2) The symmetric matrix

$$\nabla^2\eta(u) - G(u) \cdot \nabla^2 A(u)$$

is positive definite for all $u \in X$.

(H3) The vector $A(u)$ and the fluxes $F_\alpha(u)$ are bounded by the entropy, i.e.

$$|A(u)| \leq C(\eta(u) + 1), \quad |F_\alpha(u)| \leq C(\eta(u) + 1), \qquad \alpha = 1, ..., d.$$

Define for a strong solution U taking values in a compact set $D \subset X$ the relative entropy

$$\begin{aligned} \eta(u|U) &:= \eta(u) - \eta(U) - \nabla\eta(U) \cdot \nabla A(U)^{-1}(A(u) - A(U)) \\ &= \eta(u) - \eta(U) - G(U) \cdot (A(u) - A(U)) \end{aligned}$$

and the relative flux as

$$F_\alpha(u|U) := F_\alpha(u) - F_\alpha(U) - \nabla F_\alpha(U) \nabla A(U)^{-1}(A(u) - A(U))$$

for $\alpha = 1, ..., d$.

If we assume (H1) – (H3) hold and $\lim_{|u|\to\infty} \frac{A(u)}{\eta(u)} = 0$ then

$$|F_\alpha(u|U)| \leq C\eta(u|U).$$

Definition

We say that $(\nu, m_A, m_{F_\alpha}, m_\eta)$, is a dissipative measure-valued solution of system (1) if $\nu \in L^\infty_{\text{weak}}\left((0,T)\times\mathbb{T}^d;\mathcal{P}\left(\overline{X}\right)\right)$ is a parameterized measure and together with concentration measures $m_A \in (\mathcal{M}([0,T]\times\mathbb{T}^d))^n$, $m_{F_\alpha} \in (\mathcal{M}([0,T]\times\mathbb{T}^d))^{n\times n}$ satisfy

$$\int_Q \langle \nu_{t,x}, A(\lambda)\rangle \cdot \partial_t\varphi dxdt + \int_Q \partial_t\varphi \cdot m_A(dxdt)$$
$$+ \int_Q \langle \nu_{t,x}, F_\alpha(\lambda)\rangle \cdot \partial_\alpha\varphi dxdt + \int_Q \partial_\alpha\varphi \cdot m_{F_\alpha}(dxdt)$$
$$+ \int_{\mathbb{T}^d} \langle \nu_{0,x}, A(\lambda)\rangle \cdot \varphi(0)dx + \int_{\mathbb{T}^d} \varphi(0)\cdot m_A^0(dx) = 0$$

for all $\varphi \in C_c^\infty(Q)^n$.

Definition

Moreover, the total entropy balance holds for all $\zeta \in C_c^\infty([0,T))$

$$\int_Q \langle v_{t,x}, \eta(\lambda)\rangle \zeta'(t) dxdt + \int_Q \zeta'(t) m_\eta(dxdt)$$
$$+ \int_{\mathbb{T}^d} \langle v_{0,x}, \eta(\lambda)\rangle \zeta(0) dx + \int_{\mathbb{T}^d} \zeta(0) m_\eta^0(dx) \geq 0$$

with a dissipation measure $m_\eta \in \mathcal{M}^+([0,T] \times \mathbb{T}^d)$. In particular we assume that measures m_A^0 and m_η^0 are well defined.

Relations between concentration measures

Proposition

Let $f(y,u)$ be a nonnegative continuous function on $Y\times\overline{X}$ and let $g(y,u)$ be a vector-valued function, also continuous on $Y\times\overline{X}$ such that

$$\lim_{|u|\to\infty}|g(y,u)|\leq C\lim_{|u|\to\infty}f(y,u).$$

Let m_f and m_g denote the concentration measures related to $f(\cdot,u_n)$ and $g(\cdot,u_n)$ respectively, where $f(\cdot,u_n)$ is a sequence bounded in L^1. Then

$$|m_g|\leq Cm_f,$$

i.e. $|m_g|(A)\leq Cm_f(A)$ for any Borel set $A\subset Y$.

An analogue fact under more restrictive assumptions

$$\lim_{|u|\to\infty} \frac{A(u)}{\eta(u)} = \lim_{|u|\to\infty} \frac{F_\alpha(u)}{\eta(u)} = 0,$$

was proved in Christoforou & Tzavaras 2017. Note however that this condition is satisfied for polyconvex elastodynamics but is not satisfied e.g. for compressible Euler equations.

Gwiazda, Kreml, Ś.-G. 2019

Theorem

Let $(\nu, m_A, m_{F_\alpha}, m_\eta)$, $\alpha = 1, ..., d$, *be a dissipative measure-valued solution to* (1) *generated by a sequence of approximate solutions. Let* $U \in W^{1,\infty}(Q)$ *be a strong solution to* (1) *with the same initial data* $\eta(u_0) \in L^1(\mathbb{T}^d)$, *thus* $\nu_{0,x} = \delta_{u_0(x)}$, $m_A^0 = m_{F_\alpha}^0 = m_\eta^0 = 0$. *Then* $\nu_{t,x} = \delta_{U(x)}$, $m_A = m_{F_\alpha} = m_\eta = 0$ *and* $u = U$ *a.e. in* Q.

Extension

- This general framework will not cover systems of conservation laws, which may fail to be hyperbolic, typically incompressible inviscid systems.
- We propose an extension of this framework to cover the case of incompressible fluids, in case of which the assumption that ∇A is a nonsingular matrix is not satisfied.
- We distinguish from the flux the part L (Lagrange multiplier) which is perpendicular to the vector $G(U)$ (which coincides with the gradient of the entropy of the strong solution in the case $A = \mathrm{Id}$).
- Thus we assume that there exists a subspace Y, such that $G(U) \in Y$ and $L \in Y^{\perp}$, where U is a strong solution to the considered system.

Let us then consider a system in the following form

$$\partial_t A(u) + \partial_\alpha F_\alpha(u) + L = 0.$$

Examples covered by our theory:

- incompressible Euler
- incompressible magnetohydrodynamics
- inhomogeneous incompressible Euler
- incompressible inhomogeneous magnetohydrodynamics

Euler-Poisson system/Euler alignment system

Euler alignment system

$$\partial_t \varrho(t,x) + \operatorname{div}\Big(\varrho(t,x)\mathbf{u}(t,x)\Big) = 0,$$

$$\partial_t\Big(\varrho(t,x)\mathbf{u}(t,x)\Big) + \operatorname{div}\Big(\varrho(t,x)\mathbf{u}(t,x)\otimes\mathbf{u}(t,x)\Big) + \nabla p(\varrho(t,x))$$

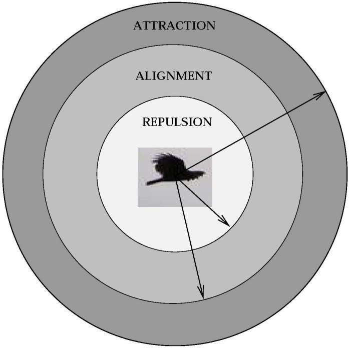
$$= \left(1 - H\left(|\mathbf{u}(t,x)|^2\right)\right)\varrho(t,x)\mathbf{u}(t,x) - \varrho(t,x)\int_\Omega \nabla K(x-y)\varrho(t,y)\,\mathrm{d}y$$

$$+\varrho(t,x)\int_\Omega \psi(x-y)\Big(\mathbf{u}(t,y)-\mathbf{u}(t,x)\Big)\varrho(t,y)\,\mathrm{d}y,$$

- the kernel K includes the repulsive-attractive interaction force between individuals and ψ gives the local averaging to compute the consensus in orientation among individuals
- If we skip the blue terms and $K * \varrho = \Delta^{-1}(\varrho - \int_\Omega \varrho)$, then we get **Euler-Poisson** system.

Origins of the system

Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way. Highly developed social organization: insects (ants, bees ...), fish, birds, micro-organisms.



Origins of the model

- Starting point: Basic particle model

$$\frac{dx_i}{dt} = v_i \,,$$
$$\frac{dv_i}{dt} = \underbrace{v_i - \alpha v_i |v_i|^2}_{\text{propulsion}-\text{friction}} - \underbrace{\sum_{j\neq i} \nabla_x K(x_i - x_j)}_{\text{attraction}-\text{repulsion}} + \underbrace{\sum_i \psi(x_i - x_j)(v_j - v_i)}_{\text{alignment}}$$

where $i \in \{1, \dots, n\}$ and $\alpha > 0$.

- By passing to the limit with number of particles one gets Vlasov-like kinetic equation
- By hydrodynamic (mono-kinetic) ansatz one gets Euler alignment system

For details see

J. A. Carrillo, Y.-P. Choi, E. Tadmor, and C. Tan, Critical thresholds in 1D Euler equations with nonlocal forces, *Math. Mod. Meth. in Appl. Sci.*, 2015.

Infinitely many admissible weak solutions to Euler alignment system

Theorem

Let $T > 0$ be given and let $N = 2, 3$. Let the initial data ϱ_0, $\mathbf{u}_0$ be given,

$$\varrho_0 \in C^2(\Omega),\ \ \varrho_0 \geq \underline{\varrho} > 0 \ \mathit{in}\ \Omega,\ \ \mathbf{u}_0 \in C^3(\Omega; R^N).$$

Then the initial value problem admits infinitely many weak solutions in the space-time cylinder $(0, T) \times \Omega$ belonging to the class

$$\varrho \in C^2([0, T] \times \Omega),\ \ \varrho > 0,$$

$$\mathbf{u} \in C_{\mathrm{weak}}([0, T]; L^2(\Omega; R^N)) \cap L^\infty((0, T) \times \Omega; R^N).$$

J. A. Carrillo, E. Feireisl, P. Gwiazda, A. Ś-G. Weak solutions for Euler systems with non-local interactions, *J. Lond. Math. Soc* 2017

Sketch of the proof

The proof consists of the following steps:

- rewrite the momentum equation as "incompressible Euler equation" with variable coefficients and with source term continuous w.r.t. weak topology.
- choose the kinetic energy to be big enough to have non-empty set of subsolutions
- apply variable coefficient version of oscillatory lemma
- by Baire category argument the points of continuity of some functional (which are solutions to our system) are of infinite cardinality.

Theorem: Existence of admissible weak solutions

Under the same assumptions, given $T > 0$ and $\varrho_0 \in C^2(\Omega)$, $\varrho_0 > 0$, there exists $\mathbf{u}_0 \in L^\infty(\Omega; R^3)$ such that the considered initial value problem admits infinitely many admissible weak solutions in the space-time cylinder $(0, T) \times \Omega$.

Weak-strong uniqueness

Theorem

Let $\varrho, \mathbf{u}$ *be an admissible weak solution in* $(0, T) \times \Omega$. *Let* $r, \mathbf{U}, r > 0$, *be a globally Lipschitz (strong) solution of the same problem, with*

$$\varrho_0 = r(0, \cdot), \ \mathbf{u}_0 = \mathbf{U}(0, \cdot).$$

Then

$$\varrho = r, \ \mathbf{u} = \mathbf{U} \ a.e. \ in \ (0, T) \times \Omega.$$

Pressureless system. The case of Poisson kernel

For strong solutions $[r, \mathbf{U}]$ it is convenient to rewrite system in a non-conservative form

$$
\begin{aligned}
\partial_t r(t,x) + \mathrm{div}\Big(r(t,x)\mathbf{U}(t,x)\Big) &= 0, \\
\partial_t \mathbf{U}(t,x) + \mathbf{U}(t,x)\cdot\nabla\mathbf{U}(t,x) &= \left(1 - H\left(|\mathbf{U}(t,x)|^2\right)\right)\mathbf{U}(t,x) \\
&\quad - \nabla\Phi_r(t,x) + \int_\Omega \psi(x-y)\Big(\mathbf{U}(t,y) - \mathbf{U}(t,x)\Big) r(t,y)\,\mathrm{d}y, \\
-\Delta_x \Phi_r(t,x) &= r(t,x) - \bar{r}.
\end{aligned}
$$

- In the case there appears no vacuum, the systems are equivalent for smooth solutions.
- Choosing $[r, \mathbf{U}]$ as a solution to the second systems does not require to exclude compactly supported smooth solutions

Weak-strong uniqueness

Let $[\varrho, \mathbf{u}]$ be an admissible weak solution to the first system in $(0, T) \times \Omega$ with initial data $\varrho_0, \mathbf{u}_0$ and $\varrho_0 \geq 0$ and let $[r, \mathbf{U}]$ be a Lipschitz (strong) solution to the second system with

$$\varrho_0 = r(0, \cdot), \ \mathbf{u}_0 = \mathbf{U}(0, \cdot).$$

Then

$$\varrho = r, \ \varrho\mathbf{u} = r\mathbf{U} \text{ a.e. in } (0, T) \times \Omega.$$

Euler alignment system. Relative energy.

Relative energy functional

$$\mathcal{E}\left(\varrho,\mathbf{u}\;\middle|\;r,\mathbf{U}\right)=\int_\Omega\left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2+\frac{1}{2}(r-\varrho)(K*(r-\varrho))\right]\,\mathrm{d}x,$$

$$\begin{aligned}&\left[\mathcal{E}\left(\varrho,\mathbf{u}\;\middle|\;r,\mathbf{U}\right)\right]_{t=0}^{t=\tau}\\ &\leq c\int_0^\tau\mathcal{E}\left(\varrho,\mathbf{u}\;\middle|\;r,\mathbf{U}\right)\,\mathrm{d}t+\int_0^\tau\int_\Omega(\varrho-r)\left(\nabla K*(\varrho-r)\right)\cdot\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^\tau\int_\Omega\left(H\left(|\mathbf{u}|^2\right)\varrho\mathbf{u}-H\left(|\mathbf{U}|^2\right)\varrho\mathbf{U}\right)\cdot(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^\tau\int_{\Omega\times\Omega}\psi(x-y)\varrho(t,x)(\mathbf{u}(t,x)-\mathbf{U}(t,x))(\mathbf{U}(t,y)-\mathbf{U}(t,x))(\varrho(t,y)-r(t,y))\mathrm{d}y\mathrm{d}x\mathrm{d}t\end{aligned}$$

Poisson kernel - remainder term estimates

$$\int_0^\tau \int_\Omega (\varrho - r)\left(\nabla K * (\varrho - r)\right) \cdot \mathbf{U} \,\mathrm{d}x \,\mathrm{d}t$$

Let

$$K * \varrho = \Phi_\varrho \text{ and } -\Delta_x \Phi_\varrho(t,x) = \varrho(t,x) - \overline{\varrho},$$

where $\overline{\varrho} = \int_\Omega \varrho(t,x) \,\mathrm{d}x$, then

$$\begin{aligned}(r-\varrho)\nabla(\Phi_r - \Phi_\varrho) \\ &= \nabla\left(\frac{1}{2}|\nabla(\Phi_r - \Phi_\varrho)|^2 + (\overline{r} - \overline{\varrho})(\Phi_r - \Phi_\varrho)\right) \\ &\quad - \operatorname{div}\left(\nabla(\Phi_r - \Phi_\varrho) \otimes \nabla(\Phi_r - \Phi_\varrho)\right).\end{aligned}$$

J. Giesselmann, C. Lattanzio, A. E. Tzavaras. Relative energy for the Korteweg theory and related Hamiltonian flows in gas dynamics. Arch. Ration. Mech. Anal. 2017

What about measure-valued solutions to this system?

Mv-strong uniqueness for the system with pressure term:

Březina, Jan; Mácha, Václav. Inviscid limit for the compressible Euler system with non-local interactions. *J. Differential Equations* (2019)

Euler–Poisson system

Let $\Omega \subset \mathbb{R}^d$ be a bounded smooth domain and let $T > 0$ be fixed. We consider the following Euler–Poisson system in $(0, T) \times \Omega$

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) &= -\varrho \mathbf{u} - \varrho \nabla_x \Phi_\varrho, \\ -\Delta \Phi_\varrho &= \varrho, \end{aligned} \tag{2}$$

For strong solutions of (2) we can rewrite the nonlocal term $\varrho \nabla_x \Phi_\varrho$ as

$$\varrho \nabla_x \Phi_\varrho = \frac{1}{2} \nabla_x |\nabla_x \Phi_\varrho|^2 - \operatorname{div}[\nabla_x \Phi_\varrho \otimes \nabla_x \Phi_\varrho]. \tag{3}$$

Measure–valued solutions to Euler-Poisson

We say that $(\boldsymbol{\nu}, m^{\rho}, m^{\rho\mathbf{u}}, m^{\rho\mathbf{u}\otimes\mathbf{u}}, m^{\rho\nabla\Phi})$ is a measure–valued solution of (2) with initial data $(\nu_0, m_0^{\rho}, m_0^{\rho\mathbf{u}}, m_0^{\rho\mathbf{u}\otimes\mathbf{u}}, m_0^{\rho\nabla\Phi})$ if

$$\boldsymbol{\nu} = \{\nu_{t,x}\} \in L^{\infty}_{weak}\left((0,T)\times\Omega; \mathcal{P}\left([0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\right)\right),$$

and the following hold for appropriate test functions ψ, ϕ, θ:

$$\left[\int_{\Omega}\overline{\rho}\psi\right]_{t=0}^{t=\tau} = \int_0^{\tau}\int_{\Omega}\overline{\rho}\psi + \overline{\rho\mathbf{u}}\cdot\nabla\psi \; dxdt,$$
$$\left[\int_{\Omega}\overline{\rho\mathbf{u}}\cdot\phi\right]_{t=0}^{t=\tau} = \int_0^{\tau}\int_{\Omega}\overline{\rho\mathbf{u}}\cdot\partial_t\phi + \overline{\rho\mathbf{u}\otimes\mathbf{u}} : \nabla\phi - \overline{\rho\mathbf{u}}\cdot\phi \; - \overline{\rho\nabla\Phi}\cdot\phi \; dxdt$$
$$\int_{\Omega}\overline{\nabla\Phi}\cdot\nabla\theta \; dx = \int_{\Omega}\overline{\rho}\theta \; dx \tag{4}$$

where

$$\overline{f} = \langle\nu_{t,x}(\cdot); f(t,x,\cdot)\rangle + m^f$$

e. g.

$$\overline{\rho\mathbf{u}} = \langle\nu;\lambda_1\lambda_2\rangle + m^{\rho\mathbf{u}}; \qquad \overline{\rho\nabla\Phi} = \langle\nu;\lambda_1\lambda_3\rangle + m^{\rho\nabla\Phi}; \qquad \overline{\nabla\Phi} = \langle\nu;\lambda_3\rangle$$

Admissibility

Let

$$E^{mv}(t) := \int_\Omega \frac{1}{2}\overline{\rho|\mathbf{u}|^2} + \frac{1}{2}\overline{|\nabla\Phi|^2}\, dx$$

where

$$\begin{aligned} \overline{\rho|\mathbf{u}|^2} &= \langle \nu; \lambda_1|\lambda_2|^2\rangle + m_{KE} \\ \overline{|\nabla\Phi|^2} &= \langle \nu; |\lambda_3|^2\rangle + m_{|\nabla\Phi|^2} \end{aligned} \tag{5}$$

A measure–valued solution is admissible if

$$E^{mv}(t) \le E^{mv}(0) - \int_0^t \int_\Omega \overline{\rho|\mathbf{u}|^2}\, dxdt.$$

Existence

Suppose $\Omega \subset \mathbb{R}^d$ is an open bounded domain. If the initial data $(\rho_0, \mathbf{u}_0)$ has finite energy, then there exists an admissible measure–valued solution with initial data

$$\nu_{0,x} = \delta_{\{\rho_0, \mathbf{u}_0, \nabla_x \Phi_r(0,x)\}} \quad \text{for a.e. } x \in \Omega. \tag{6}$$

Outline of proof:

- Approximate the initial problem $\rightsquigarrow$ sequence of approximate solutions
- Energy estimates $\rightsquigarrow$ compactness
- Passing to the limit $\rightsquigarrow$ a subsequence generates the generalized Young measure solving the equation

Relative energy inequality

Let $(r, \mathbf{U}, \Phi_r)$ be a strong solution of (2).
Define the relative energy by

$$E_{rel}^{mv}(t) := \int_\Omega \frac{1}{2}\overline{\rho|\mathbf{u}-\mathbf{U}|^2} + \frac{1}{2}\overline{|\nabla\Phi_\rho - \nabla\Phi_r|^2}\, dx.$$

Then the following inequality is satisfied

$$E_{rel}^{mv}(t) \leq E_{rel}^{mv}(0) + \mathcal{R}$$

Then:

- Bound the remainder term $\mathcal{R} \lesssim \int_0^t E_{rel}^{mv}\, dt$
- Use Gronwall inequality to conclude $E_{rel}^{mv} \equiv 0$ if initial data coincide.

MV–strong uniqueness

Let $(r, \mathbf{U}, \Phi_r)$, $r > 0$, be a strong solution and let $(\boldsymbol{\nu}, m^{\rho}, m^{\rho\mathbf{u}}, m^{\rho\mathbf{u}\otimes\mathbf{u}}, m^{\rho\nabla\Phi})$ be an admissible measure–valued solution to the system (2) with initial state

$$\nu_{0,x} = \delta_{\{r(0,x),\mathbf{U}(0,x),\nabla\Phi_r(0,x)\}} \quad \text{for a.e. } x \in \Omega.$$

Then the concentration measures are zero and

$$\nu_{t,x} = \delta_{\{r(t,x),\mathbf{U}(t,x),\nabla\Phi_r(t,x)\}} \quad \text{for a.e. } t \in (0,T), x \in \Omega.$$

Thank you for your attention