The stable uniqueness theorem for equivariant Kasparov theory

Topology and Measure in Dynamics and Operator Algebras, BIRS, Banff

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KU Leuven

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Question

What should this even mean when we classify up to cocycle conjugacy?

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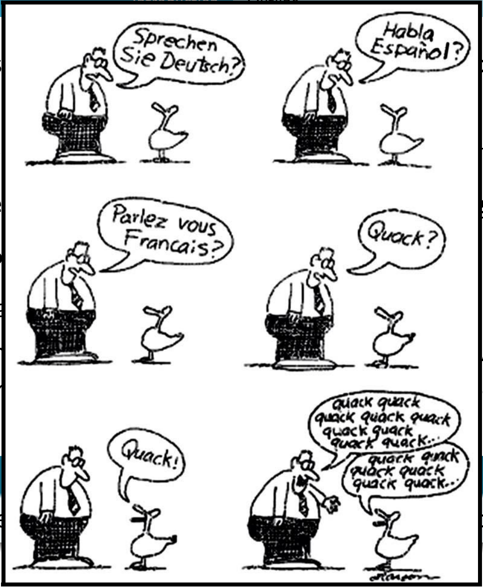

Question

What should this e ... le conjugacy?

$\rightsquigarrow$ We need the appropriate language

Definition

Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions on C^*-algebras. A cocycle representation is a pair

$$(\varphi, \mathbb{u}) : (A, \alpha) \to (\mathcal{M}(B), \beta),$$

where $\varphi : A \to \mathcal{M}(B)$ is a $*$-homomorphism, $\mathbb{u} : G \to \mathcal{U}(\mathcal{M}(B))$ is a β-cocycle, and we have $\mathrm{Ad}(\mathbb{u}_g) \circ \beta_g \circ \varphi = \varphi \circ \alpha_g$ for all $g \in G$.
If $\varphi(A) \subseteq B$, then $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ is called a cocycle morphism.

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Example

For $\mathbb{u} = \mathbf{1}$, we recover what it means for φ to be equivariant.
For $\beta = \mathrm{id}$, we recover the concept of a covariant representation for (A, α).

Definition (Composition)

Let $\alpha : G \curvearrowright A$, $\beta : G \curvearrowright B$, and $\gamma : G \curvearrowright C$ be three actions on C^*-algebras. Suppose that

$$(\varphi, \mathbb{u}) : (A, \alpha) \to (\mathcal{M}(B), \beta) \quad \text{and} \quad (\psi, \mathbb{v}) : (B, \beta) \to (\mathcal{M}(C), \gamma)$$

are two (non-degenerate) cocycle representations. Then the pair

$$(\psi \circ \varphi, \psi(\mathbb{u}_\bullet)\mathbb{v}_\bullet) =: (\psi, \mathbb{v}) \circ (\varphi, \mathbb{u})$$

is a cocycle representation from (A, α) to $(\mathcal{M}(C), \gamma)$.

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The binary operation "$\circ$" becomes associative, and on every object (A, α) the pair $(\mathrm{id}_A, \mathbf{1}) = \mathrm{id}_A$ is a neutral element. Thus we can consider the G-C^*-algebras as a category with morphisms being the cocycle morphisms.

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Observation

An isomorphism in this category is precisely a cocycle conjugacy.

Example

For a given action $\beta : G \curvearrowright B$ and a unitary $u \in \mathcal{U}(\mathcal{M}(B))$, the pair

$$\mathrm{Ad}(u) := (\mathrm{Ad}(u), u\beta_\bullet(u)^*)$$

is an inner cocycle morphism on (B, β).

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Remark

We can equip the set of cocycle morphisms $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ with the point-norm topology in the first variable, and the strict topology in the second variable, but uniformly over compact sets $K \subseteq G$. If A is separable and B is σ-unital, then this yields a Polish topology.

Definition

We say that a cocycle morphism $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ is approximately unitarily equivalent to $(\psi, \mathbb{v})$, if there exists a net $u_\lambda \in \mathcal{U}(\mathcal{M}(B))$ such that $\mathrm{Ad}(u_\lambda) \circ (\varphi, \mathbb{u}) \xrightarrow{\lambda\to\infty} (\psi, \mathbb{v})$. We write $(\varphi, \mathbb{u}) \approx_{\mathrm{u}} (\psi, \mathbb{v})$.

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Theorem (S; Elliott in unital case)

Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable C^*-algebras. Suppose that

$$(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta) \quad \textit{and} \quad (\psi, \mathbb{v}) : (B, \beta) \to (A, \alpha)$$

are two cocycle morphisms such that

$$(\psi, \mathbb{v}) \circ (\varphi, \mathbb{u}) \approx_{\mathrm{u}} \mathrm{id}_A \quad \textit{and} \quad (\varphi, \mathbb{u}) \circ (\psi, \mathbb{v}) \approx_{\mathrm{u}} \mathrm{id}_B \,.$$

Then $(\varphi, \mathbb{u})$ and $(\psi, \mathbb{v})$ are approximately unitarily equivalent to mutually inverse cocycle conjugacies between (A, α) and (B, β).

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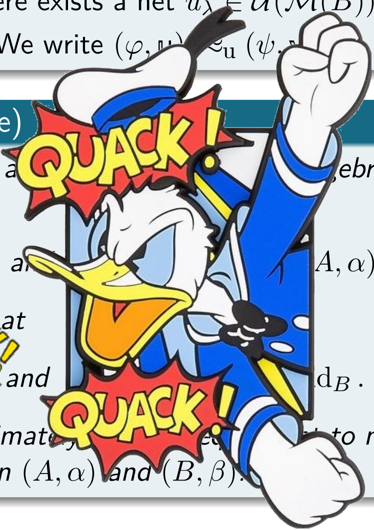

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More specifically, we focus on Kasparov's G-equivariant KK-functor as an important invariant, and investigate what information we can extract.

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The KK^G-category is universal for functors from separable G-C^*-algebras to abelian groups that are stable, half-exact, and homotopy invariant.

The key towards the proof of this is a generalization of the Cuntz picture of ordinary KK-theory. (*Cuntz–Thomsen picture*)

From now on, we will fix actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ on separable C^*-algebras, and assume (B, β) is conjugate to $(B \otimes \mathcal{K}, \beta \otimes \mathrm{id}_{\mathcal{K}})$.

Definition (Thomsen)

An (α, β)-Cuntz pair is a pair of cocycle representations

$$(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$$

such that the pointwise differences $\psi - \varphi$ and $\mathbb{v} - \mathbb{u}$ take values in B. We say that this pair is degenerate if $\varphi = \psi$.

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Definition

Pick two isometries $s_1, s_2 \in \mathcal{M}(\mathcal{K}) \subseteq \mathcal{M}(B)^\beta$ with $s_1 s_1^* + s_2 s_2^* = \mathbf{1}$. For $b_1, b_2 \in \mathcal{M}(B)$, one defines $b_1 \oplus b_2 = b_1 \oplus_{s_1, s_2} b_2 = s_1 b_1 s_1^* + s_2 b_2 s_2^*$. This element does not depend on the choice of s_1, s_2 up to conjugation with a uniquely determined unitary in $\mathcal{U}_0(\mathcal{M}(B)^\beta)$.

Definition

Given two (α, β)-Cuntz pairs $[(\varphi^{(j)}, \mathbb{u}^{(j)}), (\psi^{(j)}, \mathbb{v}^{(j)})]$ for $j = 1, 2$, we can define their sum as

$$\begin{aligned} & [(\varphi^{(1)}, \mathbb{u}^{(1)}), (\psi^{(1)}, \mathbb{v}^{(1)})] \oplus [(\varphi^{(2)}, \mathbb{u}^{(2)}), (\psi^{(2)}, \mathbb{v}^{(2)})] \\ = \;\; & [(\varphi^{(1)} \oplus \varphi^{(2)}, \mathbb{u}^{(1)} \oplus \mathbb{u}^{(2)}), (\psi^{(1)} \oplus \psi^{(2)}, \mathbb{v}^{(1)} \oplus \mathbb{v}^{(2)})]. \end{aligned}$$

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For a $(\alpha, \beta[0,1])$-Cuntz pair

$$(\Phi, \mathbb{U}), (\Psi, \mathbb{V}) : (A, \alpha) \to (\mathcal{M}(B[0,1]), \beta[0,1]),$$

the evaluation at the endpoints $0, 1 \in [0,1]$ yields two (α, β)-Cuntz pairs. This defines the homotopy relation $\sim_h$ on (α, β)-Cuntz pairs.

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The (α,β)-Cuntz pairs modulo homotopy form an abelian semigroup.

We denote $\mathbb{E}^G(\alpha,\beta) = \{(\alpha,\beta)\text{-Cuntz pairs}\}$, and $\mathbb{D}^G(\alpha,\beta)$ the subset given by degenerate elements. One defines an equivalence relation on $\mathbb{E}^G(\alpha,\beta)$ via

$$x_1 \sim_{sh} x_2 \quad :\Leftrightarrow \quad \exists\ d_1, d_2 \in \mathbb{D}^G(\alpha,\beta):\ x_1 \oplus d_1 \sim_h x_2 \oplus d_2.$$

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Question (Stable uniqueness)

If a Cuntz pair

$$(\varphi,\mathbb{u}),(\psi,\mathbb{v}) : (A,\alpha) \to (\mathcal{M}(B),\beta)$$

defines the zero element in KK^G, what does this really tell us?

Definition

For two cocycle representations

$$(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (\mathcal{M}(B), \beta),$$

let us write $(\varphi, \mathbb{u}) \sim_B (\psi, \mathbb{v})$, if there is a continuous family $\{v_t\}_{t\in\mathbb{R}}$ in $\mathcal{U}(\mathcal{M}(B))$ such that $\mathrm{Ad}(v_t) \circ \varphi \stackrel{t\to\infty}{\longrightarrow} \psi$ in point-norm, $v_t \mathbb{u}_g \beta_g(v_t)^* \stackrel{t\to\infty}{\longrightarrow} \mathbb{v}_g$ in norm uniformly over compacts, and the respective pointwise differences take value in B. If we may assume $v_t \in \mathcal{U}(\mathbf{1} + B)$, write $(\varphi, \mathbb{u}) \simeq_B (\psi, \mathbb{v})$.

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Definition

A cocycle representation $(\theta, \mathbb{x}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$ is called absorbing, if for every cocycle representation $(\varphi, \mathbb{u}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$, we have $(\theta, \mathbb{x}) \oplus (\varphi, \mathbb{u}) \sim_B (\theta, \mathbb{x})$.

Our goal is to generalize the following fundamental theorem from C^*-algebras to C^*-dynamics.

Theorem (Lin, Dadarlat–Eilers)

Let $\varphi, \psi : A \to \mathcal{M}(B)$ be a Cuntz pair of representations, and let $\theta : A \to \mathcal{M}(B)$ be an absorbing representation. Then $[\varphi, \psi] = 0$ in $KK(A, B)$ if and only if $\varphi \oplus \theta \simeq_B \psi \oplus \theta$.

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The first obstacle is that we need to transfer the theory of absorbing representations to the dynamical setup, and guarantee that we are not just talking about the empty set.

Our goal is to generalize the following fundamental theorem from C^*-algebras to C^*-dynamics.

Theorem (Lin, Dadarlat–Eilers)

Let $\varphi, \psi : A \to \mathcal{M}(B)$ be a Cuntz pair of representations, and let $\theta : A \to \mathcal{M}(B)$ be an absorbing representation. Then $[\varphi, \psi] = 0$ in $KK(A, B)$ if and only if $\varphi \oplus \theta \simeq_B \psi \oplus \theta$.

The first obstacle is that we need to transfer the theory of absorbing representations to the dynamical setup, and guarantee that we are not just talking about the empty set.

Theorem (Gabe–S, generalizing Thomsen)

For any actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ on separable C^*-algebras, there is an absorbing cocycle representation $(\theta, \mathbb{x}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$.

(The same is true w.r.t. "unitally/nuclearly absorbing" etc.)

Theorem (Gabe–S)

Suppose that

$$(\varphi, \mathbb{u}), (\psi, \mathbb{v}), (\theta, \mathbb{x}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$$

are three cocycle representations such that the first two form a (α, β)*-Cuntz pair, and* $(\theta, \mathbb{x})$ *is absorbing. Then* $[(\varphi, \mathbb{u}), (\psi, \mathbb{v})] = 0$ *in* $KK^G(\alpha, \beta)$ *if and only if* $(\varphi \oplus \theta, \mathbb{u} \oplus \mathbb{x}) \simeq_B (\psi \oplus \theta, \mathbb{v} \oplus \mathbb{x})$*.*

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Definition

Let $(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$ be two cocycle representations. We say that $(\psi, \mathbb{v})$ is weakly contained in $(\varphi, \mathbb{u})$, written $(\psi, \mathbb{v}) \precsim (\varphi, \mathbb{u})$, if for every contraction $s \in B$, $\varepsilon > 0$ and compact sets $\mathcal{F} \subset A$ and $K \subseteq G$, there exist elements $c_1, \dots, c_n \in B$ such that

$$\max_{a \in \mathcal{F}} \|s^* \psi(a) s - \sum_{j=1}^{n} c_j^* \varphi(a) c_j\| \leq \varepsilon$$

and

$$\max_{g \in K} \|b^* \mathbb{v}_g \beta_g(b) - \sum_{j=1}^{n} c_j^* \mathbb{u}_g \beta_g(c_j)\| \leq \varepsilon.$$

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This concept simultaneously generalizes two well-studied phenomena. If $G = \{1\}$, then this recovers "weak domination" of ψ by φ as u.c.p. maps. If G is non-trivial but $A = \mathbb{C}$, $B = \mathcal{K}$, $\beta = \mathrm{id}$, $\varphi = \psi = \bullet \cdot \mathbf{1}$, then this recovers weak containment of unitary representations $G \to \mathcal{U}(\ell^2)$.

Definition

Suppose (B, β) is conjugate to $(B \otimes \mathcal{K}, \beta \otimes \mathrm{id}_{\mathcal{K}})$. Choose isometries $t_n \in \mathcal{M}(B)^\beta$ with $\mathbf{1} = \sum_{n \in \mathbb{N}} t_n t_n^*$ in the strict topology. For a sequence of cocycle representations $(\varphi_n, \mathbb{u}^{(n)}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$, we define its direct sum

$$\bigoplus_{n \in \mathbb{N}} (\varphi_n, \mathbb{u}^{(n)}) = \Big(\sum_{n \in \mathbb{N}} t_n \varphi_n(\bullet) t_n^*, \sum_{n \in \mathbb{N}} t_n \mathbb{u}_\bullet^{(n)} t_n^* \Big).$$

If $(\varphi_n, \mathbb{u}^{(n)}) = (\varphi, \mathbb{u})$ is constant, we define the infinite repeat $(\varphi^\infty, \mathbb{u}^\infty)$ accordingly. Up to equivalence via a unitary in $\mathcal{M}(B)^\beta$, none of this depends on the choice of $\{t_n\}$.

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Lemma (Gabe–S; generalizing Voiculescu, Kasparov)

Let $(\varphi,\mathbb{u}),(\psi,\mathbb{v}):(A,\alpha)\to(\mathcal{M}(B),\beta)$ be two cocycle representations. Then $(\psi,\mathbb{v})\precsim(\varphi,\mathbb{u})$ if and only if $(\varphi^\infty,\mathbb{u}^\infty)\sim_B(\varphi^\infty\oplus\psi^\infty,\mathbb{u}^\infty\oplus\mathbb{v}^\infty)$.

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(The proof largely follows the old proofs, but involves lots of additional keeping track of the cocycles in the key steps.)

The existence of (unitally/nuclearly) absorbing cocycle representations is an easy corollary of the following much more general fact. We still assume that A and B are separable and (B, β) is conjugate to $(B \otimes \mathcal{K}, \beta \otimes \mathrm{id}_{\mathcal{K}})$.

Theorem (Gabe–S)

Let $\mathfrak{C}$ be a family of cocycle representations $(A, \alpha) \to (\mathcal{M}(B), \beta)$ that is closed under unitary equivalence via $\mathcal{U}(\mathcal{M}(B)^{\beta})$, and is closed under countable direct sums. Then there exists $(\theta, \mathbb{x}) \in \mathfrak{C}$ such that $(\theta, \mathbb{x}) \oplus (\varphi, \mathbb{u}) \sim_B (\theta, \mathbb{x})$ for all $(\varphi, \mathbb{u}) \in \mathfrak{C}$.

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Proof: The strict topology on the unit ball of $\mathcal{M}(B)$ is metrizable and separable. We equip the set of all cocycle representations $(\varphi,\mathbb{u}):(A,\alpha)\to(\mathcal{M}(B),\beta)$ with the point-strict topology in the first variable, and the uniform strict topology over compact sets $K\subseteq G$ in the second variable. Since A is separable and G is 2^{nd}-countable, we obtain a separable Polish space. (It is easy to write down a compatible metric.)

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Like in the work of Dadarlat–Eilers, one builds on a solid understanding of absorbing elements to show that an equivariant Cuntz pair

$$(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$$

is KK^G-trivial precisely when, after adding an absorbing cocycle representation, they become *operator homotopic* in an appropriate sense. The jump to the conclusion in the stable uniqueness theorem involves further trickery.

Next goal: Equivariant Kirchberg–Phillips theorem!

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Conjecture (S; theorem in progress (with Gabe))

Let Γ be a countable discrete amenable group. Let $\beta : \Gamma \curvearrowright B$ be an outer action on a stable Kirchberg algebra. Let $\alpha : \Gamma \curvearrowright A$ be an action on a separable exact C^*-algebra. Then the canonical map

$$\{\text{coc-hom's } (\varphi, \mathbb{u}) : (A, \alpha) \hookrightarrow (B, \beta)\} / \simeq_B \quad \longrightarrow \quad KK^\Gamma(\alpha, \beta)$$

is a bijection.(Actually this needs to be stated a bit more carefully, but let's not worry about it...)

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Corollary (assuming the above conjecture holds)

Let $\alpha : \Gamma \curvearrowright A$ and $\beta : \Gamma \curvearrowright B$ be outer actions on Kirchberg algebras.

1. *Suppose A and B are stable. Then any invertible element in $KK^\Gamma(\alpha, \beta)$ lifts to a cocycle conjugacy.*
2. *Suppose A and B are unital. Then any invertible element in $\kappa \in KK^\Gamma(\alpha, \beta)$ with $\kappa([\mathbf{1}_A]_0) = [\mathbf{1}_B]_0 \in K_0(B)$ lifts to a cocycle conjugacy.*

Thank you for your attention!



Fun fact:

Anatidaephobia is defined as a pervasive, irrational fear that one is being watched by a duck. The anatidaephobic individual fears that no matter where they are or what they are doing, a duck watches.