A Potporri of Diagrams

J. Scott Carter

De-institutionalized

Banff, BIRS, Nov 2019

Acknowledgements

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7. And of course, Alex, Jeff, and BIRS 19w5118

Goals 4 2day

1. Show a lot of pictures

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Multi-cats

- We'll work in a multi-category

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- Objects, 1-arrows, double arrows, triple arrows, etc.

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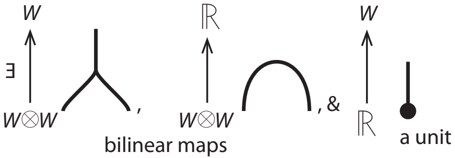
Finicky: 

Gratuitous internet cat picture.

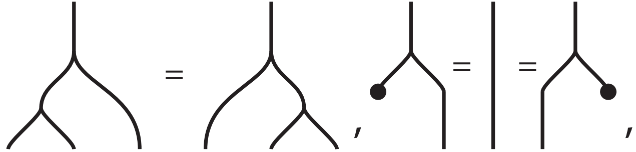
Example 1

Frobenius Algebra axioms:

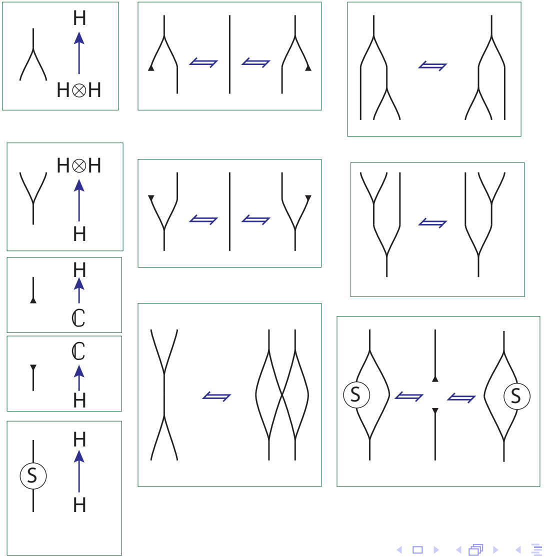
W is a vector space for which



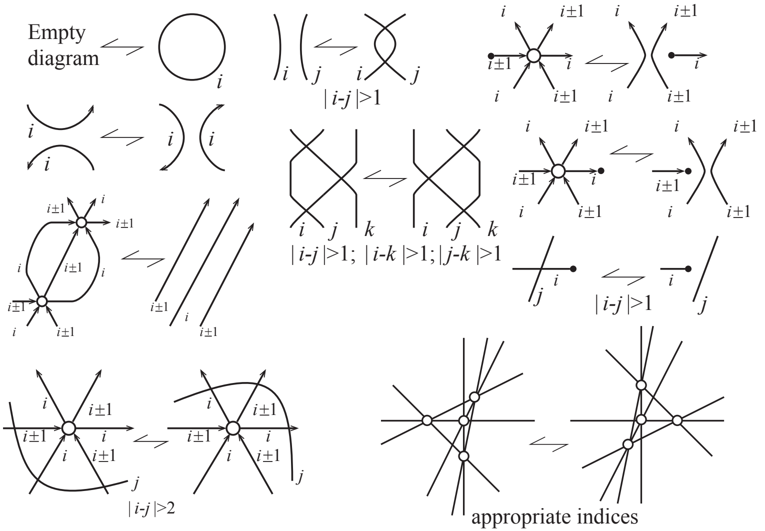
such that



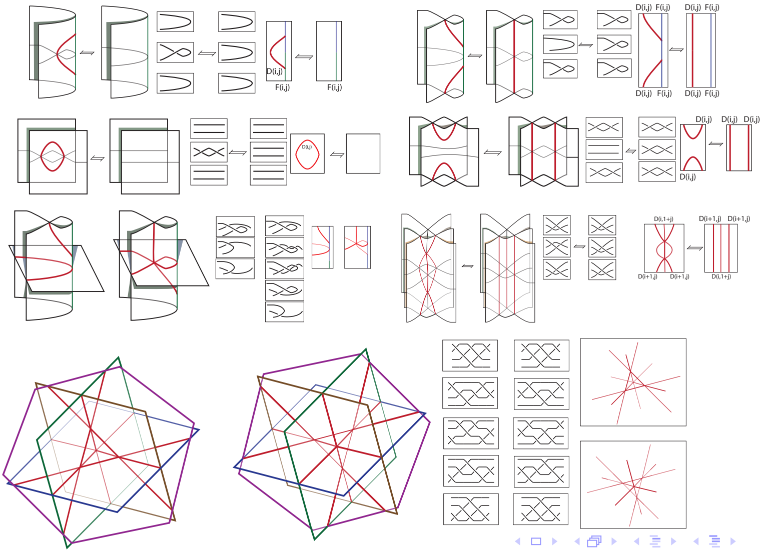
Example 2



Example 3


Empty
diagram
i
i j i j
| i-j |>1
i i±1 i i±1
i±1 i i
i i±1 i i±1
i
i i
i
i i±1 i i±1
i±1 i i±1 i
i i±1 i i±1
i±1 i
i±1 i±1
i i±1 i
i±1 i±1
i i±1 i±1 i i±1
i j k i j k
| i-j |>1; | i-k |>1; | j-k |>1
i i
j | i-j |>1 j
i i±1 i i±1
i±1 i i±1 i
i i±1 i i±1
j j
| i-j |>2
appropriate indices

Example 4


D(i,j)
F(i,j)
F(i,j)
D(i,j) F(i,j)
D(i,j) F(i,j)
D(i,j) F(i,j)
D(i,j) F(i,j)
D(i,j)
D(i,j)
D(i,j) D(i,j)
D(i,j)
D(i,1+j)
D(i+1,j) D(i+1,j)
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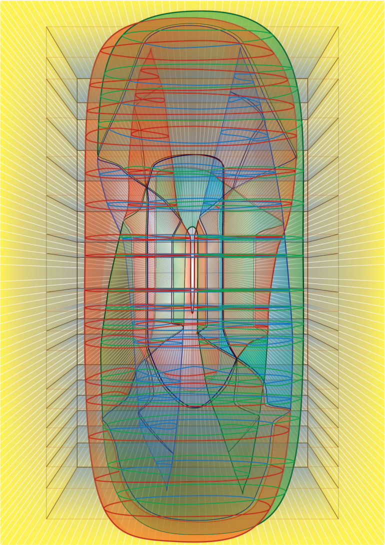
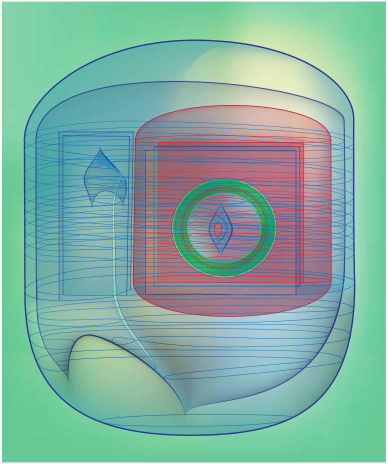
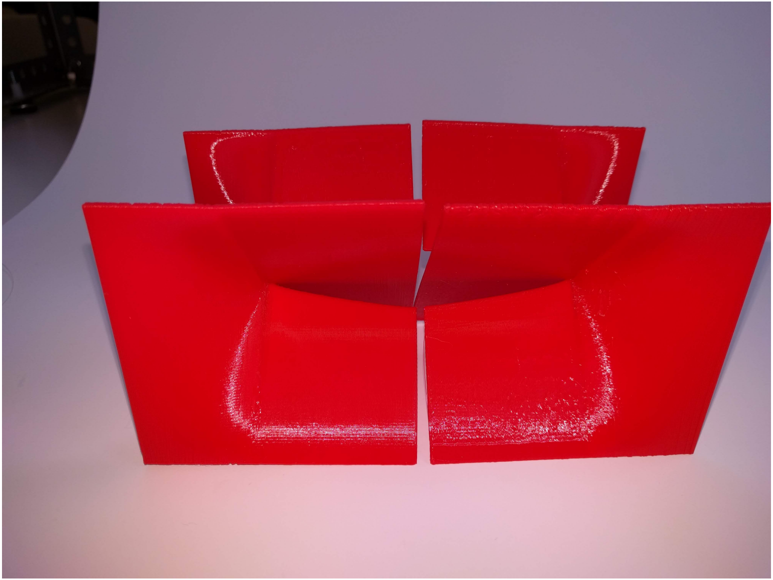
Satoh-Shima,Inoue,Kawamura



A knotted p2



Models



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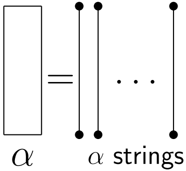

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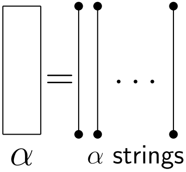
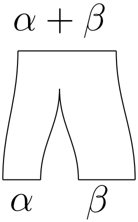


Thus



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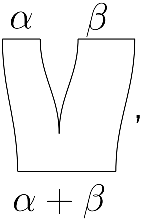
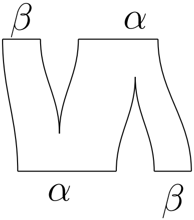


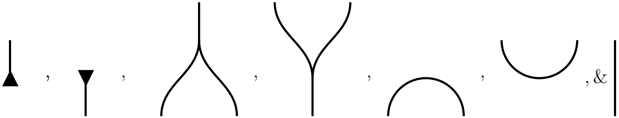
, and



all repr. id. on $\alpha + \beta$.

Def. 1-arrows, part 1

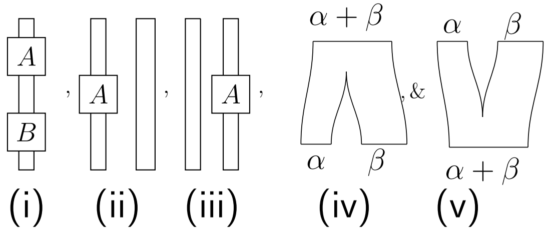
The diagrams here



are arrows.

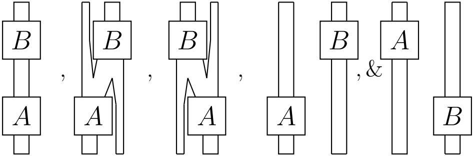
Def 1-arrows, part 2

if A and B are arrows with suitable sources and targets, then each of the diagrams here



is an arrow.

Forms of 1-arrows



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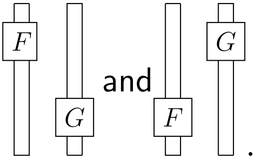
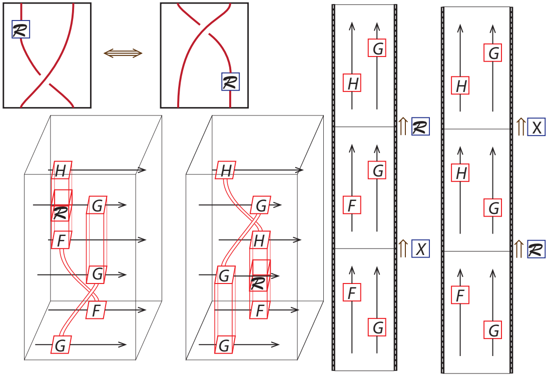
Exchanger axiom. *Suppose that* $\gamma \xleftarrow{F} \alpha$ *and* $\zeta \xleftarrow{G} \beta$ *are arrows. There is a natural family* X *of 2-arrows*

$$\mathsf{X} : (F \otimes \mathsf{I}_\zeta) \circ (\mathsf{I}_\alpha \otimes G) \Rightarrow (\mathsf{I}_\gamma \otimes G) \circ (F \otimes \mathsf{I}_\beta)$$

which are 2-isomorphisms. Here $(F \otimes \mathsf{I}_\zeta) \circ (\mathsf{I}_\alpha \otimes G)$ and $(\mathsf{I}_\gamma \otimes G) \circ (F \otimes \mathsf{I}_\beta)$ are algebraic expressions of the graphic:





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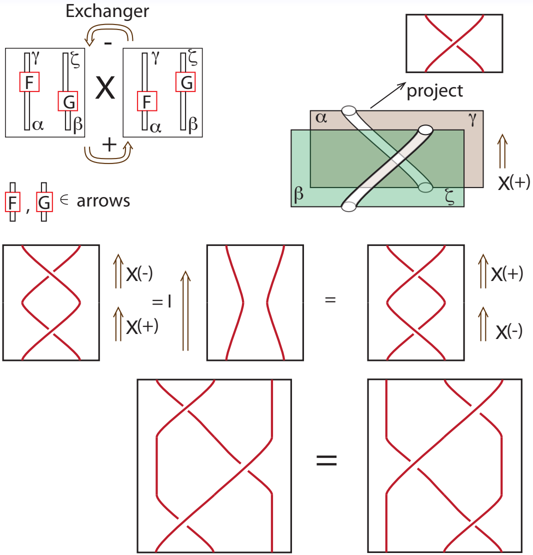

Exchanger
-
X
+
γ
ζ
F
G
α
β
F , G ∈ arrows
project
X(+)
X(-)
= I
=

The 1-arrows Y, ⅄, $\cap$, and $\cup$ are secretly 2-arrows

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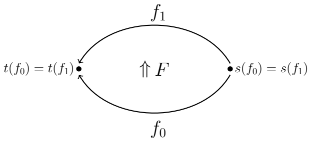
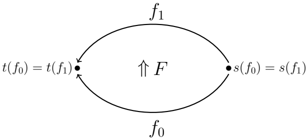
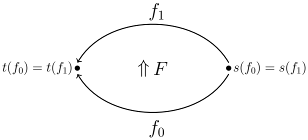
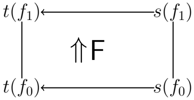

f_1
$t(f_0) = t(f_1)$
$\Uparrow F$
$s(f_0) = s(f_1)$
f_0



is the same as



is the same as



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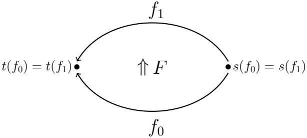

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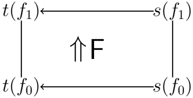
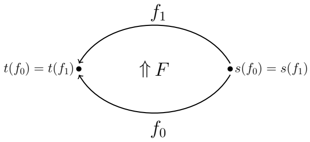


,

and such 2-arrows are composed globularly



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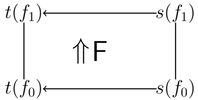

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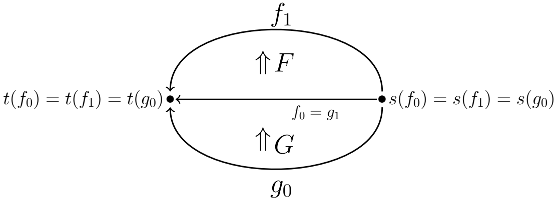

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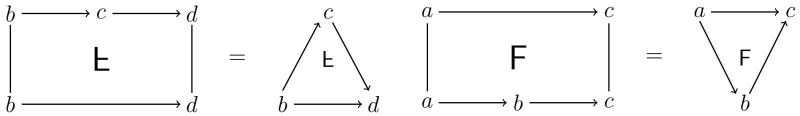

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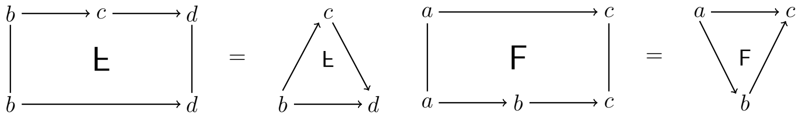

and allow

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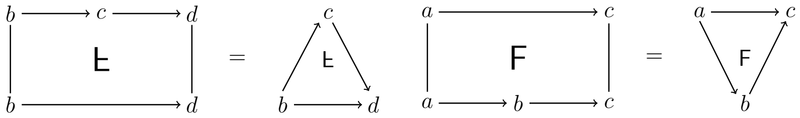
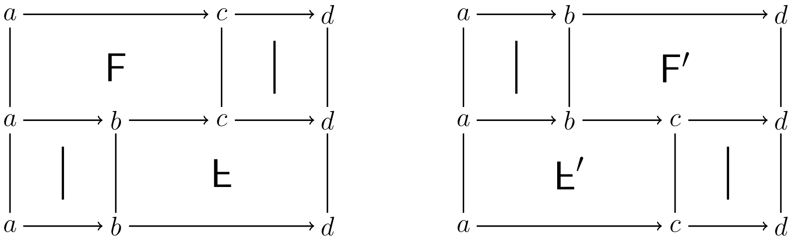


and allow



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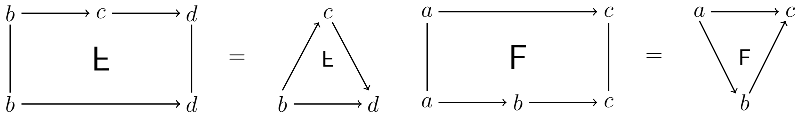

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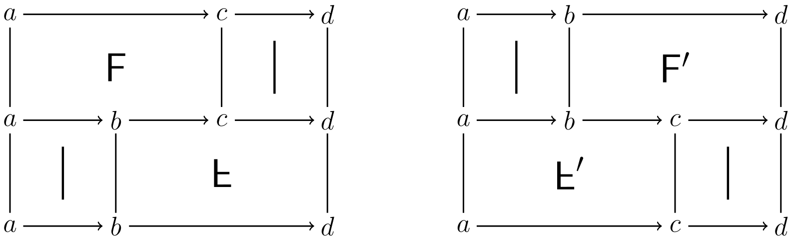

un-directed edges are identities.

These are also written as

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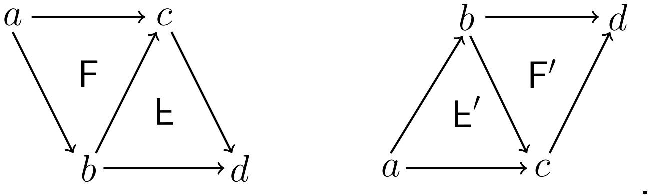
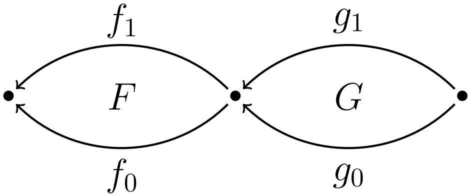
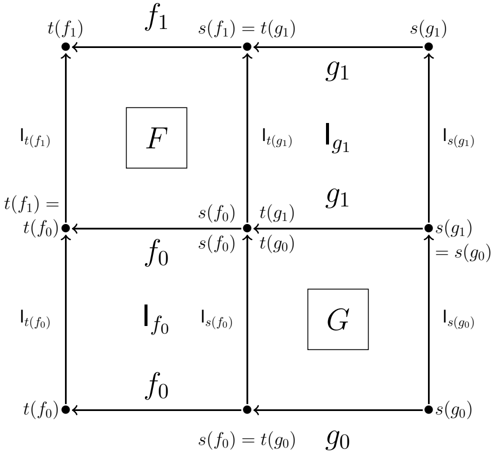


Disallow:



Replace with



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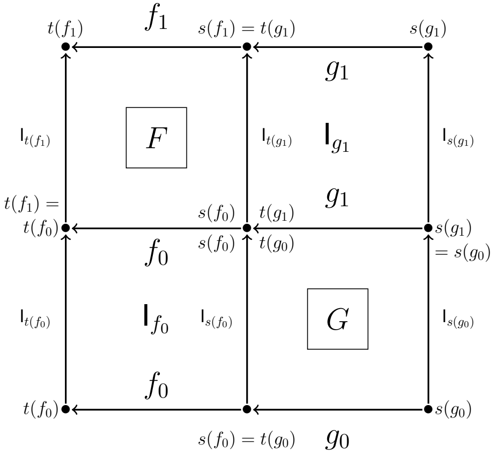

Then the exchanger X is a triple arrow.

Apology:

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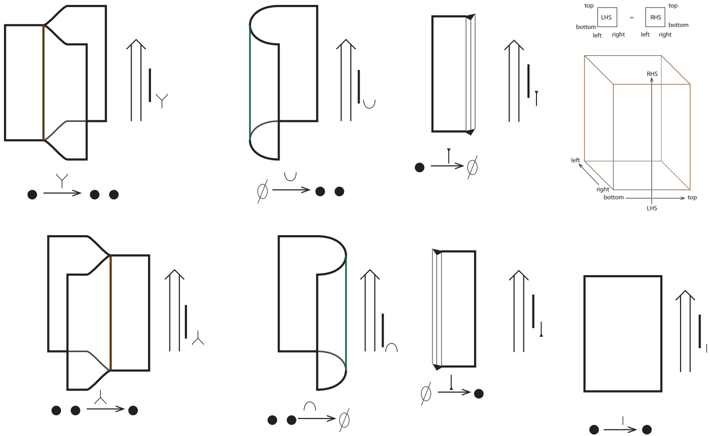

Since different things are not the same,

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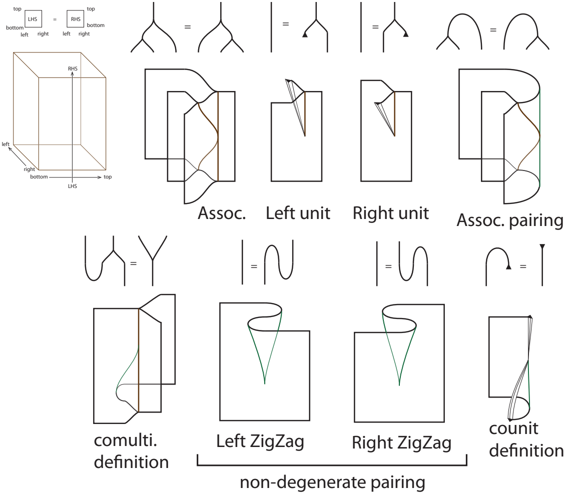
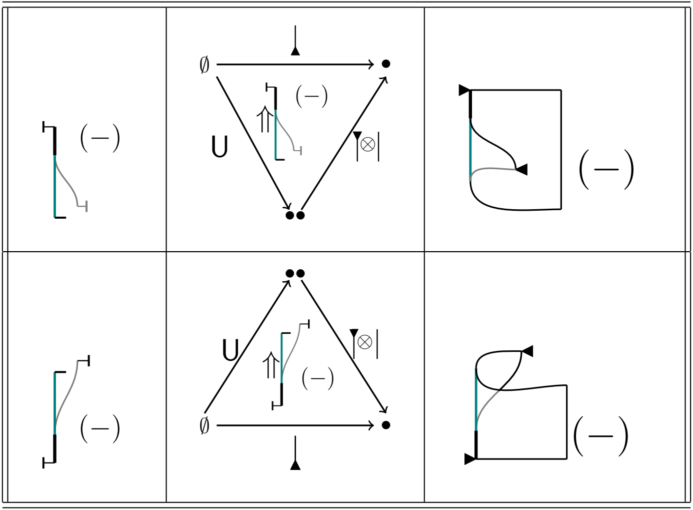
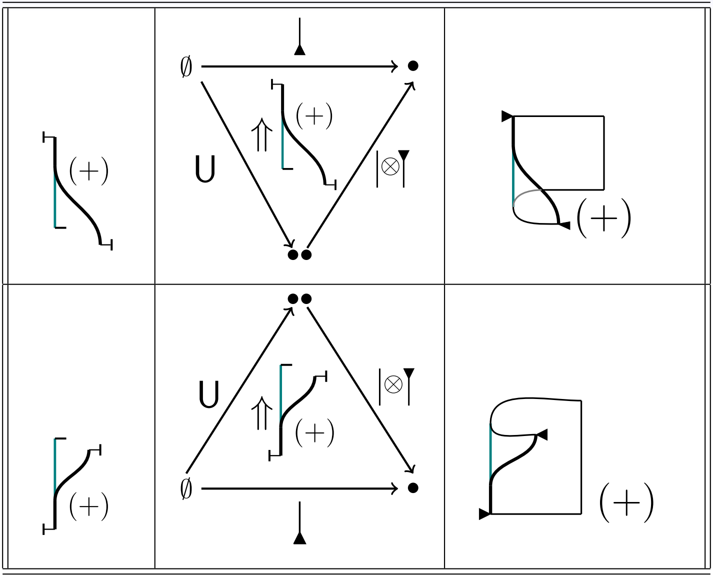
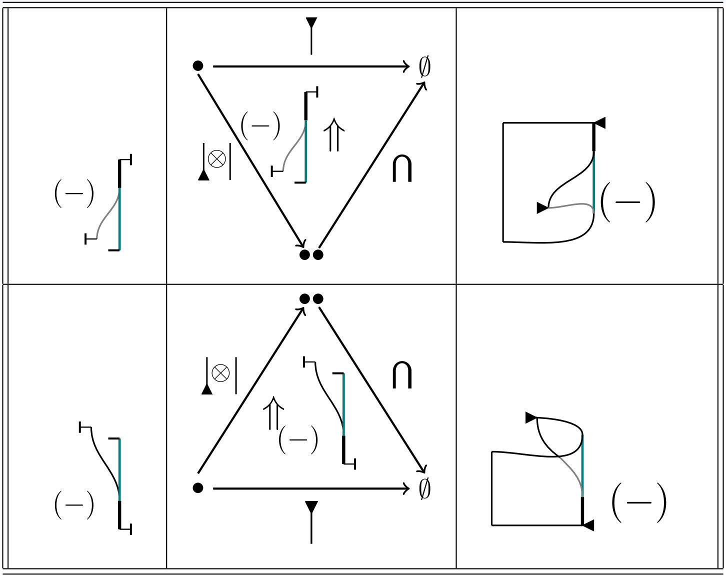
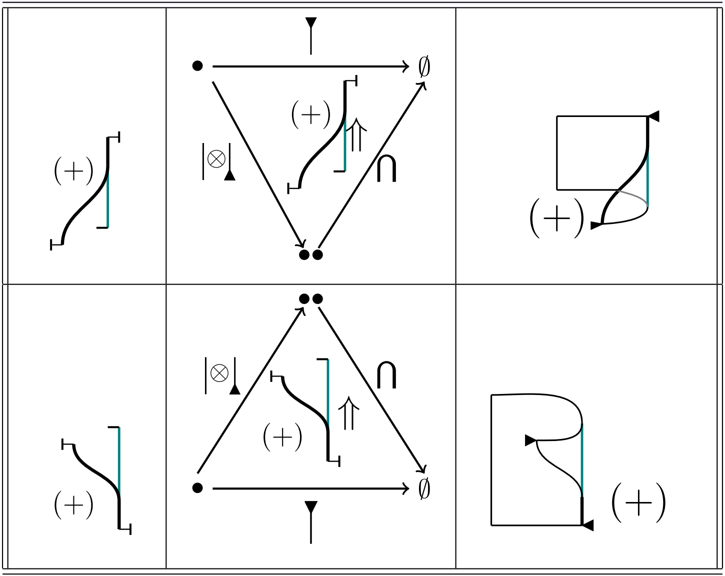


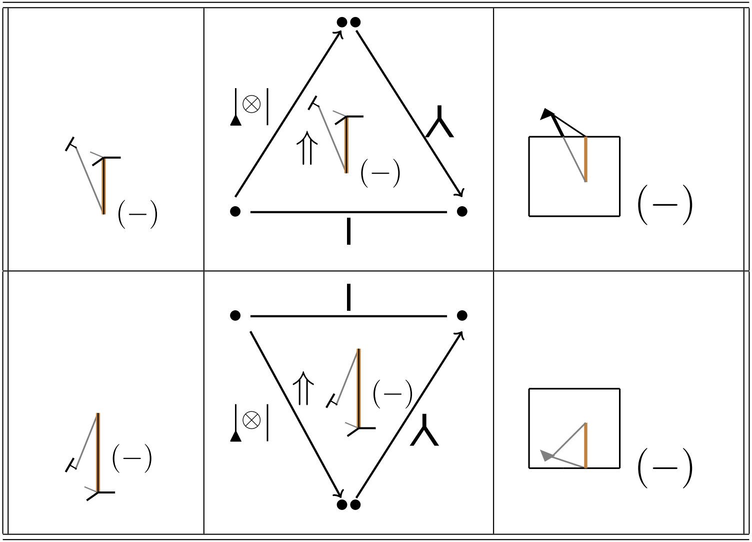
Glyphography

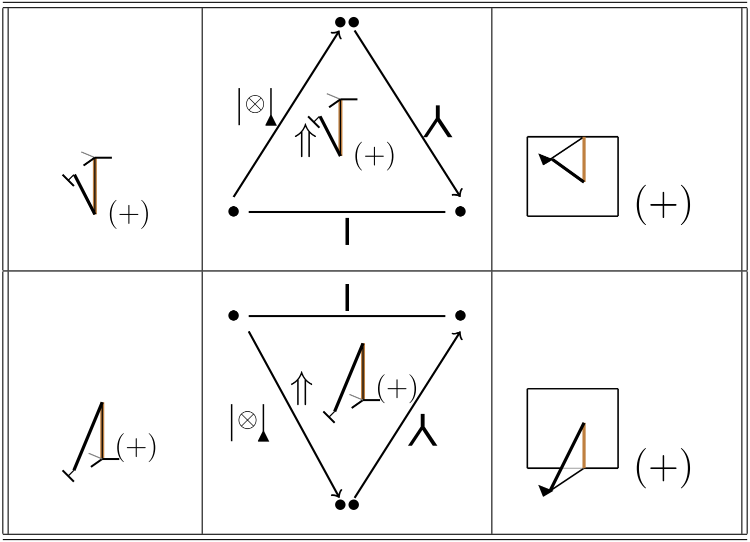

(−)
∅
U
|⊗|
(−)
(−)
(−)
U
|⊗|
(−)
∅
(−)

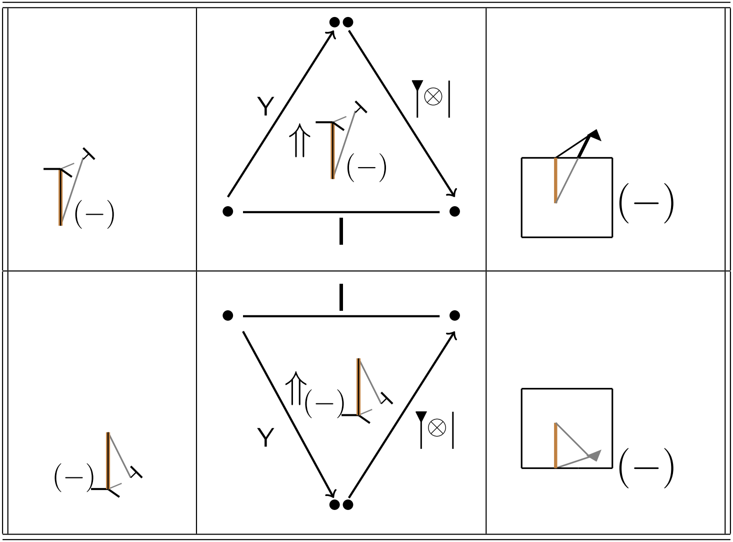

(+)
∅
U
(+)
(+)
(+)
U
(+)
∅
(+)



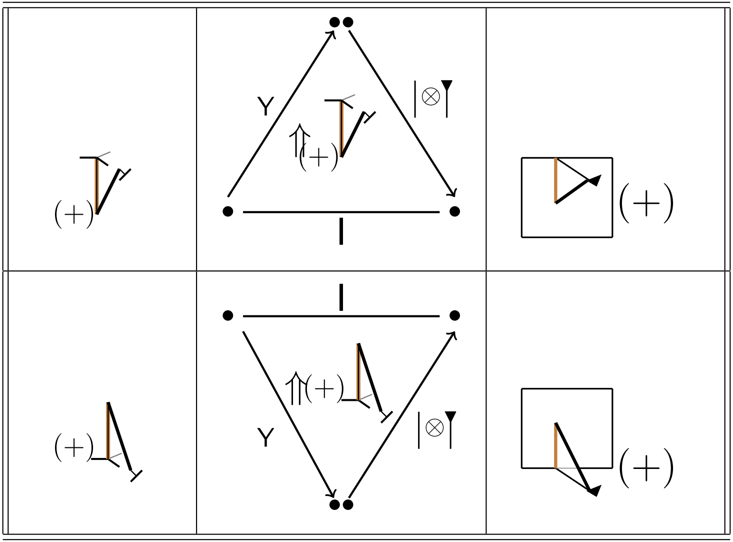


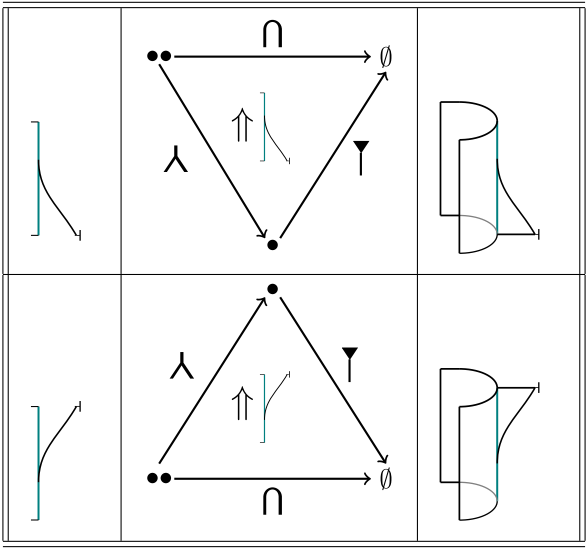


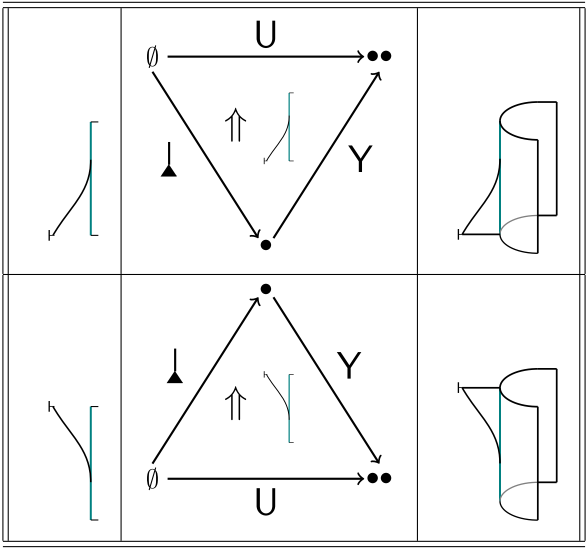


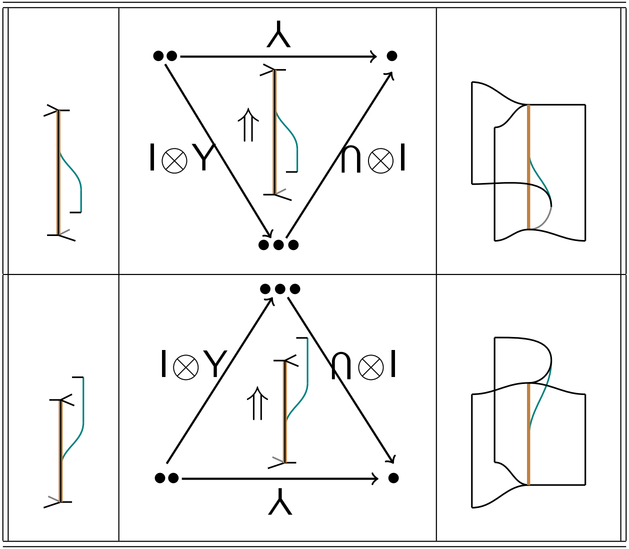


(—)
Y
I
(—)
(—)
I
(—)
Y
(—)

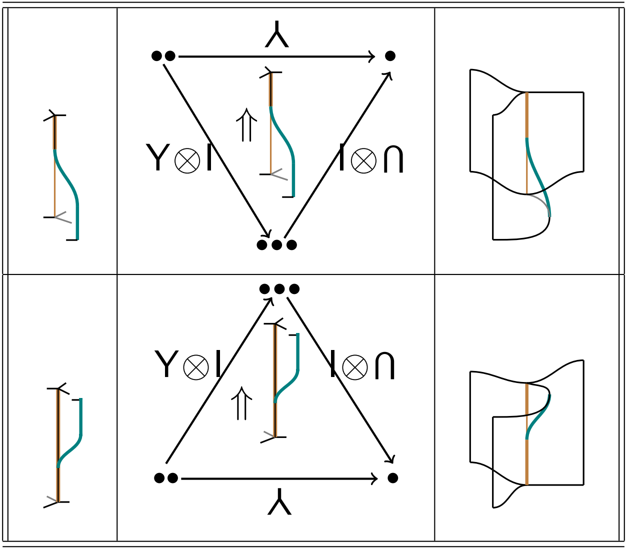

(+)
Y
⇑(+)
|⊗|
(+)
(+)
⇑(+)
Y
|⊗|
(+)

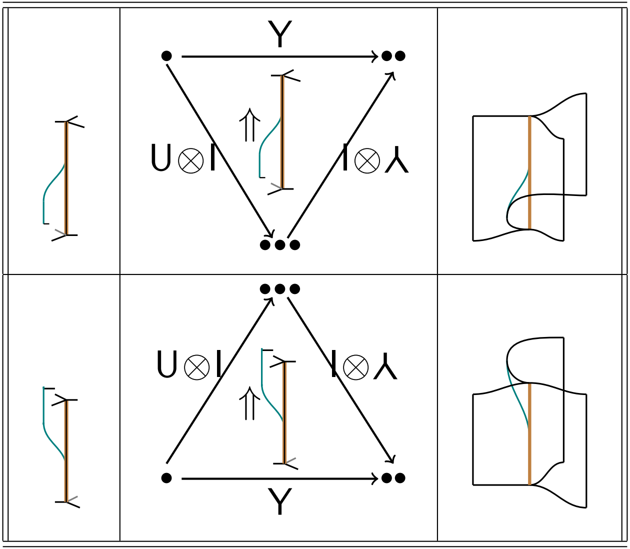


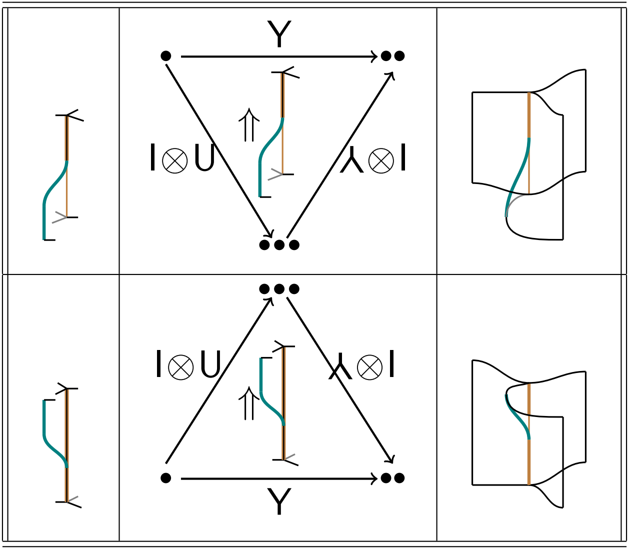

U
∅
Y
Y
∅
U



⅄
I⊗Y
∩⊗I
⇑
I⊗Y
∩⊗I
⇑
⅄


⅄
Y⊗I
I⊗∩
⇑
Y⊗I
I⊗∩
⇑
⅄


Y
U⊗I
I⊗⅄
⇑
U⊗I
I⊗⅄
⇑
Y


Y
I⊗U
⅄⊗I
I⊗U
⅄⊗I
Y

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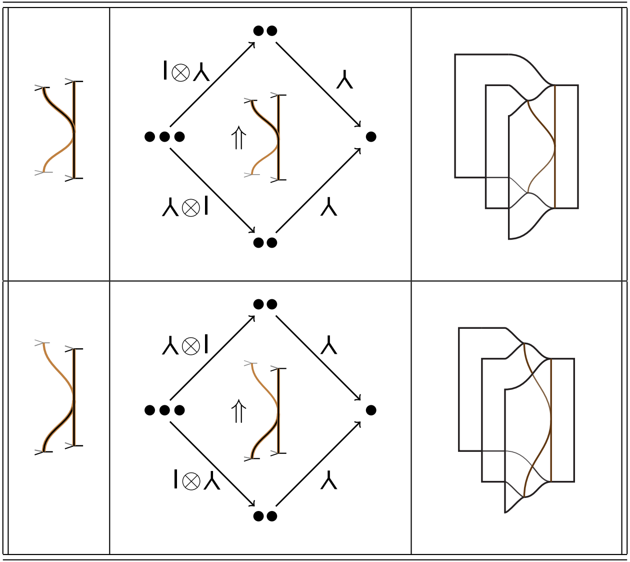
I⊗⅄
⅄
⇑
⅄⊗I
⅄
⅄⊗I
⅄
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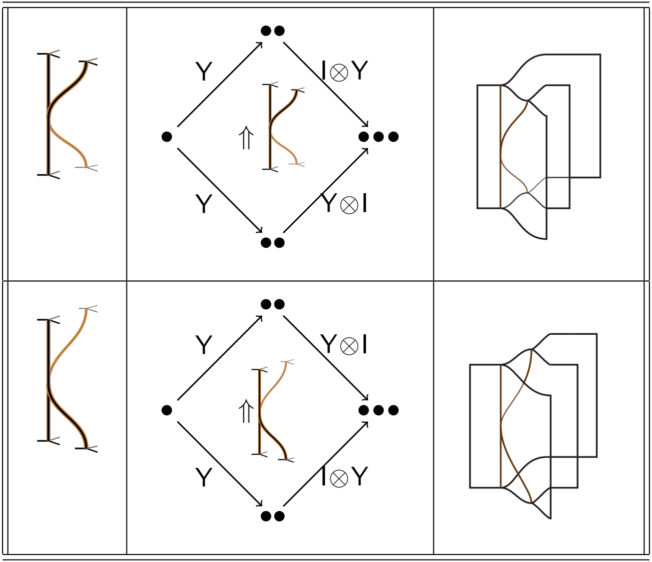
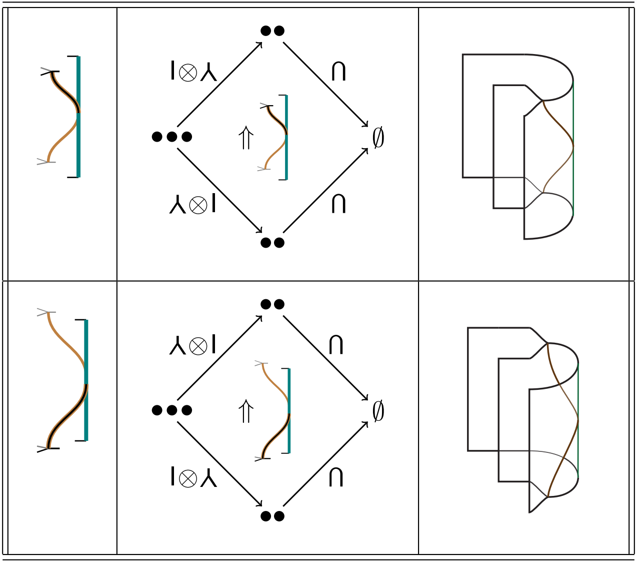
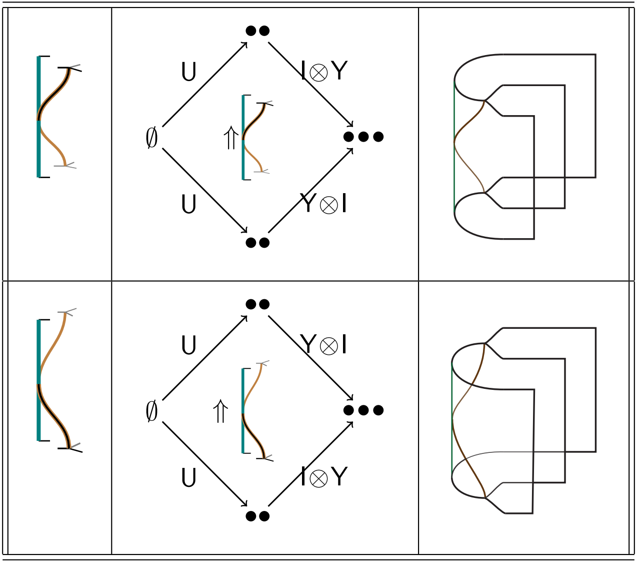
I⊗⅄
⅄

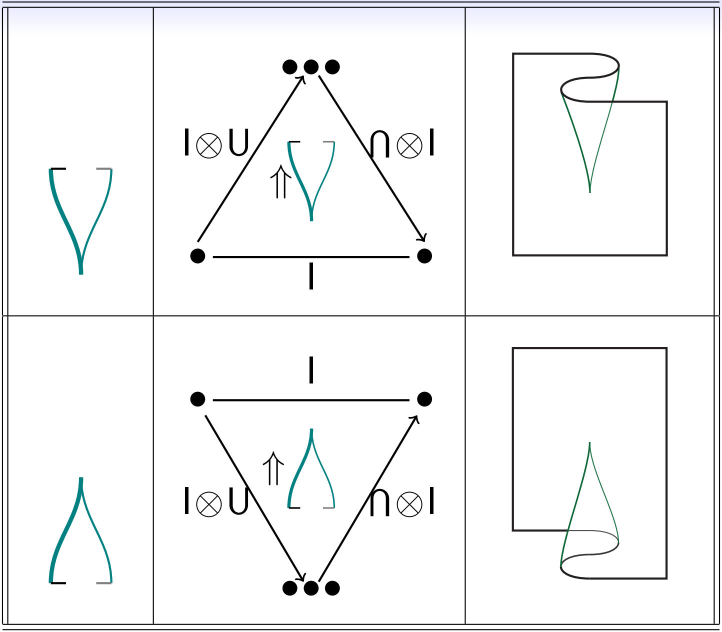


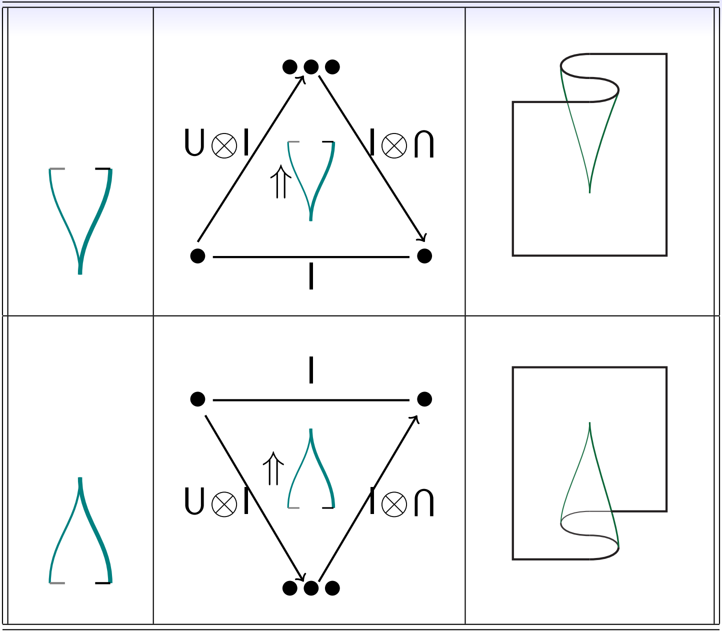
Y
I⊗Y
⇑
Y
Y⊗I
Y
Y⊗I
⇑
Y
I⊗Y

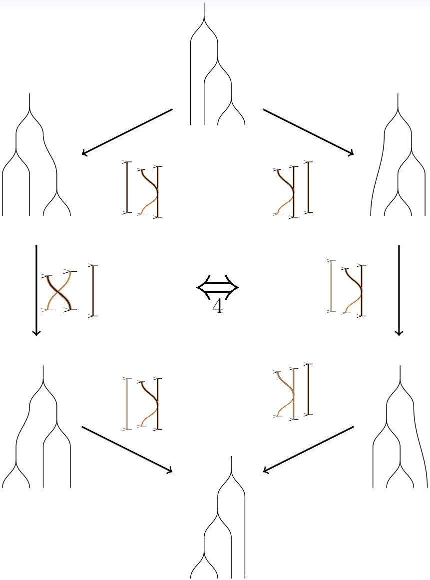

I⊗⅄
∩
⇑
∅
⅄⊗I
∩
⅄⊗I
∩
⇑
∅
I⊗⅄
∩

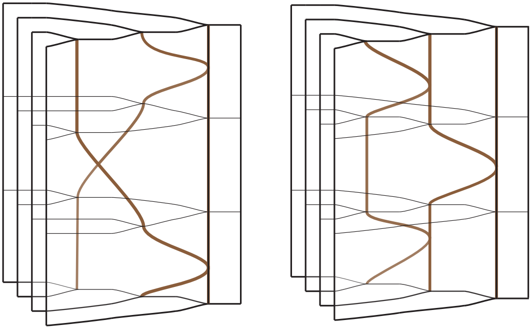


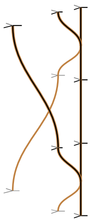
U
I⊗Y
∅
⇑
U
Y⊗I
U
Y⊗I
∅
⇑
U
I⊗Y


I⊗U
∩⊗I
I
⇑
I
⇑
I⊗U
∩⊗I

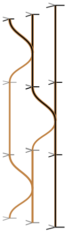

U⊗I
I⊗∩
I
U⊗I
I⊗∩
I

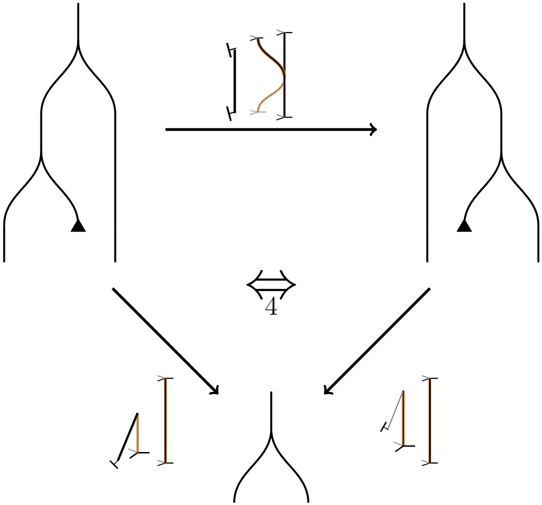

4

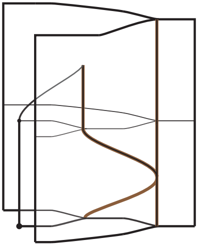
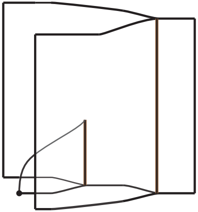


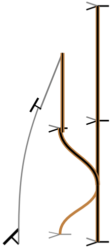


$$\underset{4}{\Longleftrightarrow}$$

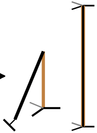



4



$$\Leftrightarrow_4$$



So, for example,

So, for example, the Joyal-Street axioms for associative unital structures can be given in a diagrammatic fashion.

So, for example, the Joyal-Street axioms for associative unital structures can be given in a diagrammatic fashion. Note that not all the unit axioms have been stated here.

Simplify and Abstract

S'pose that there are two objects $\mathtt{t}$ and $\mathtt{f}$ in a multi-cat. $\mathcal{S}$

Simplify and Abstract

S'pose that there are two objects t and f in a multi-cat. $\mathcal{S}$ In addition, there are arrows:

Simplify and Abstract

S'pose that there are two objects t and f in a multi-cat. $\mathcal{S}$ In addition, there are arrows:

f — f

Simplify and Abstract

S'pose that there are two objects t and ʇ in a multi-cat. $\mathcal{S}$ In addition, there are arrows:

ʇ — ʇ

t — t

Simplify and Abstract

S'pose that there are two objects t and ʇ in a multi-cat. $\mathcal{S}$ In addition, there are arrows:

ʇ —— ʇ

t —— t

p : ʇ $\to$ t ,

Simplify and Abstract

S'pose that there are two objects t and ʇ in a multi-cat. $\mathcal{S}$ In addition, there are arrows:

ʇ —— ʇ

t —— t

p : ʇ → t , and

b : t → ʇ.

Simplify and Abstract

S'pose that there are two objects t and f in a multi-cat. $\mathcal{S}$ In addition, there are arrows:

f — f

t — t

p : f $\rightarrow$ t , and

b : t $\rightarrow$ f.

In general, a non-id. arrow is a finite sequence

Simplify and Abstract

S'pose that there are two objects t and f in a multi-cat. $\mathcal{S}$ In addition, there are arrows:

f — f

t — t

$\mathsf{p} : \mathsf{f} \to \mathsf{t}$, and

$\mathsf{b} : \mathsf{t} \to \mathsf{f}$.

In general, a non-id. arrow is a finite sequence $\mathsf{pbpb} \cdots \mathsf{b}$,

Simplify and Abstract

S'pose that there are two objects t and ʇ in a multi-cat. $\mathcal{S}$ In addition, there are arrows:

ʇ — ʇ

t — t

p : ʇ → t , and

b : t → ʇ.

In general, a non-id. arrow is a finite sequence pbpb $\cdots$ b, pbpb $\cdots$ p,

Simplify and Abstract

S'pose that there are two objects t and f in a multi-cat. $\mathcal{S}$ In addition, there are arrows:

f —— f

t —— t

p : f $\rightarrow$ t , and

b : t $\rightarrow$ f.

In general, a non-id. arrow is a finite sequence pbpb $\cdots$ b, pbpb $\cdots$ p, bpbp $\cdots$ b, or

Simplify and Abstract

S'pose that there are two objects t and f in a multi-cat. $\mathcal{S}$ In addition, there are arrows:

f — f

t — t

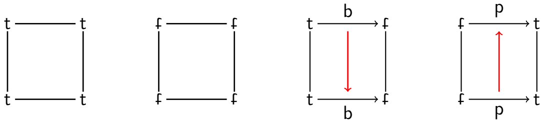
$\mathsf{p} : \mathsf{f} \to \mathsf{t}$, and

$\mathsf{b} : \mathsf{t} \to \mathsf{f}$.

In general, a non-id. arrow is a finite sequence $\mathsf{pbpb} \cdots \mathsf{b}$, $\mathsf{pbpb} \cdots \mathsf{p}$, $\mathsf{bpbp} \cdots \mathsf{b}$, or $\mathsf{bpbp} \cdots \mathsf{p}$.

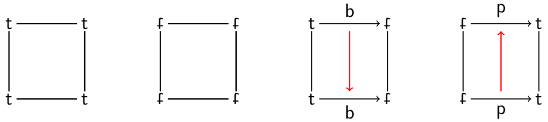
Gen. 2-arrows.

Ids:



Gen. 2-arrows.

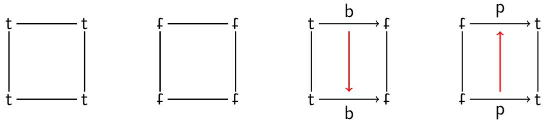
Ids:



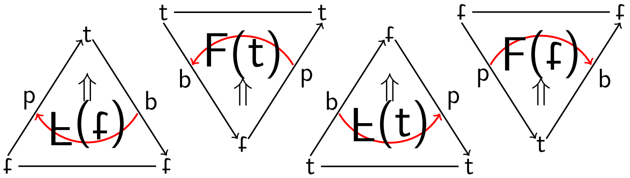
Gen. double arrows:

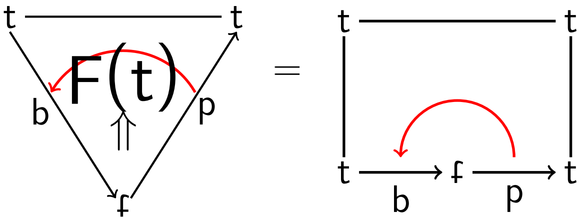
Gen. 2-arrows.

Ids:

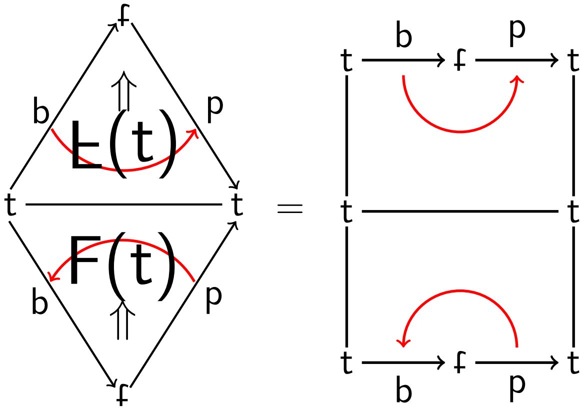


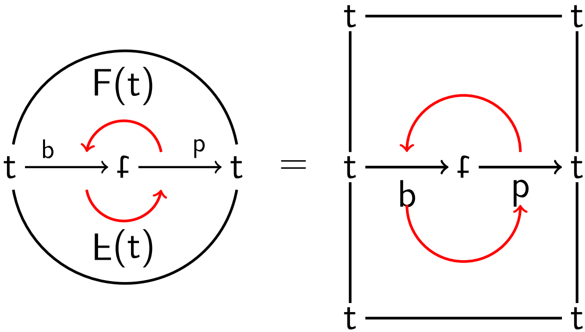
Gen. double arrows:

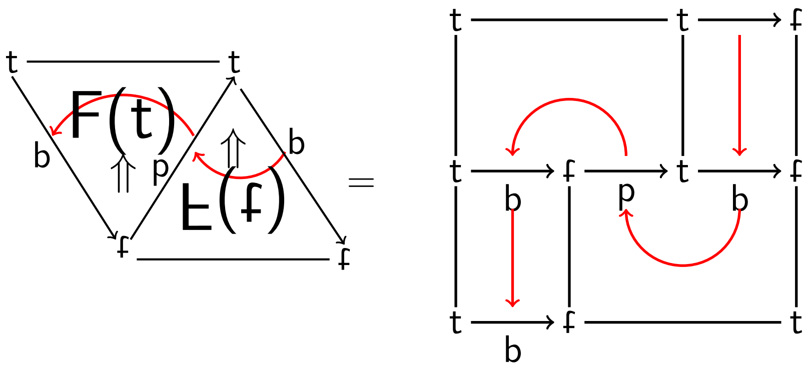


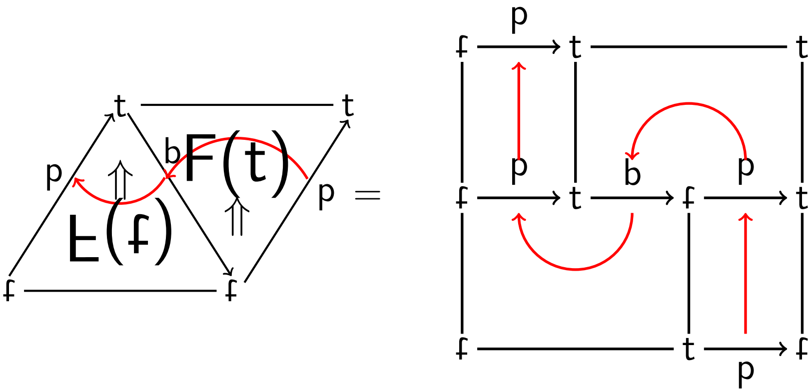

t
t
F(t)
b
p
=
t
t
t
t
b
p

Let's look at all possible 2-fold compositions.


b
p
F(t)
t
=

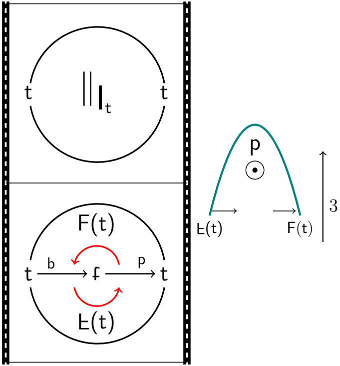

F(t)
t
b
p
t
=
t
t
t
b
p
t
t
t


F(t)
t
b
p
=

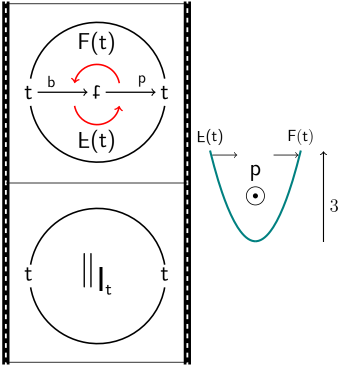

t
t
p
b
F(t)
p
=
p
t
t
p
b
p
t
t
p

These can be compared to identity double arrows.

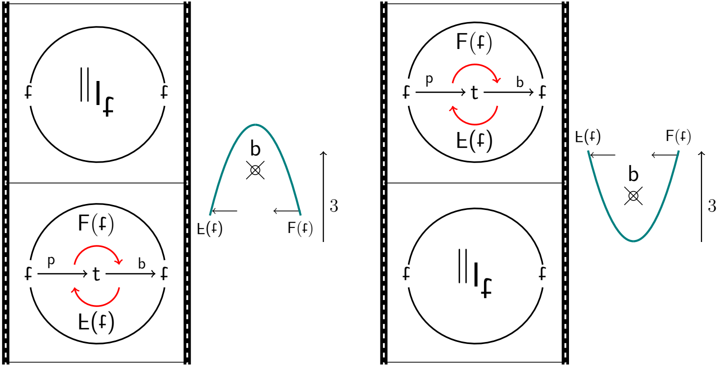
These can be compared to identity double arrows.
The comparisons are triple arrows.

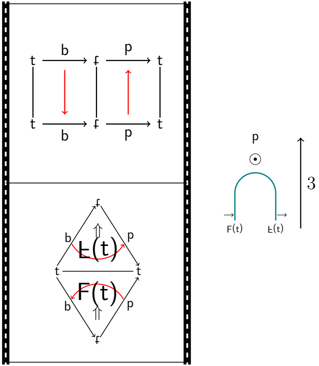


t
t
F(t)
b
ɟ
p
Ⅎ(t)
p
Ⅎ(t)
F(t)
3

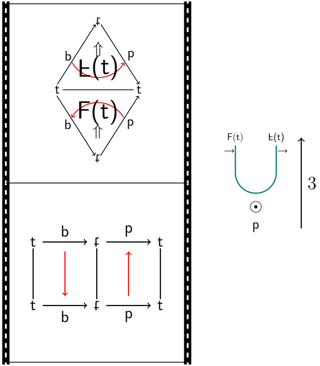


F(t)
t
b
ɟ
p
t
Ⅎ(t)
Ⅎ(t)
F(t)
p
3
t
t

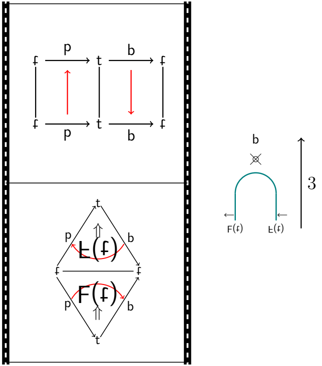

p
t
b
b
3
p
t
b
b
3

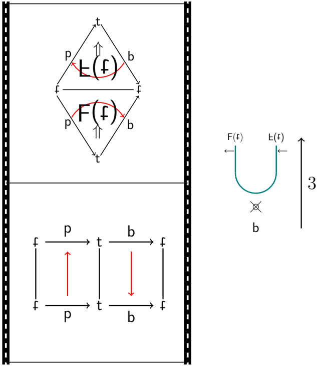


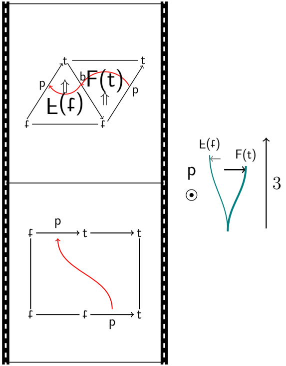
b
p
t
F(t)
3

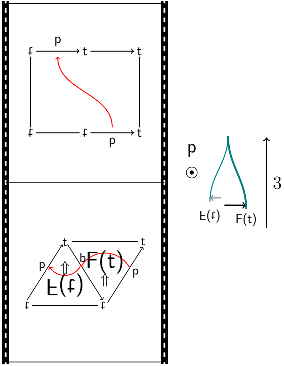


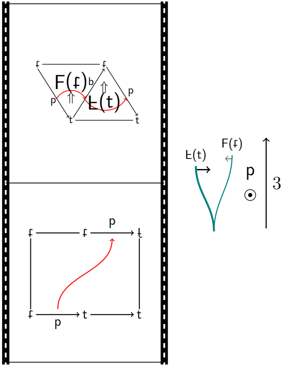
b
p
t
F(t)
3

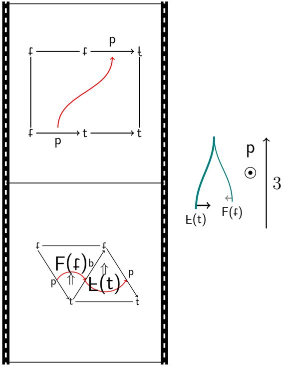

p
b
t
p
b
p
b
t
b
3

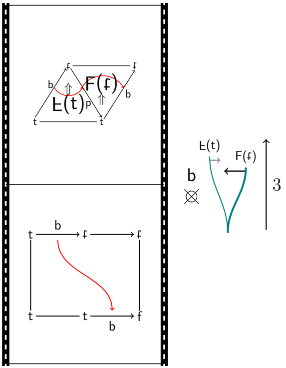

t
p
b
p
b
t
3
b
p
t
b
p
b

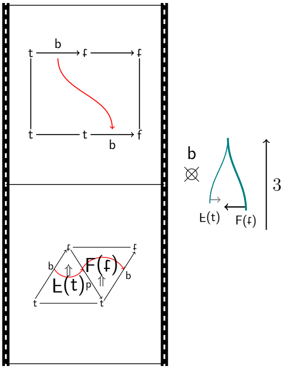

p
t
F(t)
3

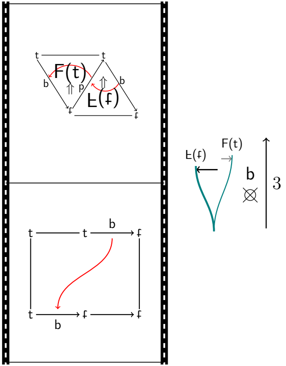

p
t
F(t)
3

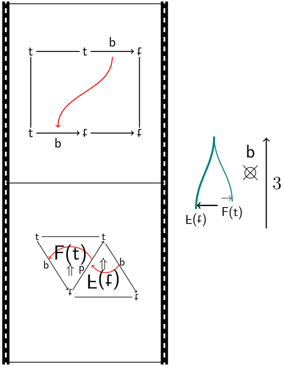

p
3


p
3

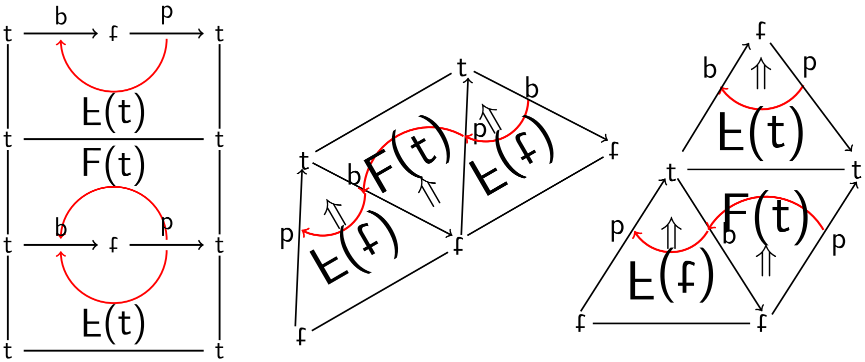

b
3

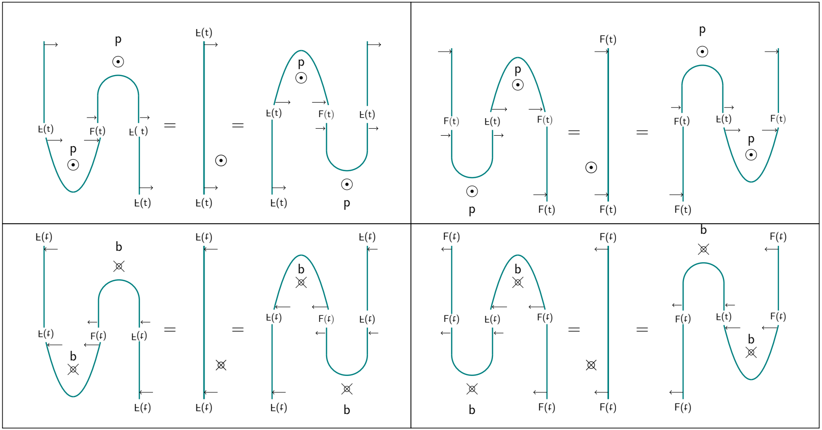

b
3

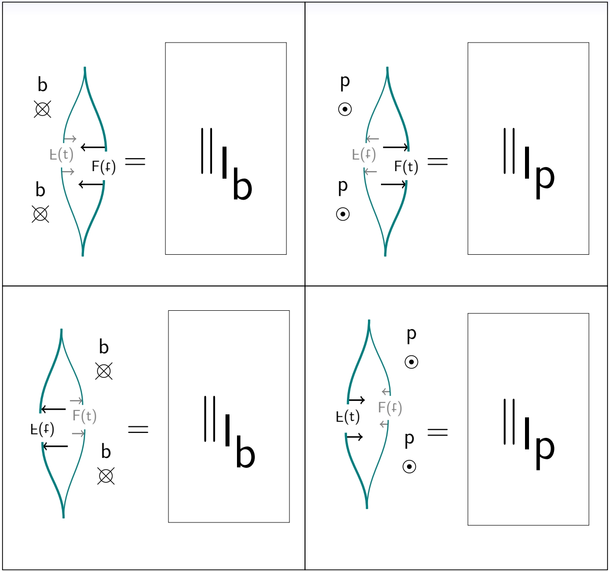

F(t)
b
3

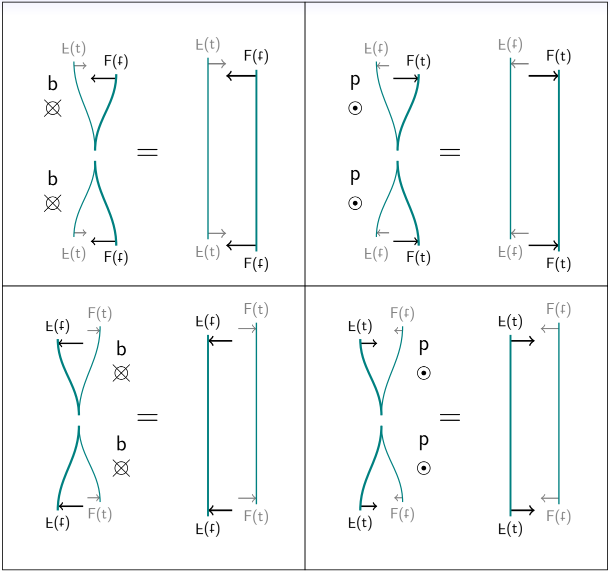

b
F(t)
3

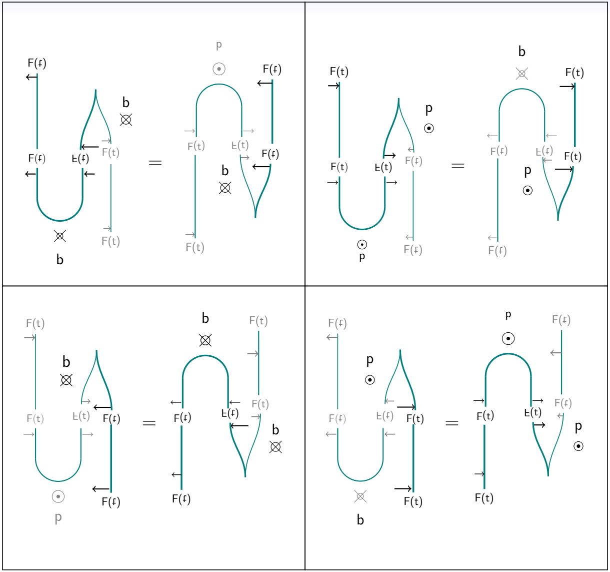
Next by examining all the three-fold compositions of double arrows, Next by examining all the three-fold compositions of double arrows, the quadruple arrows arise.


b
p
t
F(t)

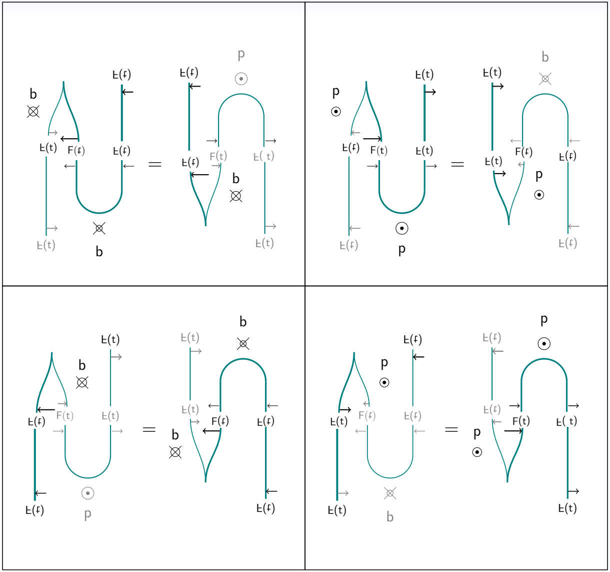

p
F(t)
b
F(ɟ)
=

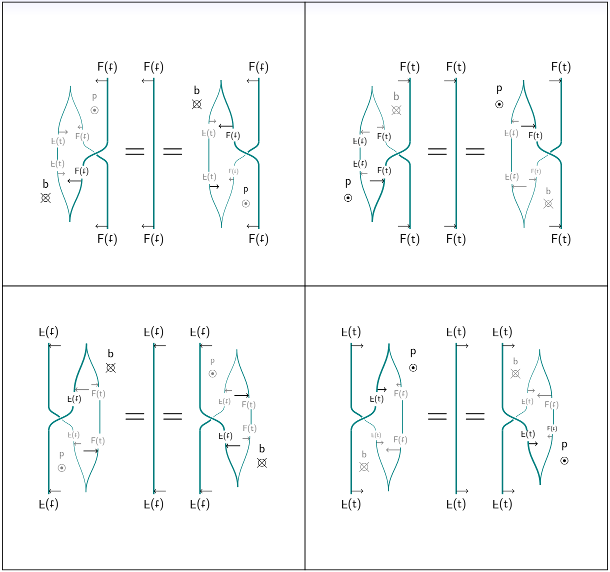

b
b
F(t)
p
p
F(t)
b
b
F(t)
p
p
F(t)

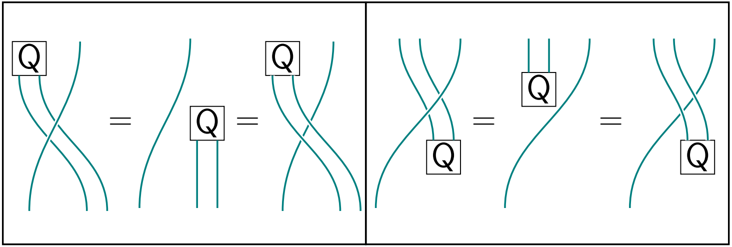

b
b
p
p
=
F(t)



F(ʇ)
b
F(t)
p
=




F(ʇ)
F(t)
Ⅎ(ʇ)
Ⅎ(t)
b
p


Q
Q
Q
Q
Q
Q

The afore constructed 4-cat $\mathcal{S}$

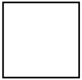
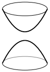
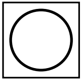
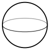
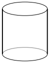
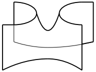
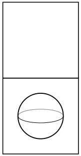
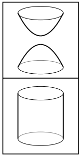
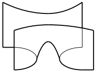
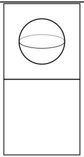
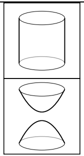
The afore constructed 4-cat $\mathcal{S}$ is the 4-cat of isotopy classes properly embedded surfaces in 3-space.

BUT

BUT

apologies to Sir Mix Alot

It's much more.

t	•	••			⊃ ⊂		
ʇ		•	••		=		
p		Y	∩				
b		⅄	∪				

$\mathsf{p} : 0 \to 1.$

$\mathsf{p} : 0 \to 1$. $\mathsf{b} : 1 \to 0$.

$\mathsf{p} : 0 \to 1$. $\mathsf{b} : 1 \to 0$.Let $\mathfrak{f}_0$ denote $0 —\ 0$.

$\mathsf{p} : 0 \to 1$. $\mathsf{b} : 1 \to 0$.Let $\mathfrak{f}_0$ denote $0 \text{ — } 0$. Let $\mathfrak{f}_1$ denote $1 \text{ — } 1$.

p : $0 \rightarrow 1$. b : $1 \rightarrow 0$.Let $\mathfrak{f}_0$ denote $0 —\ 0$. Let $\mathfrak{f}_1$ denote $1 — 1$. Let $t_0 =$ pb,

$\mathsf{p} : 0 \to 1$. $\mathsf{b} : 1 \to 0$.Let $\mathfrak{f}_0$ denote $0 — 0$. Let $\mathfrak{f}_1$ denote $1 — 1$. Let $\mathsf{t}_0 = \mathsf{pb}$, Let $\mathsf{t}_1 = \mathsf{bp}$.

$\mathsf{p} : 0 \to 1$. $\mathsf{b} : 1 \to 0$.Let f_0 denote 0 — 0. Let f_1 denote 1 — 1. Let $\mathsf{t}_0 = \mathsf{pb}$, Let $\mathsf{t}_1 = \mathsf{bp}$. For $\epsilon \in \{0, 1\}$,

$\mathsf{p} : 0 \to 1$. $\mathsf{b} : 1 \to 0$.Let f_0 denote 0 — 0. Let f_1 denote 1 — 1. Let $\mathsf{t}_0 = \mathsf{pb}$, Let $\mathsf{t}_1 = \mathsf{bp}$. For $\epsilon \in \{0, 1\}$, let $\mathsf{p}_\epsilon : \mathsf{f}_\epsilon \to \mathsf{t}_\epsilon$,

$\mathsf{p} : 0 \to 1$. $\mathsf{b} : 1 \to 0$.Let $\mathfrak{f}_0$ denote $0 \longrightarrow 0$. Let $\mathfrak{f}_1$ denote $1 \longrightarrow 1$. Let $\mathsf{t}_0 = \mathsf{pb}$, Let $\mathsf{t}_1 = \mathsf{bp}$. For $\epsilon \in \{0, 1\}$, let $\mathsf{p}_\epsilon : \mathfrak{f}_\epsilon \to \mathsf{t}_\epsilon$, and $\mathsf{b}_\epsilon : \mathsf{t}_\epsilon \to \mathfrak{f}_\epsilon$.

$\mathsf{p} : 0 \to 1$. $\mathsf{b} : 1 \to 0$.Let ʇ_0 denote 0 — 0. Let ʇ_1 denote 1 — 1. Let $\mathsf{t}_0 = \mathsf{pb}$, Let $\mathsf{t}_1 = \mathsf{bp}$. For $\epsilon \in \{0, 1\}$, let $\mathsf{p}_\epsilon : \text{ʇ}_\epsilon \to \mathsf{t}_\epsilon$, and $\mathsf{b}_\epsilon : \mathsf{t}_\epsilon \to \text{ʇ}_\epsilon$. Note: $\mathsf{p}_0 = \text{Ⅎ}(\text{ʇ})$,

$\mathsf{p} : 0 \to 1$. $\mathsf{b} : 1 \to 0$.Let f_0 denote 0 — 0. Let f_1 denote 1 — 1. Let $\mathsf{t}_0 = \mathsf{pb}$, Let $\mathsf{t}_1 = \mathsf{bp}$. For $\epsilon \in \{0, 1\}$, let $\mathsf{p}_\epsilon : \mathsf{f}_\epsilon \to \mathsf{t}_\epsilon$, and $\mathsf{b}_\epsilon : \mathsf{t}_\epsilon \to \mathsf{f}_\epsilon$. Note: $\mathsf{p}_0 =$ Ł(f), $\mathsf{p}_1 =$ Ł(t),

$\mathsf{p} : 0 \to 1$. $\mathsf{b} : 1 \to 0$.Let f_0 denote $0 — 0$. Let f_1 denote $1 — 1$. Let $\mathsf{t}_0 = \mathsf{pb}$, Let $\mathsf{t}_1 = \mathsf{bp}$. For $\epsilon \in \{0, 1\}$, let $\mathsf{p}_\epsilon : \mathsf{f}_\epsilon \to \mathsf{t}_\epsilon$, and $\mathsf{b}_\epsilon : \mathsf{t}_\epsilon \to \mathsf{f}_\epsilon$. Note: $\mathsf{p}_0 = \text{Ⅎ}(\mathsf{f})$, $\mathsf{p}_1 = \text{Ⅎ}(\mathsf{t})$, $\mathsf{b}_0 = \mathsf{F}(\mathsf{f})$,

$\mathsf{p} : 0 \to 1$. $\mathsf{b} : 1 \to 0$.Let $\mathfrak{f}_0$ denote 0 — 0. Let $\mathfrak{f}_1$ denote 1 — 1. Let $\mathsf{t}_0 = \mathsf{pb}$, Let $\mathsf{t}_1 = \mathsf{bp}$. For $\epsilon \in \{0, 1\}$, let $\mathsf{p}_\epsilon : \mathfrak{f}_\epsilon \to \mathsf{t}_\epsilon$, and $\mathsf{b}_\epsilon : \mathsf{t}_\epsilon \to \mathfrak{f}_\epsilon$. Note: $\mathsf{p}_0 = \underline{\vdash}(\mathfrak{f})$, $\mathsf{p}_1 = \underline{\vdash}(\mathsf{t})$, $\mathsf{b}_0 = \mathsf{F}(\mathfrak{f})$, and $\mathsf{b}_1 = \mathsf{F}(\mathsf{t})$.
Sp for $x = \epsilon_{k-1} \cdots \epsilon_1$,

p : $0 \to 1$. b : $1 \to 0$.Let ʇ$_0$ denote 0 — 0. Let ʇ$_1$ denote 1 — 1. Let t$_0$ = pb, Let t$_1$ = bp. For $\epsilon \in \{0, 1\}$, let p$_\epsilon$: ʇ$_\epsilon$ $\to$ t$_\epsilon$, and b$_\epsilon$: t$_\epsilon$ $\to$ ʇ$_\epsilon$. Note: p$_0$ = Ⅎ(ʇ), p$_1$ = Ⅎ(t), b$_0$ = F(ʇ), and b$_1$ = F(t). Sp for $x = \epsilon_{k-1} \cdots \epsilon_1$, $(k-1)$-arrows: ʇ$_x$, t$_x$ are def'd.

$\mathsf{p} : 0 \to 1$. $\mathsf{b} : 1 \to 0$.Let $\mathsf{ʇ}_0$ denote $0 \text{ --- } 0$. Let $\mathsf{ʇ}_1$ denote $1 \text{ --- } 1$. Let $\mathsf{t}_0 = \mathsf{pb}$, Let $\mathsf{t}_1 = \mathsf{bp}$. For $\epsilon \in \{0, 1\}$, let $\mathsf{p}_\epsilon : \mathsf{ʇ}_\epsilon \to \mathsf{t}_\epsilon$, and $\mathsf{b}_\epsilon : \mathsf{t}_\epsilon \to \mathsf{ʇ}_\epsilon$. Note: $\mathsf{p}_0 = \mathsf{Ⅎ}(\mathsf{ʇ})$, $\mathsf{p}_1 = \mathsf{Ⅎ}(\mathsf{t})$, $\mathsf{b}_0 = \mathsf{F}(\mathsf{ʇ})$, and $\mathsf{b}_1 = \mathsf{F}(\mathsf{t})$. Sp for $x = \epsilon_{k-1} \cdots \epsilon_1$, $(k-1)$-arrows: $\mathsf{ʇ}_x$, t_x are def'd. w/ k-arrows, $\mathsf{p}_x : \mathsf{ʇ}_x \to \mathsf{t}_x$ and $\mathsf{b}_x : \mathsf{t}_x \to \mathsf{ʇ}_x$ b/2 them.

$\mathsf{p} : 0 \to 1$. $\mathsf{b} : 1 \to 0$.Let ʇ_0 denote 0 — 0. Let ʇ_1 denote 1 — 1. Let $\mathsf{t}_0 = \mathsf{pb}$, Let $\mathsf{t}_1 = \mathsf{bp}$. For $\epsilon \in \{0, 1\}$, let $\mathsf{p}_\epsilon : \text{ʇ}_\epsilon \to \mathsf{t}_\epsilon$, and $\mathsf{b}_\epsilon : \mathsf{t}_\epsilon \to \text{ʇ}_\epsilon$. Note: $\mathsf{p}_0 = \text{Ⅎ}(\text{ʇ})$, $\mathsf{p}_1 = \text{Ⅎ}(\mathsf{t})$, $\mathsf{b}_0 = \mathsf{F}(\text{ʇ})$, and $\mathsf{b}_1 = \mathsf{F}(\mathsf{t})$. Sp for $x = \epsilon_{k-1} \cdots \epsilon_1$, $(k-1)$-arrows: ʇ_x, t_x are def'd. w/ k-arrows, $\mathsf{p}_x : \text{ʇ}_x \to \mathsf{t}_x$ and $\mathsf{b}_x : \mathsf{t}_x \to \text{ʇ}_x$ b/2 them. Let $I[\mathsf{s}_x]$ denote the id. k-arrow upon s for $\mathsf{s} = \mathsf{t}, \text{ʇ}, \mathsf{p}$, or b.

Then def. k-arrows

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- $\mathsf{f}_{0x} = I[\mathsf{f}_x]$,

Then def. k-arrows

- $\mathsf{f}_{0x} = I[\mathsf{f}_x]$,
- $\mathsf{f}_{1x} = I[\mathsf{t}_x]$,

Then def. k-arrows

- $\mathsf{f}_{0x} = I[\mathsf{f}_x]$,
- $\mathsf{f}_{1x} = I[\mathsf{t}_x]$,
- $\mathsf{t}_{0x} = \mathsf{p}_x \mathsf{b}_x$,

Then def. k-arrows

- $\mathsf{f}_{0x} = I[\mathsf{f}_x]$,
- $\mathsf{f}_{1x} = I[\mathsf{t}_x]$,
- $\mathsf{t}_{0x} = \mathsf{p}_x \mathsf{b}_x$, and
- $\mathsf{t}_{1x} = \mathsf{b}_x \mathsf{p}_x$.

Then def. k-arrows

- $\mathsf{f}_{0x} = I[\mathsf{f}_x]$,
- $\mathsf{f}_{1x} = I[\mathsf{t}_x]$,
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- $\mathsf{t}_{1x} = \mathsf{b}_x \mathsf{p}_x$.

Def $(k+1)$-arrows

Then def. k-arrows

- $\mathsf{f}_{0x} = I[\mathsf{f}_x]$,
- $\mathsf{f}_{1x} = I[\mathsf{t}_x]$,
- $\mathsf{t}_{0x} = \mathsf{p}_x \mathsf{b}_x$, and
- $\mathsf{t}_{1x} = \mathsf{b}_x \mathsf{p}_x$.

Def $(k+1)$-arrows

- $\mathsf{p}_{0x} : \mathsf{f}_{0x} \to t_{0x} = \mathsf{E}(\mathsf{f}_x)$;

Then def. k-arrows

- $\mathsf{f}_{0x} = I[\mathsf{f}_x]$,
- $\mathsf{f}_{1x} = I[\mathsf{t}_x]$,
- $\mathsf{t}_{0x} = \mathsf{p}_x \mathsf{b}_x$, and
- $\mathsf{t}_{1x} = \mathsf{b}_x \mathsf{p}_x$.

Def $(k+1)$-arrows

- $\mathsf{p}_{0x} : \mathsf{f}_{0x} \to t_{0x} = \mathsf{E}(\mathsf{f}_x)$;
- $\mathsf{p}_{1x} : \mathsf{f}_{1x} \to t_{1x} = \mathsf{E}(\mathsf{t}_x)$;

Then def. k-arrows

- $\mathsf{f}_{0x} = I[\mathsf{f}_x]$,
- $\mathsf{f}_{1x} = I[\mathsf{t}_x]$,
- $\mathsf{t}_{0x} = \mathsf{p}_x \mathsf{b}_x$, and
- $\mathsf{t}_{1x} = \mathsf{b}_x \mathsf{p}_x$.

Def $(k+1)$-arrows

- $\mathsf{p}_{0x} : \mathsf{f}_{0x} \to t_{0x} = \mathsf{Ⅎ}(\mathsf{f}_x)$;
- $\mathsf{p}_{1x} : \mathsf{f}_{1x} \to t_{1x} = \mathsf{Ⅎ}(\mathsf{t}_x)$;
- $\mathsf{b}_{0x} : \mathsf{t}_{0x} \to \mathsf{f}_{0x} = \mathsf{F}(\mathsf{f}_x)$;

Then def. k-arrows

- $\mathsf{f}_{0x} = I[\mathsf{f}_x]$,
- $\mathsf{f}_{1x} = I[\mathsf{t}_x]$,
- $\mathsf{t}_{0x} = \mathsf{p}_x \mathsf{b}_x$, and
- $\mathsf{t}_{1x} = \mathsf{b}_x \mathsf{p}_x$.

Def $(k+1)$-arrows

- $\mathsf{p}_{0x} : \mathsf{f}_{0x} \to t_{0x} = \underline{\vdash}(\mathsf{f}_x)$;
- $\mathsf{p}_{1x} : \mathsf{f}_{1x} \to t_{1x} = \underline{\vdash}(\mathsf{t}_x)$;
- $\mathsf{b}_{0x} : \mathsf{t}_{0x} \to \mathsf{f}_{0x} = \mathsf{F}(\mathsf{f}_x)$; and
- $\mathsf{b}_{1x} : \mathsf{t}_{1x} \to \mathsf{f}_{1x} = \mathsf{F}(\mathsf{t}_x)$;

Inductive Step

$x = \epsilon_k \epsilon_{k-1} \ldots \epsilon_1$; $\epsilon_j \in \{0,1\}$, for $j \in \{1, \ldots, k\}$.

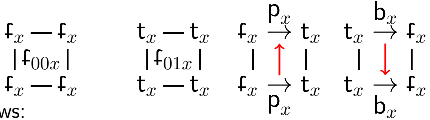
Obj: $\mathsf{t}_x, \mathsf{f}_x$

Generating 1-arrows: $\mathsf{p}_x : \mathsf{f}_x \to \mathsf{t}_x$; $\mathsf{b}_x : \mathsf{t}_x \to \mathsf{f}_x$.

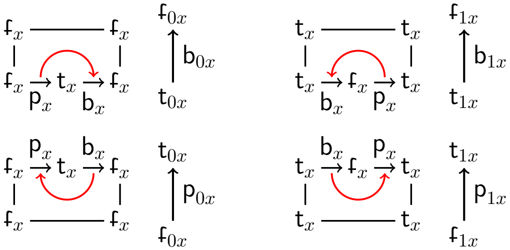
Inductively define

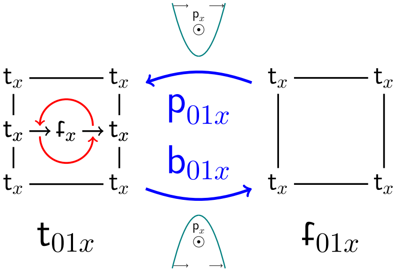
$\mathsf{f}_{0x} : \mathsf{f}_x — \mathsf{f}_x$, $\mathsf{f}_{1x} : \mathsf{t}_x — \mathsf{t}_x$; $\mathsf{t}_{0x} = \mathsf{p}_x \mathsf{b}_x$; $\mathsf{t}_{1x} = \mathsf{b}_x \mathsf{p}_x$.

Id. 2-arrows:

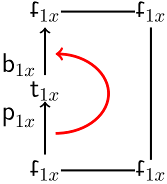


Generating 2-arrows:

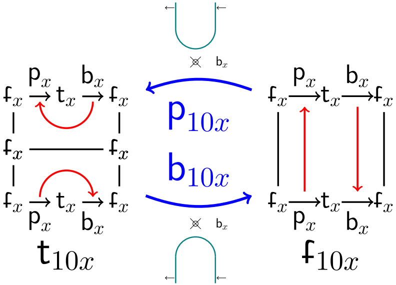




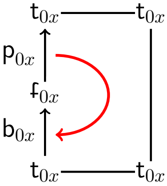
p_x
t_x
f_x
p_{01x}
b_{01x}
t_{01x}
f_{01x}



f_{1x}
b_{1x}
t_{1x}
p_{1x}



$\mathfrak{f}_x$ p_x t_x b_x
b_x
p_{10x}
b_{10x}
t_{10x}
$\mathfrak{f}_{10x}$

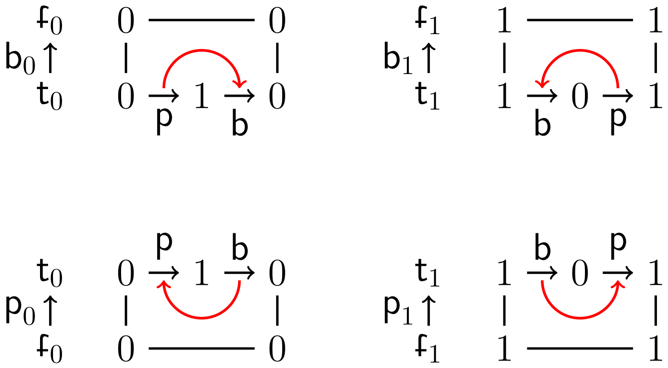


t_{0x}
p_{0x}
$\mathfrak{f}_{0x}$
b_{0x}

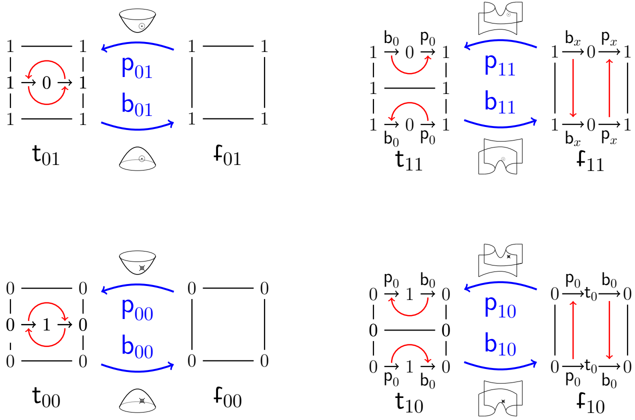
$$\mathsf{t}_0 : 0 \xrightarrow{\mathsf{p}} 1 \xrightarrow{\mathsf{b}} 0 \qquad\qquad \mathsf{t}_1 : 1 \xrightarrow{\mathsf{b}} 0 \xrightarrow{\mathsf{p}} 1$$

$$\scalebox{1}[-1]{t}_0 : 0 \;\underline{\hspace{4em}}\; 0 \qquad\qquad \scalebox{1}[-1]{t}_1 : 1 \;\underline{\hspace{4em}}\; 1$$

arrows as objects



2-arrows as 1-arrows



3-arrows as 2-arrows

- The b's and p's are all critical points

- The b's and p's are all critical points or IOW handle attachments.
- Cusps correspond to critical (handle) cancelations.

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- The construction grows exponentially.

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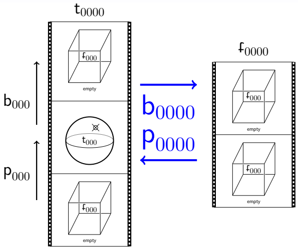
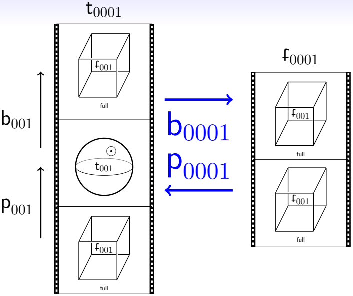
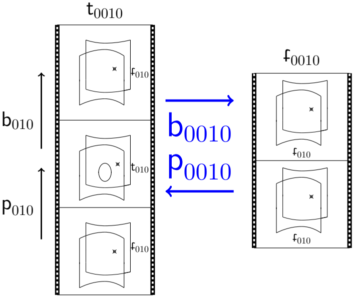
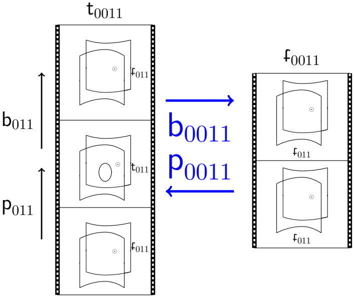
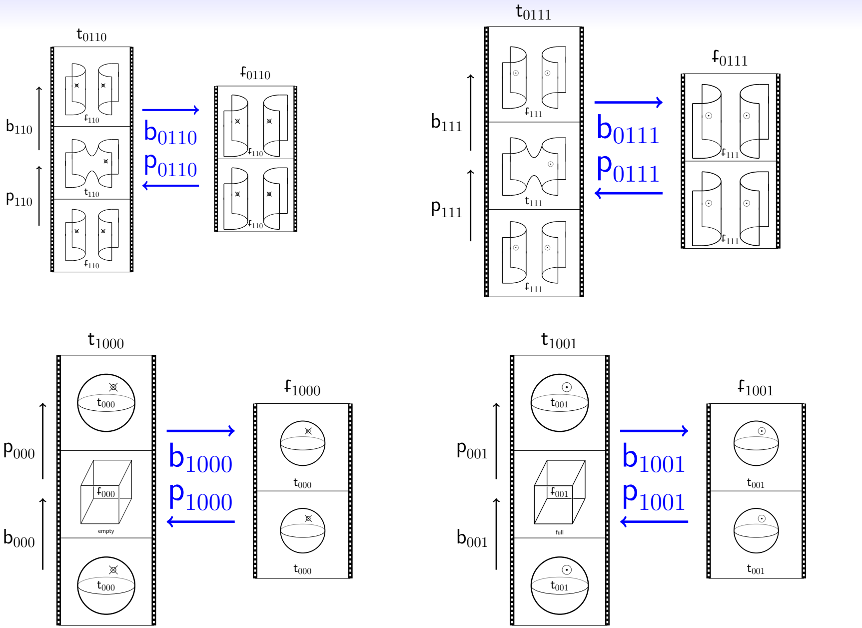
- Cusps correspond to critical (handle) cancelations.
- Swallow-tails and horizontal cusps are always interesting. Last picture.
- The construction grows exponentially.
- I'll skip a step and go to the highest level that we have computed.


t_{0000}
f_{0000}
b_{000}
p_{000}
b_{0000}
p_{0000}
f_{000}
t_{000}
empty

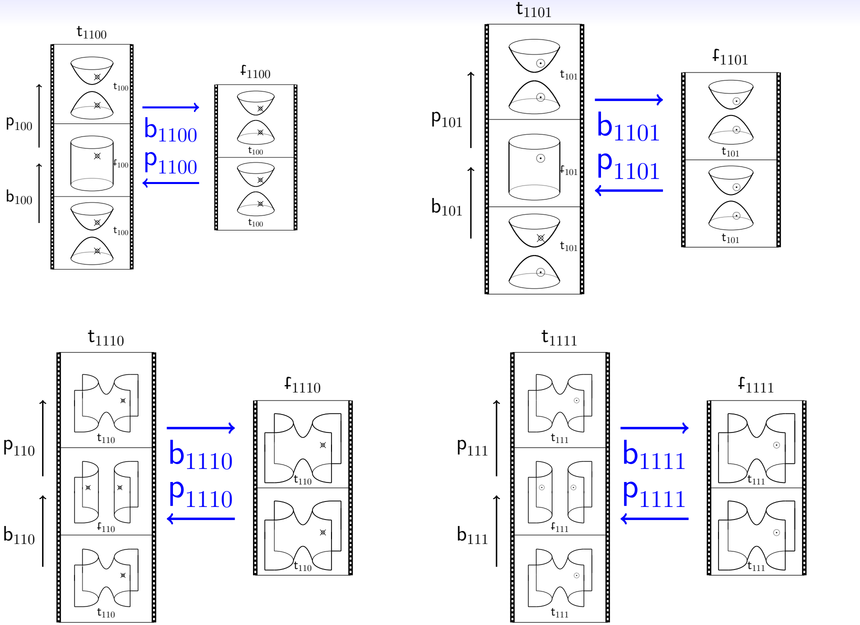

t_{0001}
f_{0001}
b_{001}
p_{001}
b_{0001}
p_{0001}
f_{001}
t_{001}
full


t_{0010}
f_{0010}
b_{010}
p_{010}
b_{0010}
p_{0010}
f_{010}
t_{010}


t_{0011}
f_{0011}
b_{011}
p_{011}
b_{0011}
p_{0011}
f_{011}
t_{011}



t0110
b110
p110
f110
t110
f0110
b0110
p0110
t0111
b111
p111
f111
t111
f0111
b0111
p0111
t1000
p000
b000
t000
f000
empty
f1000
b1000
p1000
t1001
p001
b001
t001
f001
full
f1001
b1001
p1001



t1100
p100
b100
t100
f100
t100
b1100
p1100
f1100
t100
t100
t1101
p101
b101
t101
f101
t101
b1101
p1101
f1101
t101
t101
t1110
p110
b110
t110
f110
t110
b1110
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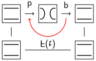
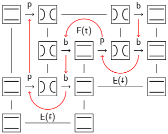
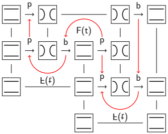
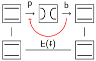
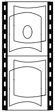
t111
b1111
p1111
f1111
t111
t111

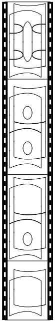

p
b

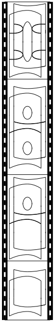

p
b
F(t)

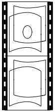

p
b
F(t)


p
b









Epilogue

- There is more glyphography to come.

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- There is more glyphography to come.
- Components of glyphs are vertices types of edges

Epilogue

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- bookmarks can be associated to serifs and vertices that indicate composability.
- That's my story and I'm sticking to it.

Thank you