A short proof of the discontinuity of phase transition in the planar random-cluster model with $q > 4$

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Joint with Gourab Ray

University of British Columbia

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The random-cluster model Definition

- Finite graph $G = (V, E)$
- Two parameters: $p \in [0,1]$ and $q > 0$
- Configurations: $\omega \in \{0,1\}^E$

$$\mathbb{P}_{p,q}(\omega) \;\propto\; p^{\#\{\text{open edges}\}}(1-p)^{\#\{\text{closed edges}\}}q^{\#\text{clusters}}$$

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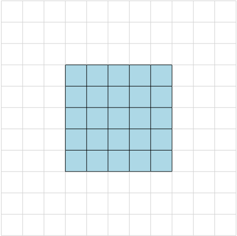
1. The **free** measure



- Closed boundary conditions
- The "**smallest**" measure

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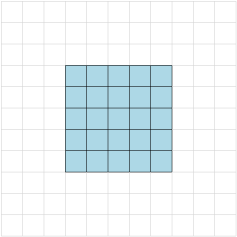
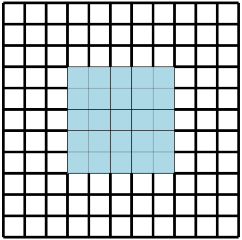


- Closed boundary conditions
- The "**smallest**" measure

2. The **wired** measure



- Open boundary conditions
- The "**largest**" measure

The random-cluster model

Phase transition

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Does ω have an infinite open cluster?

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There exists $p_c = p_c(q, d) \in (0, 1)$ such that

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The random-cluster model — Phase transition

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$$\theta^{\textsf{free}}(p) := \mathbb{P}^{\textsf{free}}_{p,q}(\text{the origin is in an infinite open cluster of } \omega)$$

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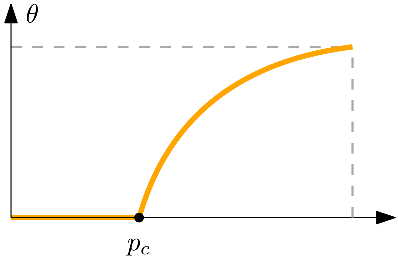
1. $\theta^{\text{wired}}(p) = \theta^{\text{free}}(p) = 0$ if $p < p_c$
2. $\theta^{\text{wired}}(p) \geq \theta^{\text{free}}(p) > 0$ if $p > p_c$

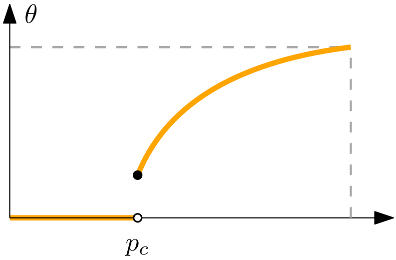
The random-cluster model — Phase transition

The random-cluster model on $\mathbb{Z}^d$ – one of two possibilities:

1. **Continuous** phase transition



2. **Discontinuous** phase transition



The random-cluster model — Phase transition

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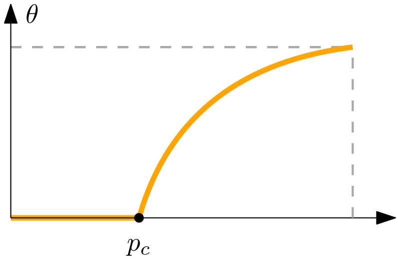
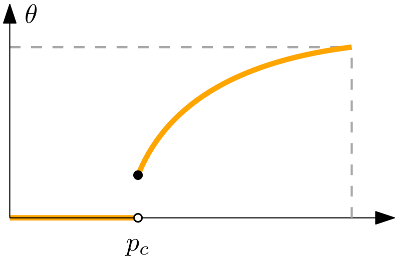


- $\theta^{\text{wired}}(p_c) = 0$

2 Discontinuous phase transition



- $\theta^{\text{wired}}(p_c) > 0$

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   - Short proof [Ray–Spinka 2019+]

Main tools

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   - The random-cluster model
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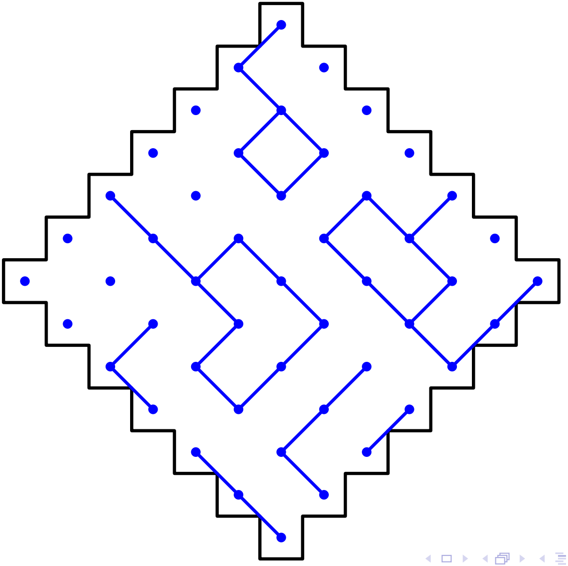
2. Height function representation for the six-vertex model

From the random-cluster model to loop configurations



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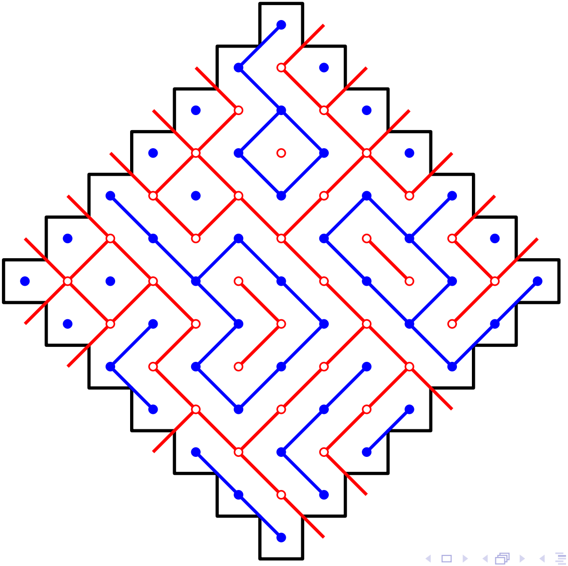

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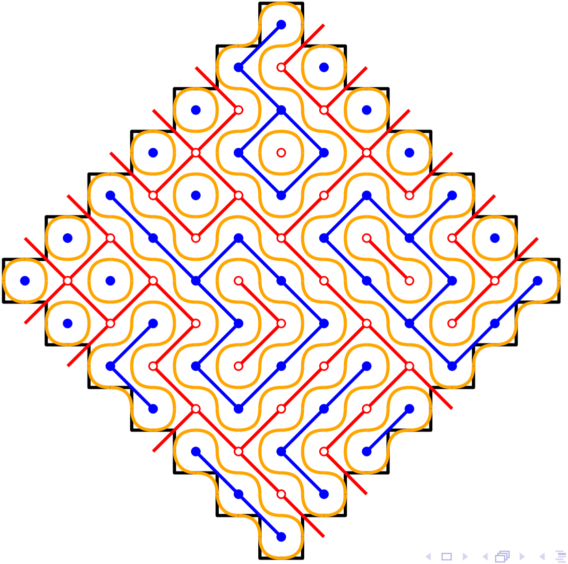

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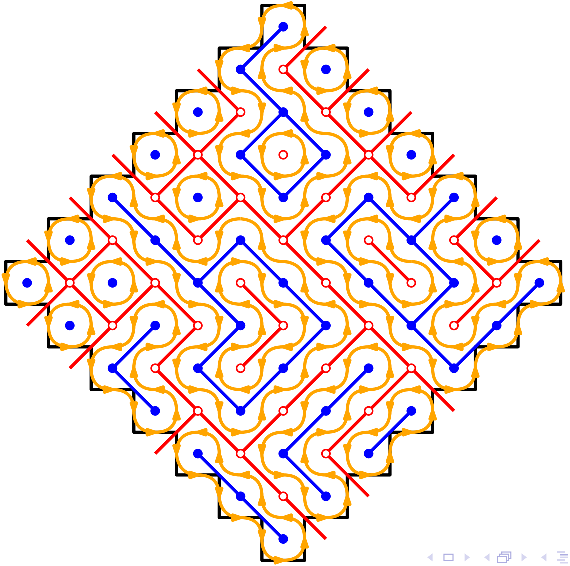

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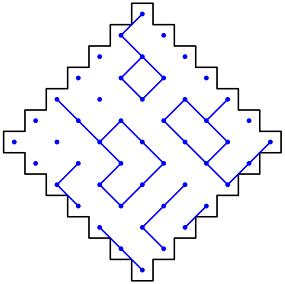

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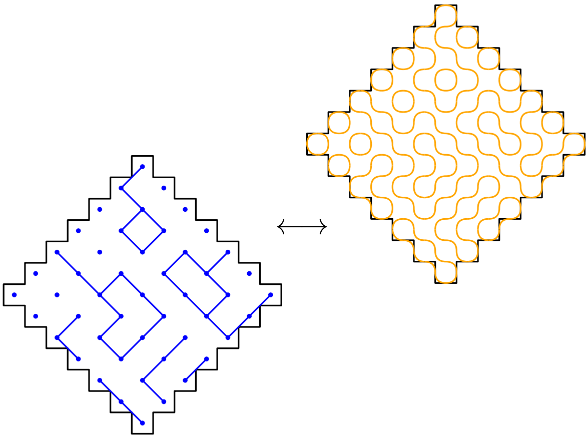

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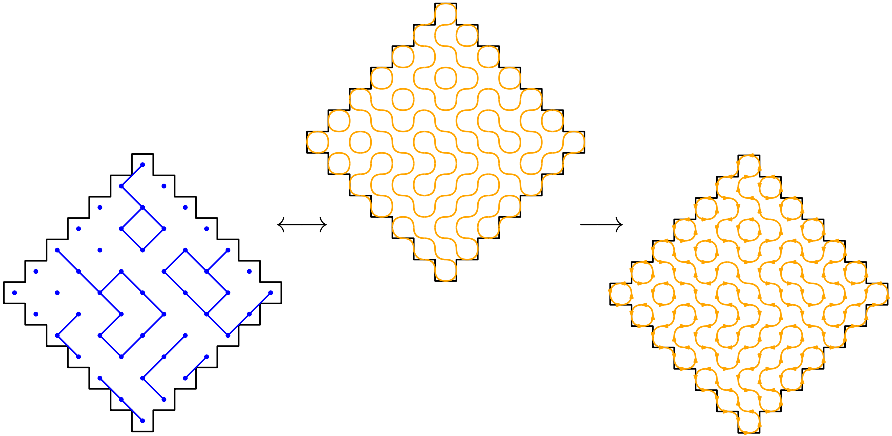

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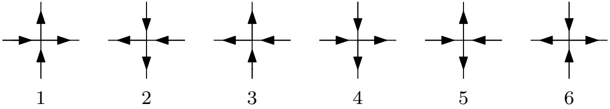

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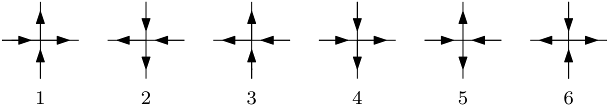

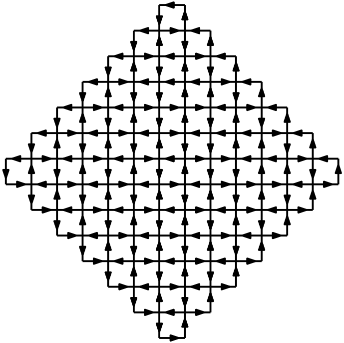

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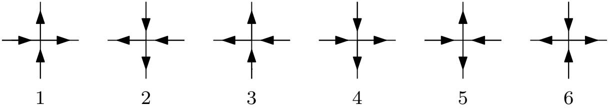

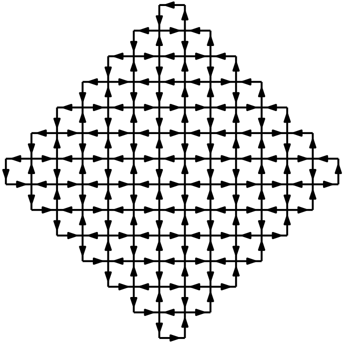

$$\mathbb{P}_c(\sigma) \ \propto \ c^{\#\{\text{type 5 and 6 vertices}\}}$$

From the six-vertex model to loop configurations

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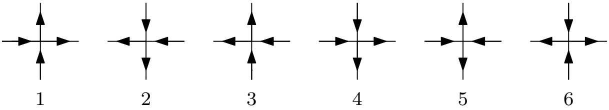

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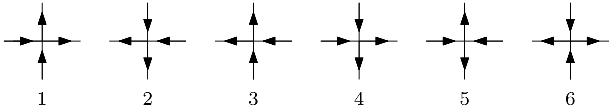

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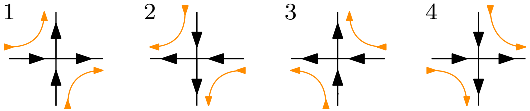

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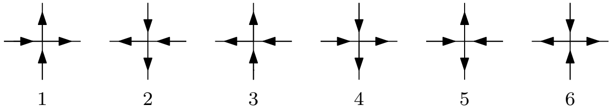

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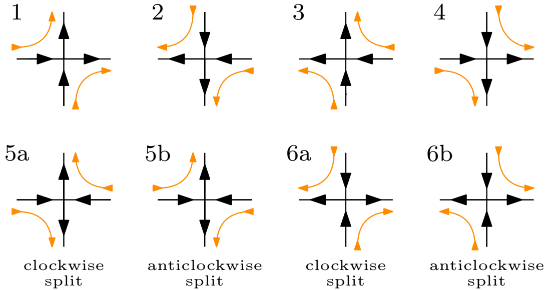

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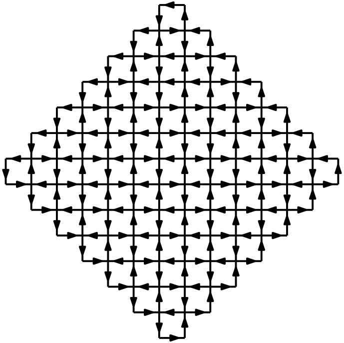

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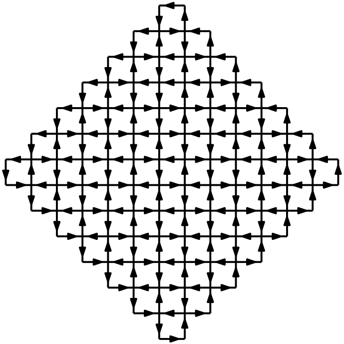
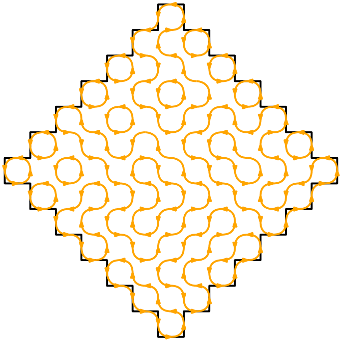
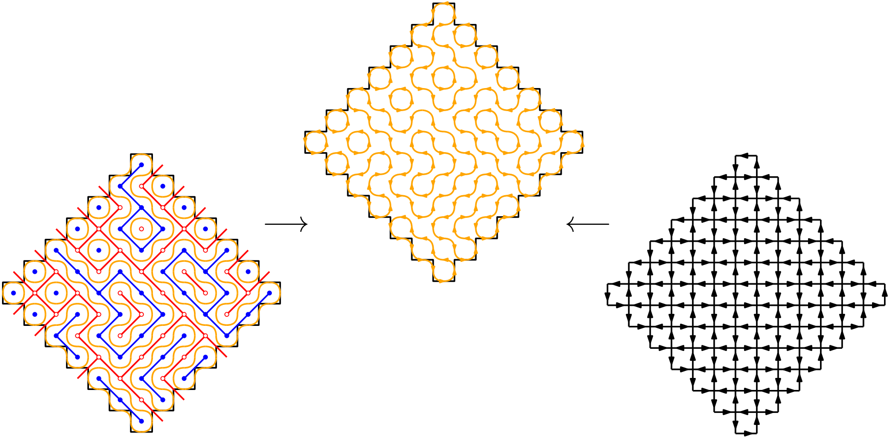


$\longrightarrow$



The BKW coupling



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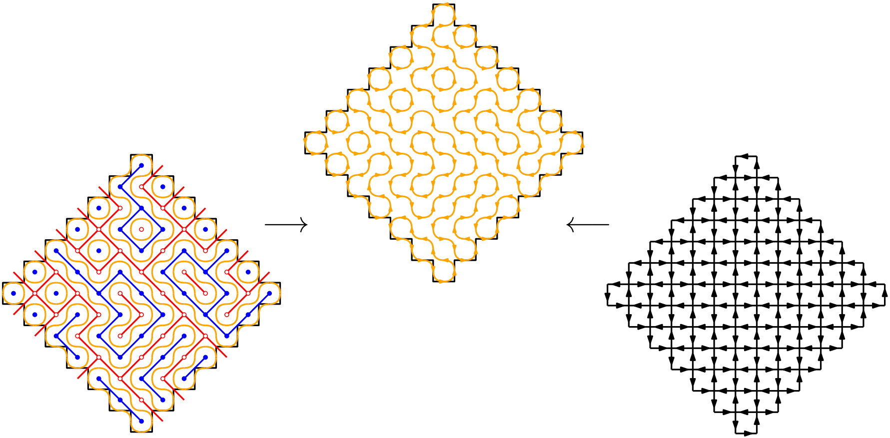

$p^{\#\text{open edges}}(1-p)^{\#\text{closed edges}}q^{\#\text{clusters}}$

$c^{\#\{\text{type 5 and 6}\}}$

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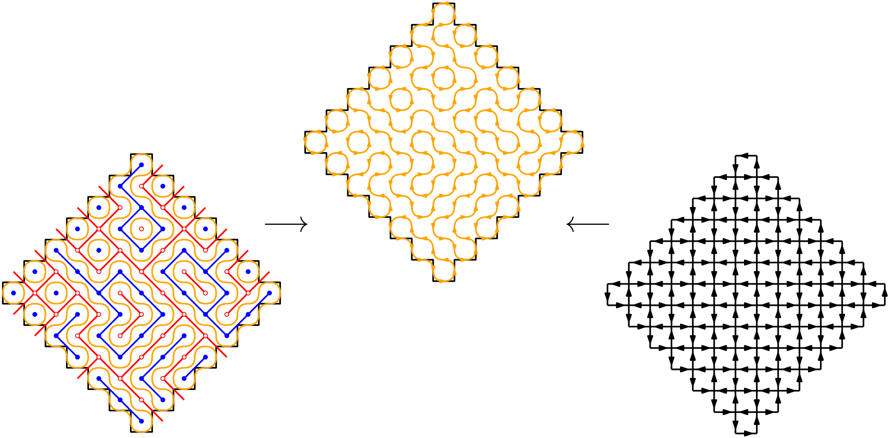

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$p = p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$ and **Euler's formula**

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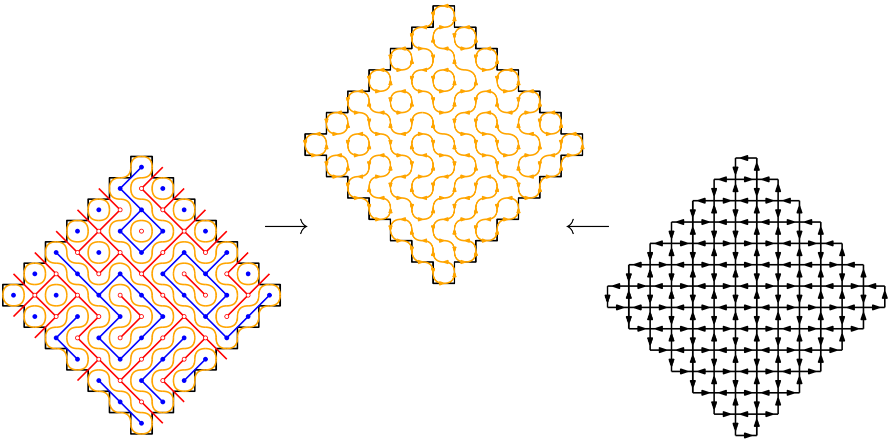

$\sqrt{q}^{\#\text{loops}}$ $\qquad$ $c^{\#\{\text{type 5 and 6}\}}$

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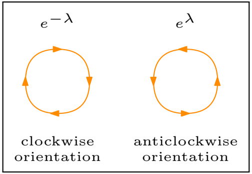

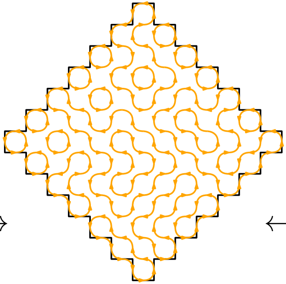

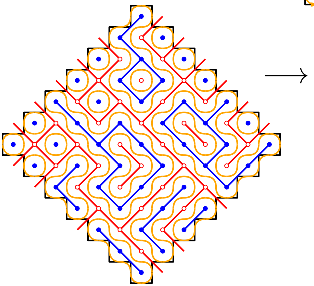

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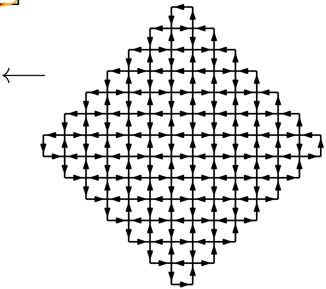

$c^{\#\{\text{type 5 and 6}\}}$

$$\sqrt{q} = e^{-\lambda} + e^{\lambda}$$

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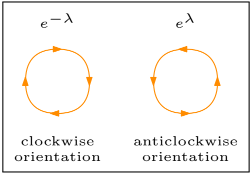

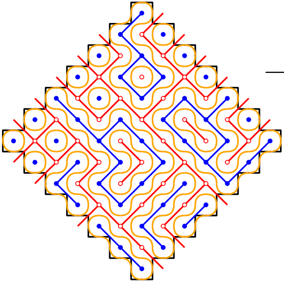

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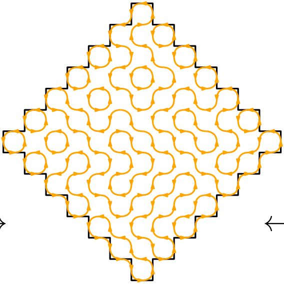
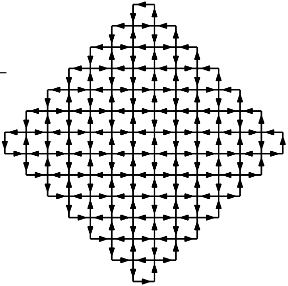


$e^{\lambda \#\{\text{signed loops}\}}$



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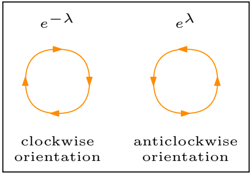

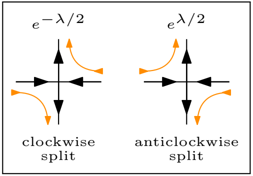

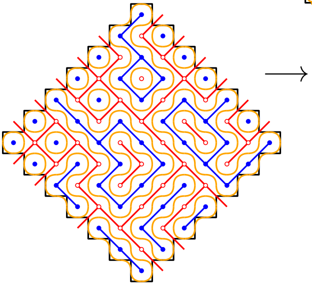

$\sqrt{q}^{\#\text{loops}}$

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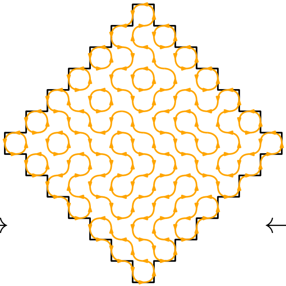

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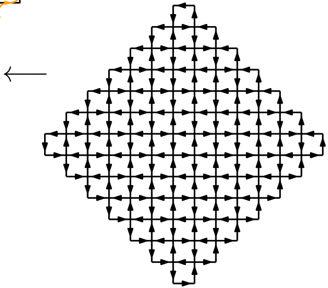

$c^{\#\{\text{type 5 and 6}\}}$

$$c = e^{-\lambda/2} + e^{\lambda/2}$$

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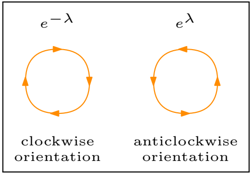
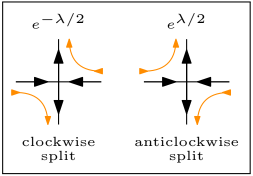
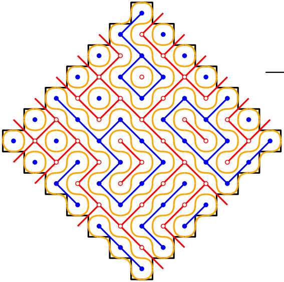


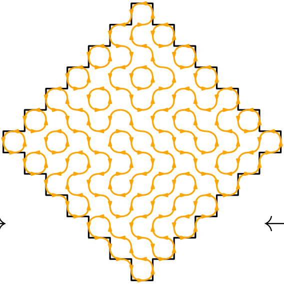




$\sqrt{q}^{\#\text{loops}}$

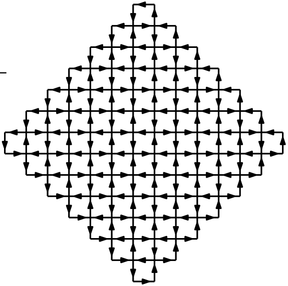
$\sqrt{q} = e^{-\lambda} + e^{\lambda}$

→



$e^{\lambda \#\{\text{signed loops}\}}$
$= e^{\frac{\lambda}{2}\#\{\text{signed splits}\}}$

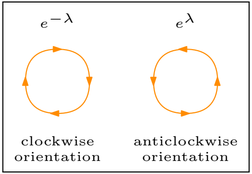
←

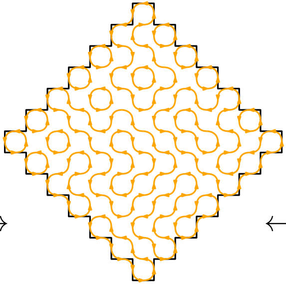


$c^{\#\{\text{type 5 and 6}\}}$

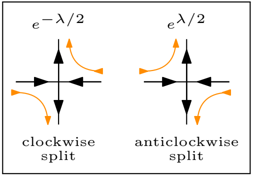
$c = e^{-\lambda/2} + e^{\lambda/2}$

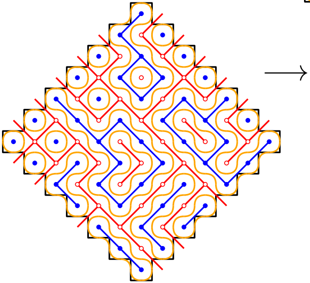
The BKW coupling





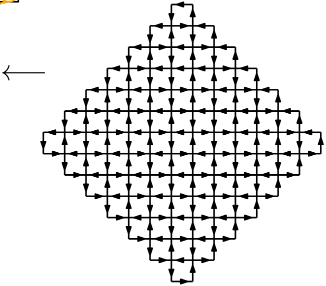
$e^{\lambda \#\{\text{signed loops}\}}$
$= e^{\frac{\lambda}{2}\#\{\text{signed splits}\}}$
$= e^{\frac{\lambda}{2\pi}\{\text{total winding}\}}$





$\sqrt{q}^{\#\text{loops}}$

$\sqrt{q} = e^{-\lambda} + e^{\lambda}$



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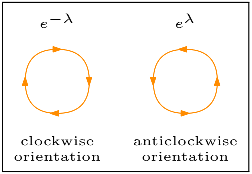
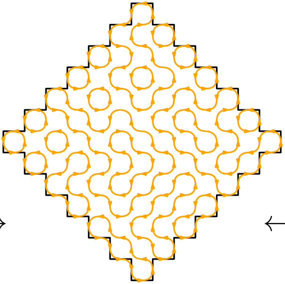
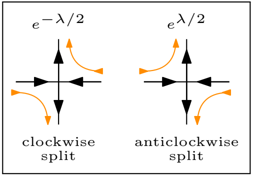
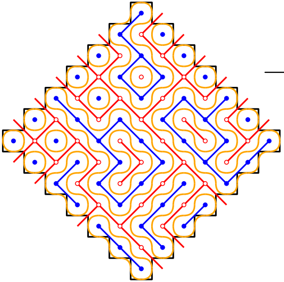
The BKW coupling









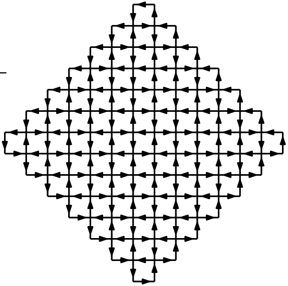
$\sqrt{q}^{\#\{\text{inner loops}\}} e^{\lambda\#\{\text{boundary loops}\}}$

$\sqrt{q} = e^{-\lambda} + e^{\lambda}$

$\longrightarrow$

$$e^{\lambda\#\{\text{signed loops}\}} = e^{\frac{\lambda}{2}\#\{\text{signed splits}\}} = e^{\frac{\lambda}{2\pi}\{\text{total winding}\}}$$

$\longleftarrow$



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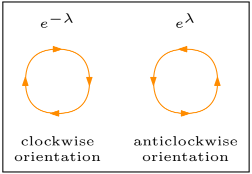
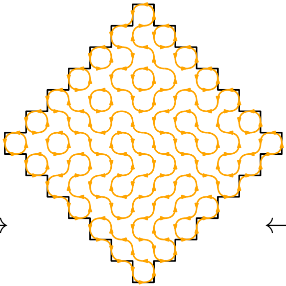
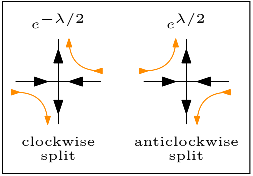
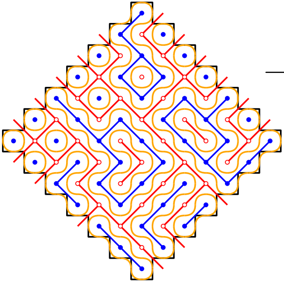
The BKW coupling

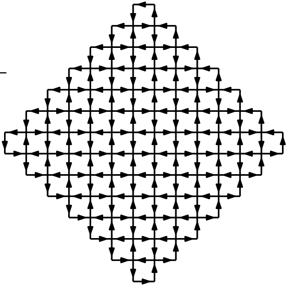








$\longrightarrow$ $\longleftarrow$



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$$c^{\#\{\text{type 5 and 6}\}}$$

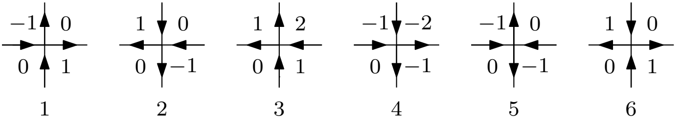
$$\sqrt{q} = e^{-\lambda} + e^{\lambda}$$

[Glazman–Peled 2019+]

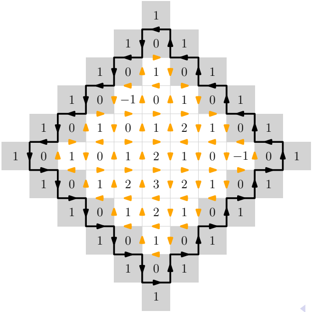
$$c = e^{-\lambda/2} + e^{\lambda/2}$$

The height function

- A six-vertex config is the **gradient** of a height function:

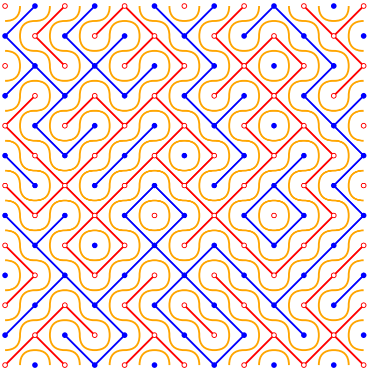


- The height function is defined up to a global additive constant



Proof of discontinuity by contradiction

- Fix $q > 4$ and consider the random-cluster model.



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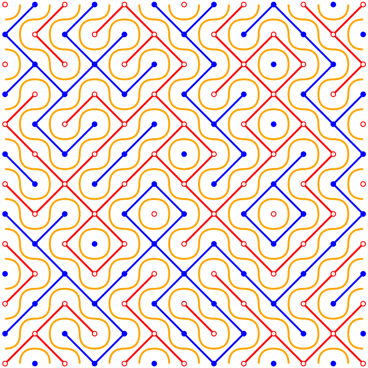
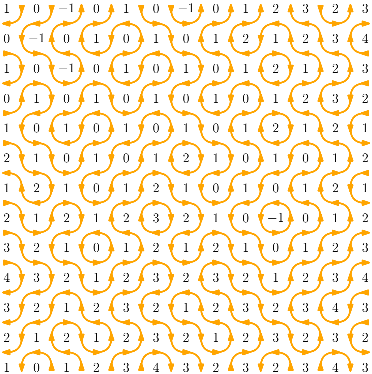
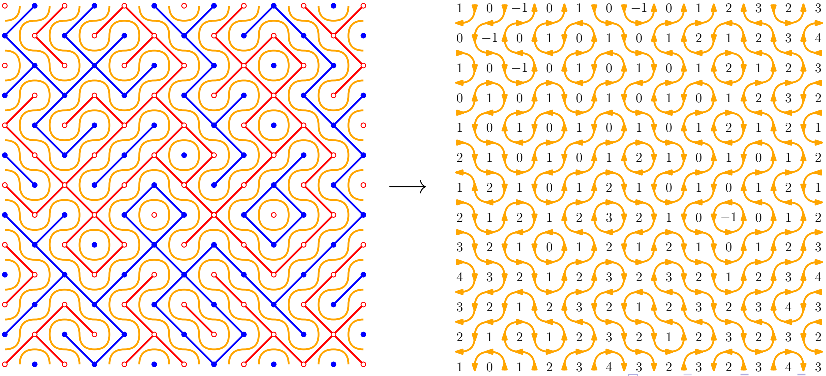


$\longrightarrow$



Proof of discontinuity by contradiction

- Fix $q > 4$ and consider the random-cluster model.
- Suppose it undergoes a **continuous** phase transition.



Proof of discontinuity by contradiction

- Fix $q > 4$ and consider the random-cluster model.
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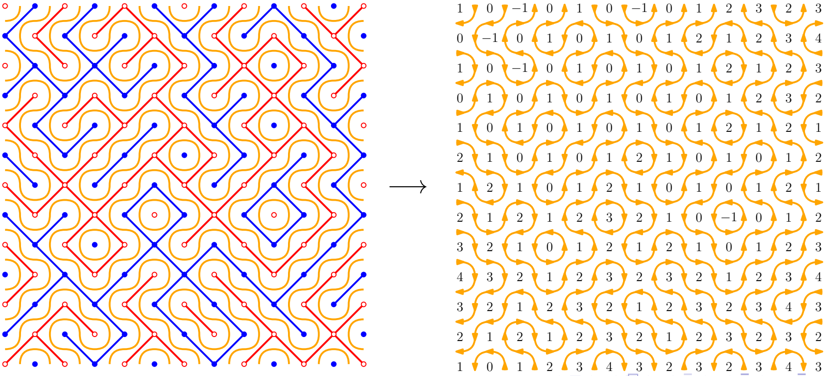
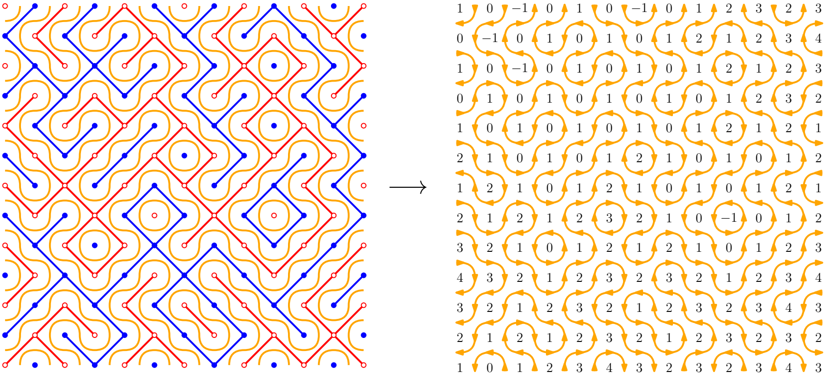
- All clusters (primal and dual) are **finite**.



Proof of discontinuity by contradiction

- Fix $q > 4$ and consider the random-cluster model.
- Suppose it undergoes a **continuous** phase transition.
- All clusters (primal and dual) are **finite**.
- Every vertex is surrounded by **infinitely many loops**.



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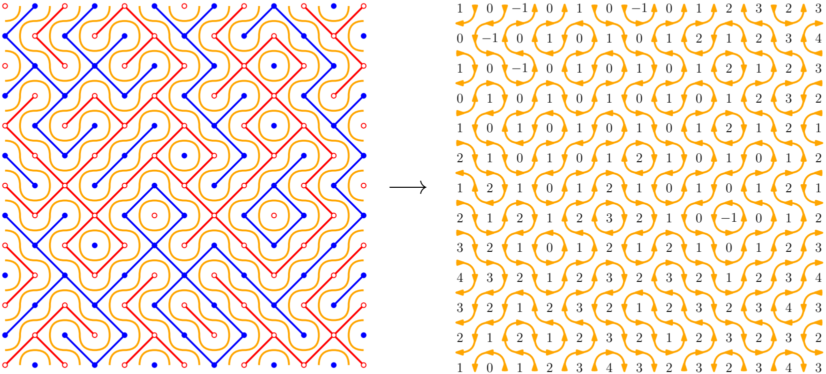

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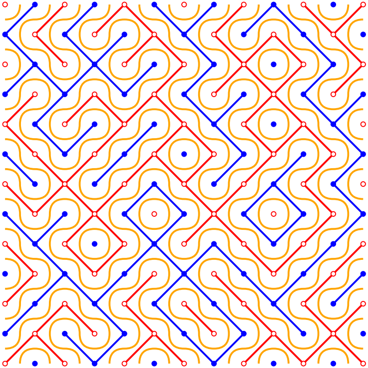

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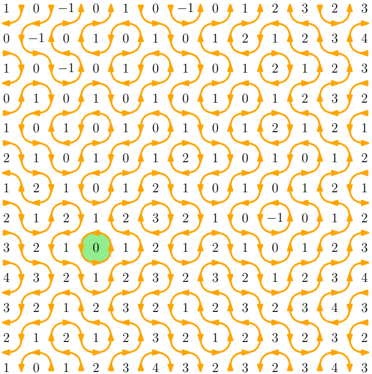

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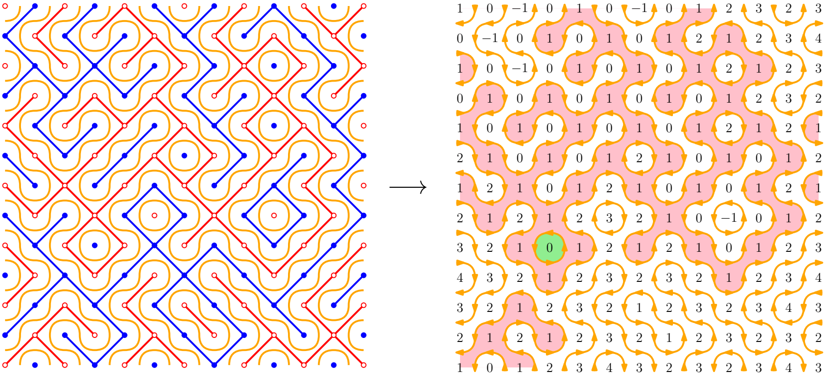

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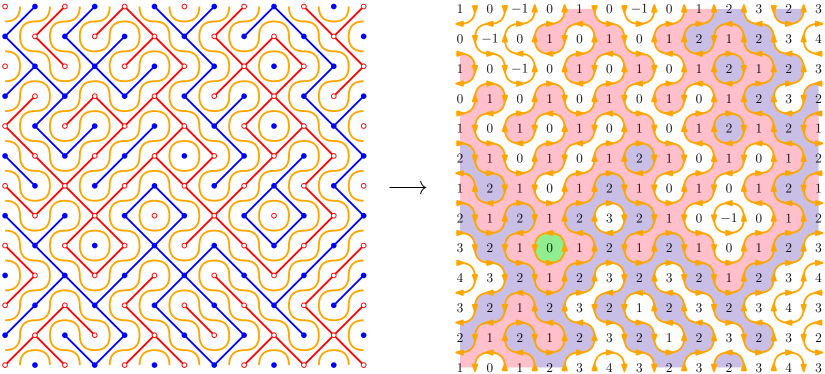
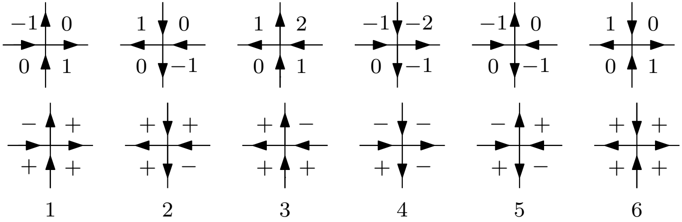
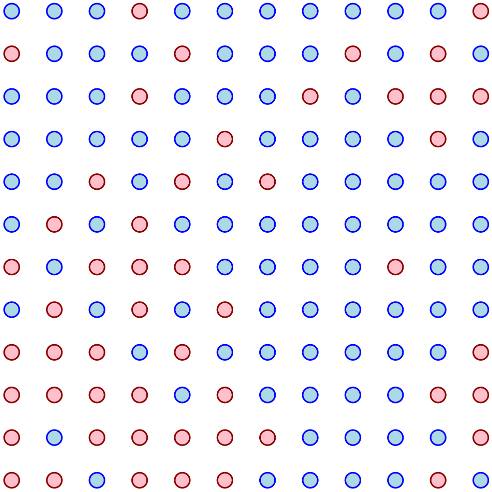


The spin representation of the six-vertex model

- Height function gives spin configuration via **mod 4**.



The FK-spin representation of the six-vertex model



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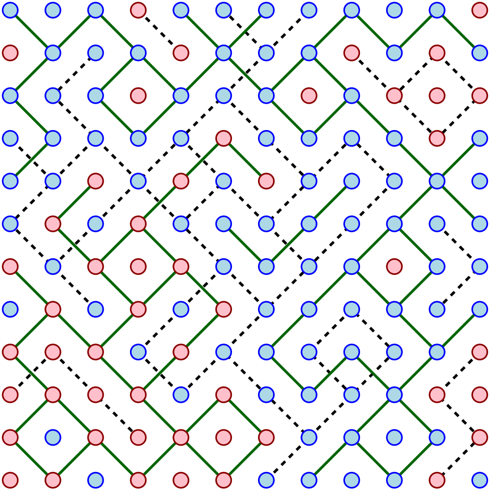
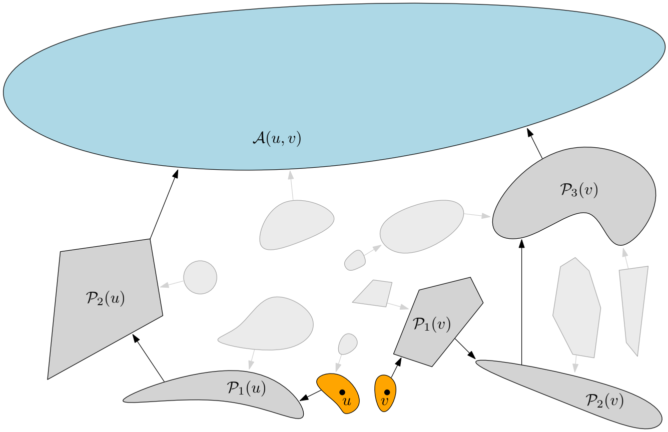


The gradient as a finitary factor of iid



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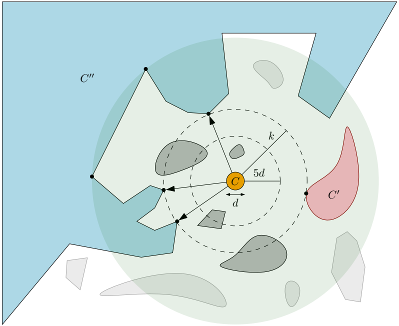

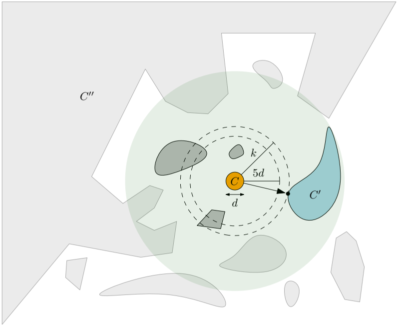

Thank you!