Resource theories of quantum channels

Andreas Winter (ICREA &
Universitat Autònoma de Barcelona)

[With Zi-Wen Liu, arXiv:1904.04201]

Outline

1. Resource paradigm
2. Resource theories of states
3. Quantum channels as resources
4. Channels, superchannels & circuits
5. Resource theories of channels
6. Remarks on multiple resources
7. Resource erasure
8. Conclusions

1. Resource paradigm

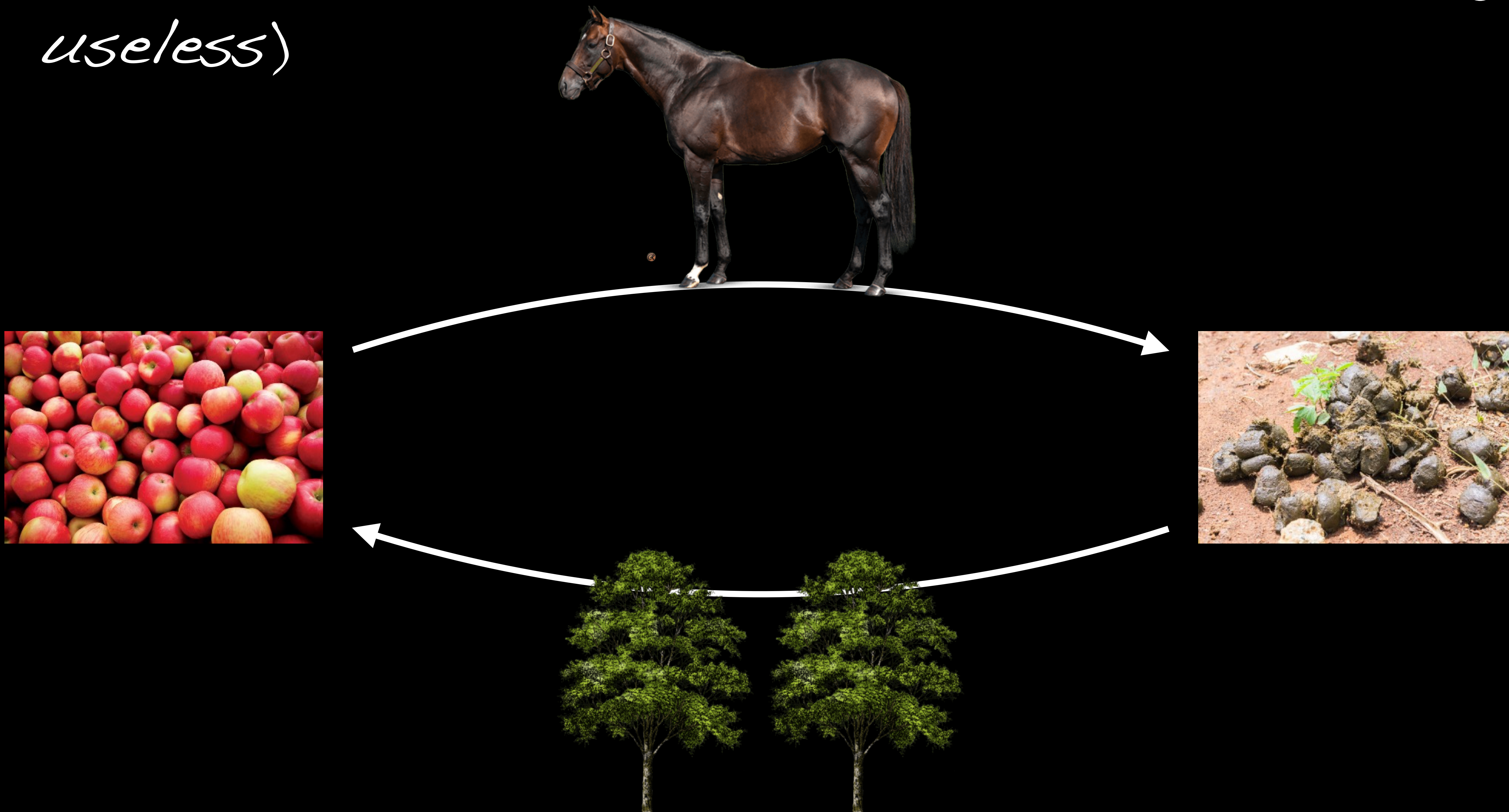
Resource [rɪ'sɔːs]: something that can be used to help you (in contrast to something useless)

1. Resource paradigm

Resource [rɪˈsɔːs]: something that can be used to help you (in contrast to something useless)

1. Resource paradigm

Resource [rɪ'sɔ:s]: something that can be used to help you (in contrast to something useless)



Example: Bipartite entangled states are useful, as opposed to separable states

- Quantum privacy
- MBQC
- Teleportation

To view it as a resource theory, need "free" operations, that do not create entanglement:
LOCC = local operations & classical comm.

[Horodecki^3, Rev. Mod. Phys. 2009]

2. Resource theories of states

In general, for a resource theory whose objects are quantum states, we need:

* for every system A, a set $\mathbb{F}(A)$ of "free" states (=useless states);
* for every two systems A, B, a set $\mathbb{F}(A \to B)$ of free quantum channels (cptp maps);
* ...such that free channels map free states to free states.

[Brandão/Gour, PRL 2015]

2. Resource theories of states

Purposes of a resource theory of quantum states:

- Resource measures on states = monotones under free channels
- Is the theory reversible? (This means that the free operations induce a linear order on the states.)

[Brandão/Gour, PRL 2015]

2. Resource theories of states

Purposes of a resource theory of quantum states:

- Resource measures on states = monotones under free channels
- Is the theory reversible? (This means that the free operations induce a linear order on the states.)
- Implementing tasks (not another state!)

[Brandão/Gour, PRL 2015]

3. Channels as resources

We should consider as possible resource any object in our theory. Thus, not only quantum states but also channels...

How? Example: Shannon theory – resources are channels $\mathcal{N}$ from Alice to Bob; local channels are free. Transform channels by encoding and decoding, i.e. composition with free channels: $\mathcal{N}' = \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}$, for instance to turn a noisy channel into a less noisy one...

[Cf. Devetak/Harrow/AW, IEEE-IT 2008]

3. Channels as resources

Denote the subset of channels from A to B that are free by $\mathbb{F}(A \to B)$. For our purposes, we need three essential axioms:

1) Doing nothing is free: $id_A \in \mathbb{F}(A \to A)$.
2) The free sets $\mathbb{F}(A \to B)$ are topologically closed, i.e. limits of free maps are free.
3) Composition and tensor product of free channels are free: $\mathbb{F}(B \to C) \circ \mathbb{F}(A \to B) \subset \mathbb{F}(A \to C)$, $\mathbb{F}(A \to B) \otimes \mathbb{F}(A' \to B') \subset \mathbb{F}(AA' \to BB')$.

[Gour, 1808.02808; Liu/Yuan, 1904.02680, Liu/AW, 1904.04201]

3. Channels as resources

Denote the subset of channels from A to B that are free by $\mathbb{F}(A \to B)$.

Note that this includes free states: $\mathbb{F}(A) = \mathbb{F}(\mathbb{C} \to A)$

[Gour, 1808.02808; Liu/Yuan, 1904.02680, Liu/AW, 1904.04201]

3. Channels as resources

Some additional properties that may or may not hold:

4) Trace/partial trace is free: $\mathrm{Tr}_A \in \mathbb{F}(A \to \mathbb{C})$.
5) Every system has some free states, i.e. $\mathbb{F}(B) = \mathbb{F}(\mathbb{C} \to B)$ is nonempty.
6) The free sets $\mathbb{F}(A \to B)$ are convex.
7) In system composed of n identical parts, the permutations are free, i.e. for $A^n = A^{\otimes n}$, $U_\pi \cdot U_\pi^\dagger \in \mathbb{F}(A^n \to A^n)$, for all π.

[Gour, 1808.02808; Liu/Yuan, 1904.02680, Liu/AW, 1904.04201]

4. Channels, superchannels & quantum circuits

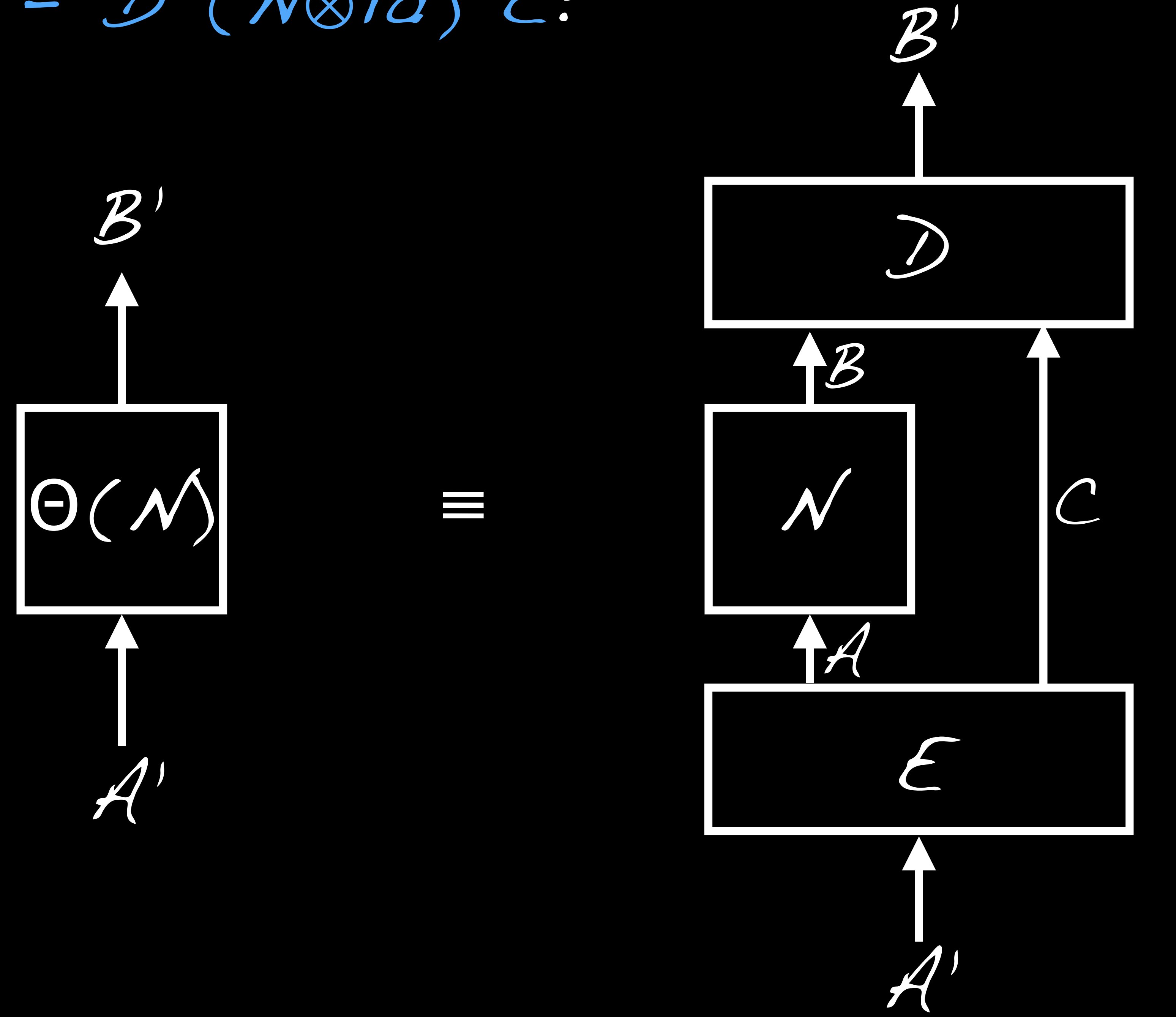
To build a theory of channels as resources, we need to understand how to transform one into another.

4. Channels, superchannels & quantum circuits

To build a theory of channels as resources, we need to understand how to transform one into another.

Axiomatically, we want a superchannel: a map Θ that takes quantum channels (cptp maps) to quantum channels (cptp maps on a potentially different system); it should be linear and its extensions $\mathrm{id}\otimes\Theta$ should behave the same.

Lemma: A map Θ on quantum channels is a superchannel iff it can be written as
$\Theta(\mathcal{N}) = \mathcal{D}\circ(\mathcal{N}\otimes\mathrm{id})\circ\mathcal{E}$:



[Chiribella/D'Ariano/Perinotti, PRL 2008]

5. Resource theories of quantum channels

To make a resource theory, we need to identify the free objects and the free transformations – in the present case, they will turn out to be essentially the same.

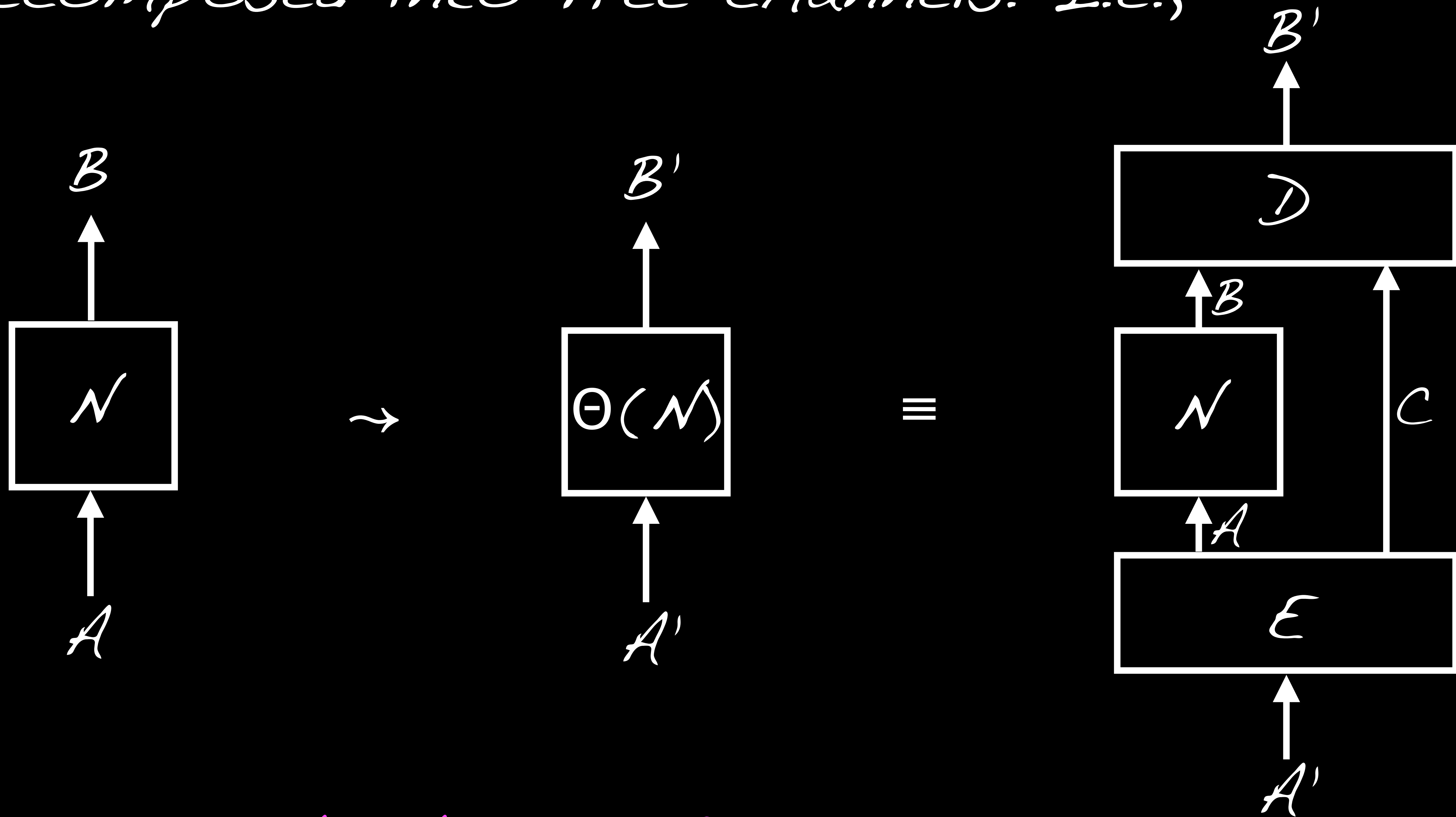
From the previous examples, we are used to the idea that the free channels are given.

5. Resource theories of quantum channels

Recall the axioms:

1) Doing nothing is free: $\mathrm{id}_A \in \mathbb{F}(A \to A)$.
2) The free sets $\mathbb{F}(A \to B)$ are topologically closed, i.e. limits of free maps are free.
3) Composition and tensor product of free channels are free: $\mathbb{F}(B \to C) \circ \mathbb{F}(A \to B) \subset \mathbb{F}(A \to C)$, $\mathbb{F}(A \to B) \otimes \mathbb{F}(A' \to B') \subset \mathbb{F}(AA' \to BB')$.

Definition: A free transformation of channels is a superchannel Θ that can be decomposed into free channels. I.e.,



s.t. $\mathcal{E} \in \mathbb{F}(A' \to AC)$ and $\mathcal{D} \in \mathbb{F}(BC \to B')$.

Definition: A free transformation of channels is a superchannel Θ that can be decomposed into free channels.

Observation: Given free superchannels Θ and Ξ, their composition $\Theta \circ \Xi$ and tensor product $\Theta \otimes \Xi$ are free, too.
(This is because the free channels are closed under composition and tensor products: "completely free" - Gilad Gour)

Definition: A free transformation of channels is a superchannel Θ that can be decomposed into free channels.

Observation: Given free superchannels Θ and Ξ, their composition $\Theta \circ \Xi$ and tensor product $\Theta \otimes \Xi$ are free, too.
(This is because the free channels are closed under composition and tensor products: "completely free" – Gilad Gour)

Note: We care for the (free) realisation of free superchannels. More than simply asking that they map free channels to free ones.

Now, the resource theory is about possible free channel transformations, $\mathcal{N} \to \mathcal{N}' = \Theta(\mathcal{N})$.

Often we are happy with approximation:
$\mathcal{N} \to \mathcal{N}' \approx \Theta(\mathcal{N})$, w.r.t the diamond norm on cptp maps.

Now, the resource theory is about possible free channel transformations, $\mathcal{N} \to \mathcal{N}' = \Theta(\mathcal{N})$.

Often we are happy with approximation: $\mathcal{N} \to \mathcal{N}' \approx \Theta(\mathcal{N})$, w.r.t the diamond norm on cptp maps.

To decide transformability, we seek to classify all monotones, i.e. real-valued functions f on channels s.t. $f(\mathbb{F})=0$ and for all free superchannels Θ, $f(\mathcal{N}) \geq f(\Theta(\mathcal{N}))$.

Constructions of monotones

1. Generating power: Let ω be a resource monotone on states, then

$$\Omega(\mathcal{N}) = \sup_{\rho} \omega(\mathcal{N}\otimes id(\rho)) - \omega(\rho)$$

defines a monotone on channels, which extends ω.

Constructions of monotones

1. *Generating power:* Let ω be a resource monotone on states, then

$$\Omega(\mathcal{N}) = \sup_{\rho} \omega(\mathcal{N}\otimes id(\rho)) - \omega(\rho)$$

defines a monotone on channels, which extends ω.

2. *Distance measures:* Let d be contractive on states (a metric or divergence), then

$$\Delta(\mathcal{N}) = \inf_{\mathcal{L}\in\mathbb{F}} \sup_{\rho} d(\mathcal{N}\otimes id(\rho), \mathcal{L}\otimes id(\rho))$$

defines a monotone on channels.

Constructions of monotones

3. Robustness is defined as

$$LR(N) = \inf_{L \in \mathbb{F}} D_{max}(N \| L), \text{ where}$$

$D_{max}(N \| L) = \log \min \lambda$ s.t. $N \leq \lambda L$ is the max-relative entropy, extended from states to channels.

Constructions of monotones

3. Robustness is defined as

$$LR(\mathcal{N}) = \inf_{\mathcal{L}\in\mathbb{F}} D_{max}(\mathcal{N}\|\mathcal{L}), \text{ where}$$

$D_{max}(\mathcal{N}\|\mathcal{L}) = \log \min \lambda$ s.t. $\mathcal{N} \leq \lambda\mathcal{L}$ is the max-relative entropy, extended from states to channels.

There is also a smooth version:

$$LR^{\varepsilon}(\mathcal{N}) = \inf_{\mathcal{N}'} \inf_{\mathcal{L}\in\mathbb{F}} D_{max}(\mathcal{N}'\|\mathcal{L}),$$

where $\mathcal{N}'$ runs over channels with $\mathcal{N}' \overset{\varepsilon}{\approx} \mathcal{N}$.

Why do we have so many monotones? In fact, often there will be many inequivalent ones.

This is related to the fact that $\mathcal{N} \to \mathcal{N}'$ under free superchannels is usually not a linear, only a partial order. Irreversibility!

Example: Pure bipartite states under LOCC or SEP. Infinite set of majorisation conditions necessary and sufficient...

[Nielsen, PRL 1999]

In some theories, reversibility (i.e. linear ordering) is restored in an asymptotic limit of large number of copies.
Many examples either way:

1) Mixed entangled states with LOCC:
bound entanglement, zoo of e-measures
[Cf. Christandl, PhD thesis 2006]

In some theories, reversibility (i.e. linear ordering) is restored in an asymptotic limit of large number of copies.
Many examples either way:

1) Mixed entangled states with LOCC:
bound entanglement, zoo of e-measures
[Cf. Christandl, PhD thesis 2006]

2) Mixed entangled states with $\overline{SEPP}$
(=almost separability-preserving channels):
Reversible with unique quantifier $E_r^\infty(\rho)$.
[Brandao/Plenio Nature Phys. 2008,
Brandão/Gour, PRL 2015]

In some theories, reversibility (i.e. linear ordering) is restored in an asymptotic limit of large number of copies.
Many examples either way:

1) Mixed entangled states with LOCC:
bound entanglement, zoo of e-measures
[Cf. Christandl, PhD thesis 2006]

2) Mixed entangled states with $\overline{SEPP}$
(=almost separability-preserving channels):
Reversible with unique quantifier $E_r^{\infty}(\rho)$.

⚡ Free operations not closed under $\otimes$! ⚡

3) (Thermodynamics toy model) Systems with Hamiltonian at temperature T, and under Gibbs-preserving channels:

- Work is a special resource, a state $|E\rangle$ of a battery.
- Work extractable from many copies of $\mathcal{N}$ is

$$W(\mathcal{N}) = \sup_{\rho} kTS(\mathcal{N}(\rho)) - \mathrm{Tr}\,\mathcal{N}(\rho)\mathcal{H} - kTS(\rho) + \mathrm{Tr}\,\rho\mathcal{H}$$

Free energy difference after–before :-)

[Navascués/García-Pintos, PRL 2015, Faist/Berta/Brandão, 1807.05610]

3) (Thermodynamics toy model) Systems with Hamiltonian at temperature T, and under Gibbs-preserving channels:

- Work is a special resource, a state $|E\rangle$ of a battery.
- Work extractable from many copies of N is $W(N) = \sup_\rho kTS(N(\rho)) - \mathrm{Tr}\, N(\rho)H - kTS(\rho) + \mathrm{Tr}\,\rho H$
- ...turns out to be the asymptotic cost of implementing many copies of N!

[Navascués/García-Pintos, PRL 2015, Faist/Berta/Brandão, 1807.05610]

4) *Entanglement-assisted communication between Alice and Bob*: Interesting because all states are free, but only the local channels are free.

[Bennett/Devetak/Harrow/Shor/AW, IEEE-IT 2014]

4) Entanglement-assisted communication between Alice and Bob: Interesting because all states are free, but only the local channels are free.

Subtheory of channels from Alice to Bob is reversible, and the rate of converting N into a perfect binary classical channel is the entanglement-assisted quantum capacity:

$$C_E(N) = \max_{|\varphi\rangle} I(A:B)_\rho \quad \text{s.t. } \rho=(id\otimes N)\varphi$$

[Bennett/Devetak/Harrow/Shor/AW, IEEE-IT 2014]

6. On multiple resources

For states, if you understand one, you understand many: having access to resource states ρ_1 , ρ_2, ..., ρ_n is the same as having access to $\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_n$ – just another state.

6. On multiple resources

For states, if you understand one, you understand many: having access to resource states $\rho_1, \rho_2, \ldots, \rho_n$ is the same as having access to $\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_n$ – just another state.

For channels, if you have N and M, you can clearly build $N \otimes M$, but also the compositions $N \circ F \circ M$ and $M \circ F' \circ N$, with free channels F and F'.

6. On multiple resources

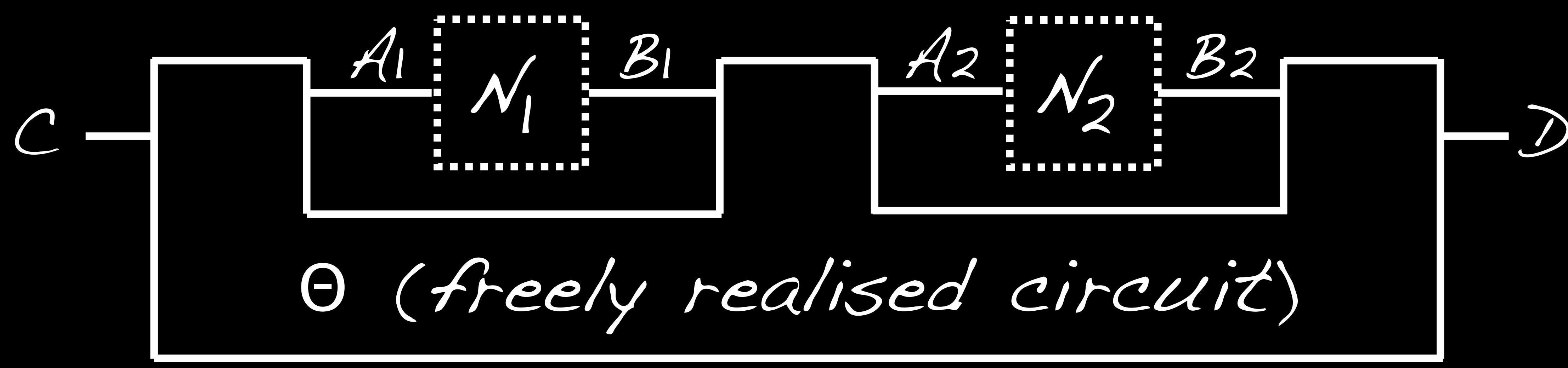
For states, if you understand one, you understand many: having access to resource states ρ_1, ρ_2, ..., ρ_n is the same as having access to $\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n$ – just another state.

For channels, if you have N and M, you can clearly build $N \otimes M$, but also the compositions $N \circ F \circ M$ and $M \circ F' \circ N$, with free channels F and F'.
Axiomatic way: No-signalling channels, and supermaps between them... Free supermaps?

[Cf. Bisio/Perinotti, arXiv:1806.09554]

6. On multiple resources

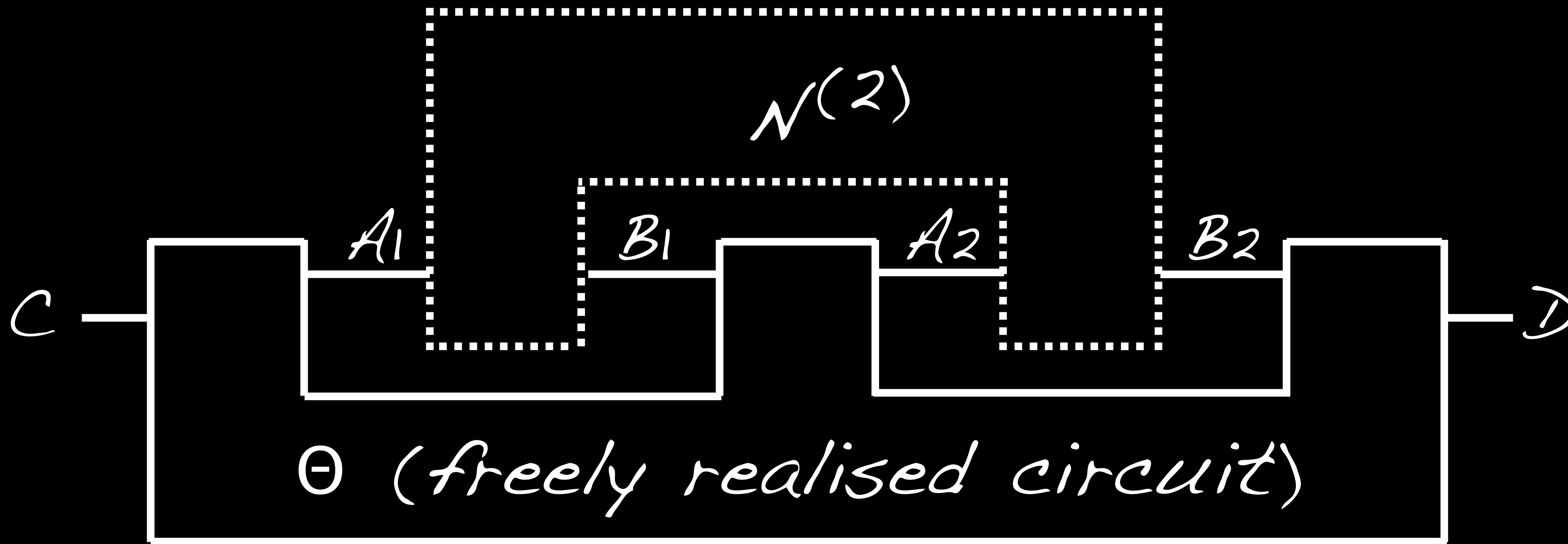
To avoid thorny issues (which however may have to be confronted eventually), let's stick with free transformations as being those realised by a free quantum circuit, with slots in which the resource channels are to be inserted in a given causal order:



[Chiribella/D'Ariano/Perinotti, PRL 2008]

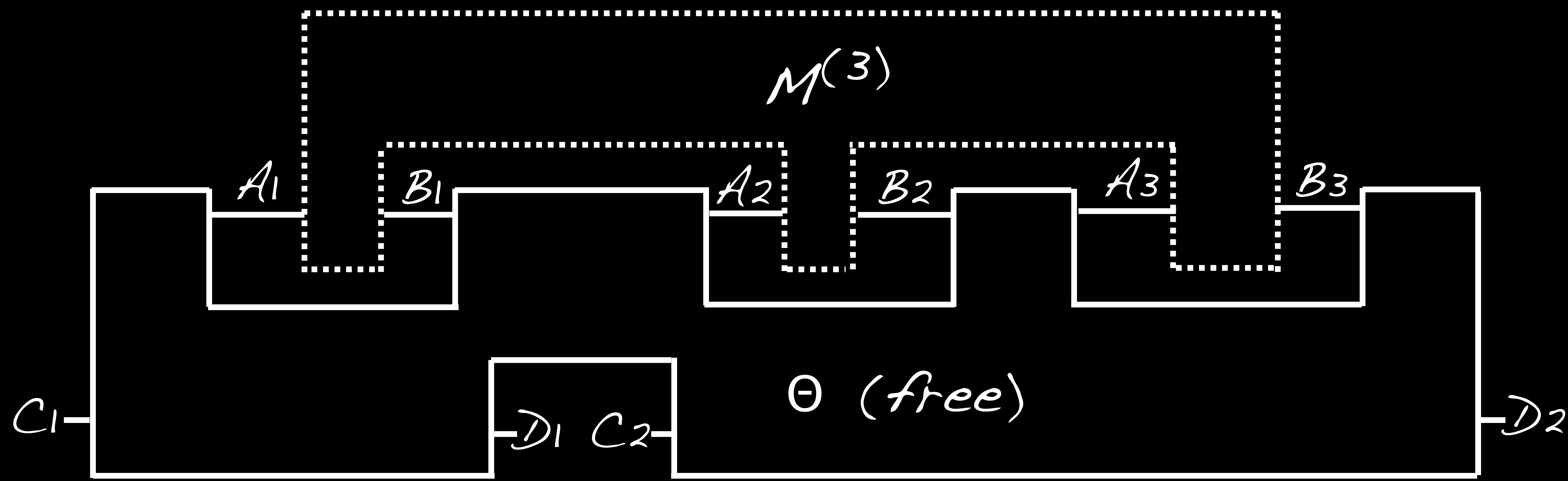
6. On multiple resources

Actually transforms a memory channel into a channel, by means of a free memory channel:



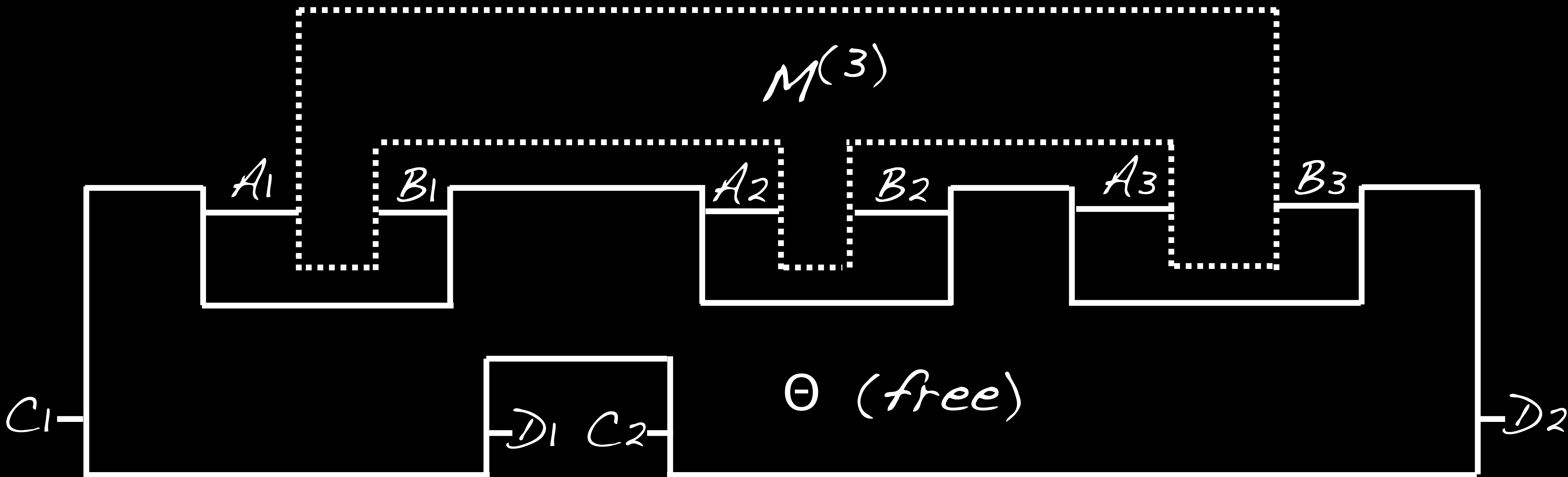
[Chiribella/D'Ariano/Perinotti, PRL 2008]

...And more generally, memory channels to memory channels:

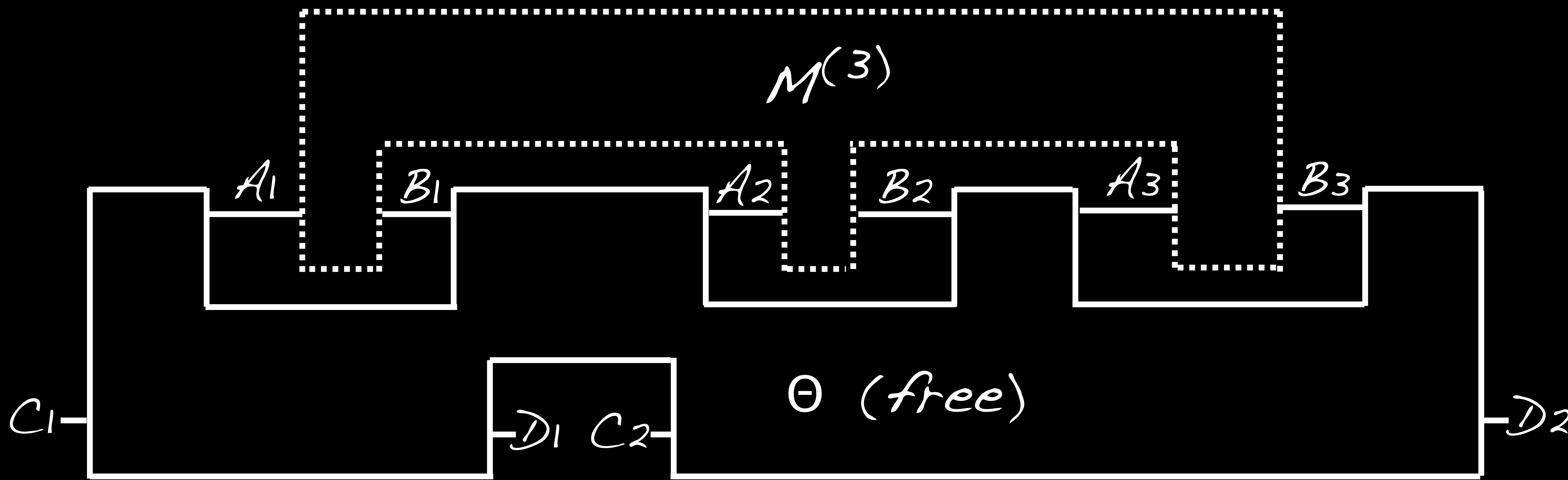


[Chiribella/D'Ariano/Perinotti, PRL 2008]

...And more generally, memory channels to memory channels:



Note: Even when the input $M^{(3)}$ is a product of independent channels, the output $N^{(2)} = \Theta \circ M^{(3)}$ is in general a memory channel! [Chiribella/D'Ariano/Perinotti, PRL 2008]



Means: We want a resource theory not of channels, but of memory channels (combs), transformed to other such objects via freely realised memory channels (combs).

Natural metric: comb-extension of ◇-norm.

[Chiribella/D'Ariano/Perinotti, Europhys. Lett. 2008]

Means: We want a resource theory not of channels, but of memory channels (combs), transformed to other such objects via freely realised memory channels (combs).

Natural metric: comb-extension of ◇-norm.

Outstanding project/work in progress:

Define monotones for memory channels, or extend them from states and channels.

For product channels, amortised measures are good candidates (see Mark Wilde's talk).

Outstanding project/work in progress:

Define monotones for memory channels, or extend them from states and channels.

Generalised amortised channel divergences:

For a divergence D on states, and degree-t memory channels $N = N^{(t)}$ and $M = M^{(t)}$, let

$$D^A(N\|M) := \sup_{\Theta,\rho,\sigma} D((\Theta\circ N)\rho\|(\Theta\circ N)\sigma) - D(\rho\|\sigma)$$

Test combs (deg. $t+1$) → Θ

Test states → ρ,σ

(...)

[Berta/Hirche/Kaur/Wilde, 1808.01498; Wang/Wilde, 1907.06306]

7. Resource erasure

Universal quantifier of resourceness: How much randomness is required to destroy a given resource channel N?

[Groisman/Popescu/AW, PRA 2005; Berta/Majenz, 1708.00360; Anshu/Hsieh/Jain, 1708.00381]

7. Resource erasure

Universal quantifier of resourceness: How much randomness is required to destroy a given resource channel $\mathcal{N}$?

Assume that there is a free $\mathcal{F} \in \mathbb{F}(A' \to B')$ and an ensemble $\{p_i, \mathcal{U}_i, \mathcal{V}_i\}$ of free unitaries, s.t.

$$\sum_{i=1}^{k} p_i \, \mathcal{V}_i \circ (\mathcal{N} \otimes \mathcal{F}) \circ \mathcal{U}_i \overset{\varepsilon}{\approx} \mathcal{M} \in \mathbb{F}(AA' \to BB').$$

[Groisman/Popescu/AW, PRA 2005; Berta/Majenz, 1708.00360; Anshu/Hsieh/Jain, 1708.00381]

7. Resource erasure

Universal quantifier of resourceness: How much randomness is required to destroy a given resource channel $\mathcal{N}$?

Assume that there is a free $\mathcal{F} \in \mathbb{F}(A' \to B')$ and an ensemble $\{p_i, \mathcal{U}_i, \mathcal{V}_i\}$ of free unitaries, s.t.

$$\sum_{i=1}^{k} p_i \, \mathcal{V}_i \circ (\mathcal{N} \otimes \mathcal{F}) \circ \mathcal{U}_i \overset{\varepsilon}{\approx} \mathcal{M} \in \mathbb{F}(AA' \to BB').$$

Then by forgetting i (i.e. $H(p) \leq \log k$ bits) we destroy the resource.

[Groisman/Popescu/AW, PRA 2005; Berta/Majenz, 1708.00360; Anshu/Hsieh/Jain, 1708.00381]

7. Resource erasure

Assume that there is a free $F \in \mathbb{F}(A' \rightarrow B')$ and an ensemble $\{p_i, U_i, V_i\}$ of free unitaries, s.t.

$$\sum_{i=1}^{k} p_i\, V_i \circ (N \otimes F) \circ U_i \overset{\varepsilon}{\approx} M \in \mathbb{F}(AA' \rightarrow BB').$$

Then by forgetting i ($\log k$ bits) we destroy the resource, approximately:

$$\mathrm{COST}_{\varepsilon}(N) := \min \log k.$$

[Groisman/Popescu/AW, PRA 2005; Berta/Majenz, 1708.00360; Anshu/Hsieh/Jain, 1708.00381]

7. Resource erasure

Assume that there is a free $F \in \mathbb{F}(A' \to B')$ and an ensemble $\{p_i, U_i, V_i\}$ of free unitaries, s.t.

$$\sum_{i=1}^{k} p_i \, V_i \circ (N \otimes F) \circ U_i \overset{\varepsilon}{\approx} M \in \mathbb{F}(AA' \to BB').$$

Equivalent to N if the resource theory has free states and free partial trace! Assume from now..

[Groisman/Popescu/AW, PRA 2005; Berta/Majenz, 1708.00360; Anshu/Hsieh/Jain, 1708.00381]

7. Resource erasure

Assume that there is a free $F \in \mathbb{F}(A' \to B')$ and an ensemble $\{p_i, \mathcal{U}_i, \mathcal{V}_i\}$ of free unitaries, s.t.

$$\sum_{i=1}^{k} p_i \, \mathcal{V}_i \circ (\mathcal{N} \otimes F) \circ \mathcal{U}_i \overset{\varepsilon}{\approx} \mathcal{M} \in \mathbb{F}(AA' \to BB').$$

$$\mathrm{COST}_\varepsilon(\mathcal{N}) := \min \log k.$$

Theorem: $\mathrm{COST}_\varepsilon(\mathcal{N}) \approx LR^\zeta(\mathcal{N}) + O(1)$, with

$$\varepsilon/2 < \zeta < 2\sqrt{\varepsilon}$$

(Assuming theory has free permutations. Proof uses generalised "convex-split lemma")

[Liu/AW, arXiv:1904.04201. Extends Anshu/Hsieh/Jain, 1708:00381 for states!]

8. Conclusion

- Channels not enough: Memory channels for a minimal self-consistent theory!
- General questions are hard to answer, but there are some common features: log-robustness plays a universal role both for resource destruction (extends to general memory channels ✓), and for channel simulation... [Cf. García Díaz et al., 1805.04045 for coherence; Faist/Berta/Brandão, 1807.05610 for thermodynamics]

8. Conclusion

- Question about asymptotics: Rate of randomness to destroy resource $N^{\otimes n}$?

$$\mathrm{COST}^{\infty}(N) = \sup_{\varepsilon} \lim_{n}(\inf/\sup)\ \mathrm{COST}_{\varepsilon}(N^{\otimes n})/n$$
$$= \sup_{\varepsilon} \lim_{n}(\inf/\sup)\ LR^{\varepsilon}(N^{\otimes n})/n$$
$$= ???$$

8. Conclusion

- Question about asymptotics: Rate of randomness to destroy resource $N^{\otimes n}$?

$$COST^{\infty}(N) = \sup_{\varepsilon} \lim_{n}(\inf/\sup)\, COST_{\varepsilon}(N^{\otimes n})/n$$
$$= \sup_{\varepsilon} \lim_{n}(\inf/\sup)\, LR^{\varepsilon}(N^{\otimes n})/n$$
$$= ???$$

For states [Brandao/Gour, PRL 2015; Anshu/Hsieh/Jain, 1708.00381]:

$$COST^{\infty}(\rho) = D_{\mathbb{F}}^{\infty}(\rho) = \lim_{n} \min_{\sigma\in\mathbb{F}} D(\rho^{\otimes n}\|\sigma)/n$$

Quantum asymptotic equipartition property: From states to channels?!