t-holomorphic functions

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Recap of Sanjay's talk

- We saw a new type of embedding or a least realisation for a bipartite graph using circle patterns.
- It generalises known embedding but is completely general : any weighted graph as a representation. Any weighted periodic graph has en embedding for each liquid phase.
- The geometry is compatible with the dimer model : weights can be read from the geometry. It is compatible with urban renewal, ...

Goal for the next two talks

Present a project with Chelkak and Russkikh about height fluctuations.

Aim

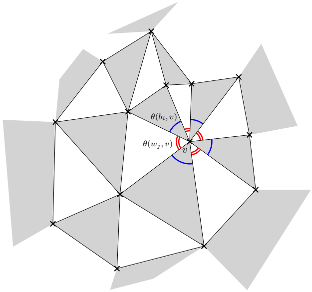
Prove GFF convergence without microscopic assumption on the boundary or constraint on the underlying graph.

- We need assumptions on a "perfect t-embedding" which should be weak but are still hard to check in a specific case.
- We use a discrete holomorphicity approach adapted to t-embeddings/circle patterns.
- We can be general because the embeddings are general.

Setup

We will only use the embedding of the dual graph by circle centers, which we call t-embedding. We will not use the circle pattern.

- Fixed finite or infinite triangulation with bipartite dual.
- Fixed embedding in the plane with straight edges and non-zero angles.
- Angle condition : $\sum_i \theta(b_i, v) = \sum_j \theta(w_j, v)$.



No assumption on the boundary yet because in this talk we focus on the bulk behaviour.

Reading K from the embedding

We will call the embedding $\mathcal{T}$. It is a function from the combinatorial structure to $\mathbb{C}$.
Given adjacent black and white faces b, w, we let $(bw)^*$ be the adjacent edge oriented with black on the right. We set

$$K(b, w) = d\mathcal{T}(bw^*).$$

- edge length = dimer weight
- edge orientation = Kasteleyn sign

Remark : To go in the other direction and find the embedding from the combinatorics, we find two functions $F^\bullet$ and F° in the kernel K and integrate their product.

Origami map

Definition

We fix a function $\eta \in \mathbb{S}_1$, satisfying the relation :

$$\eta_b \overline{\eta}_w = \frac{d\mathcal{T}(bw^*)}{|d\mathcal{T}(bw^*)|}.$$

The origami map $\mathcal{O}$ is then defined by

$$d\mathcal{O}(z) = \begin{cases} \eta_w^2 \, dz & \text{if } z \text{ is in the white face } \mathcal{T}(w) \\ \eta_b^2 \, d\bar{z} & \text{if } z \text{ is in the black face } \mathcal{T}(b). \end{cases}$$

Geometrically, $\mathcal{O}$ describes the folding of the plane along all edges of the embedding.
$d\mathcal{O}(bw^*) = \eta_w^2 \, d\mathcal{T}(bw^*) = \eta_b^2 \, \overline{d\mathcal{T}}(bw^*)$

t-holomorphic functions

Definition

A function $F_{\mathfrak{w}}$ is said to be t-white-holomorphic if

$$\begin{cases} F^{\bullet}_{\mathfrak{w}}(b) \in \overline{\eta}_b \mathbb{R}, \\ \Pr(F^{\circ}_{\mathfrak{w}}(w), \overline{\eta}_b \mathbb{R}) = F^{\bullet}_{\mathfrak{w}}(b). \end{cases}$$

Similarly, we say that $F_{\mathfrak{b}}$ is t-black-holomorphic if

$$\begin{cases} F^{\circ}_{\mathfrak{b}}(w) \in \eta_w \mathbb{R}, \\ \Pr(F^{\bullet}_{\mathfrak{b}}(b), \eta_w \mathbb{R}) = F^{\circ}_{\mathfrak{b}}(w). \end{cases}$$

The values on one color determine everything.

Discussion

- An example of t-white-holomorphic function is $b \to \bar{\eta}_w K^{-1}(w, b)$ for some fixed w.
- Given the first condition, the second is equivalent to $F^{\bullet}_{\mathfrak{w}} K = 0$ or $K F^{\circ}_{\mathfrak{b}} = 0$.
- Existence of an extension is analogous to the definition pf S-holomorphic functions.
- The choice of name is because as we will see the "nice" parts of the functions are the restrictions $F^{\circ}_{\mathfrak{w}}$ and $F^{\bullet}_{\mathfrak{b}}$.
- The first condition only reflects the fact that K has fixed directions $\eta_b \bar{\eta}_w$.

Contour integrals

Proposition

If $F_{\mathfrak{w}}$ and $F_{\mathfrak{b}}$ are t-holomorphic, then the following forms are closed away from the boundary :

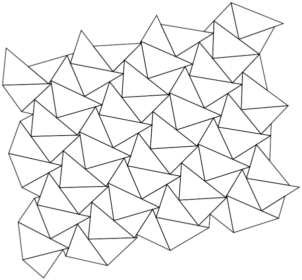
$$2 \cdot F_{\mathfrak{w}}^{\bullet}\, d\mathcal{T} = F_{\mathfrak{w}}^{\circ}\, d\mathcal{T} + \overline{F}_{\mathfrak{w}}^{\circ}\, d\overline{\mathcal{O}}$$

$$2 \cdot F_{\mathfrak{b}}^{\circ}\, d\mathcal{T} = F_{\mathfrak{b}}^{\bullet}\, d\mathcal{T} + \overline{F}_{\mathfrak{b}}^{\bullet}\, d\mathcal{O}$$

$$2 \cdot F_{\mathfrak{b}}^{\circ} F_{\mathfrak{w}}^{\bullet}\, d\mathcal{T} = \mathrm{Re}\left(F_{\mathfrak{b}}^{\bullet} F_{\mathfrak{w}}^{\circ}\, d\mathcal{T} + \overline{F}_{\mathfrak{b}}^{\bullet} F_{\mathfrak{w}}^{\circ}\, d\mathcal{O} \right)$$

- For K^{-1}, the forms are closed up to the boundary.
- We will call $\mathrm{I}_{\mathbb{C}}(F)$ the integral of these forms.
- If $\mathcal{O}$ is small, then any subsequential limit of $F_{\mathfrak{w}}^{\circ}$ or $F_{\mathfrak{b}}^{\bullet}$ is holomorphic. We need some regularity theory to take limits.

From t-embedding to T-graph



Proposition

For any α with $|\alpha| = 1$, the set $\mathcal{T} + \alpha^2\mathcal{O}$ is a T-graph, possibly with degenerate faces. In this T-graph:
(i) $(\mathcal{T} + \alpha\mathcal{O})(w) = (1 + \alpha^2\eta_w^2)\mathcal{T}(w) + cst$
(ii) $(\mathcal{T} + \alpha\mathcal{O})(b) = 2Pr(\mathcal{T}(b), \alpha\eta_b\mathbb{R}) + cst.$
For a generic choice of α, no face is degenerate.

First interpretation of t-holomorphicity

t-holomorphic function $\Leftrightarrow$ derivative of continuous piecewise affine real function :

- Inside each face of $\mathcal{T} + \mathcal{O}$, we can encode a linear real map by one complex number.
- Along each segment (of direction η_b), the derivative is naturally in $\bar{\eta}_b\mathbb{R}$.
- Projection have to match for it to be compatible.

Affine on each segment $\Leftrightarrow$ harmonic for a martingale random walk.

Formal proposition

Definition

For any interior vertex $v \in \mathcal{T} + \mathcal{O}$, there exists a unique segment (v^-, v^+) such that $v \in (v^-, v^+)$. We set transition rates $p(v \to v^\pm) = \frac{1}{|v^\pm - v| \cdot |v^+ - v^-|}$.

Proposition

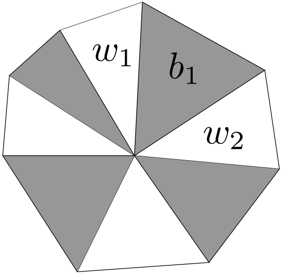
$F_{\mathfrak{w}}$ is t-white-holomorphic function if and only if it is the derivative of a harmonic function $\mathrm{I}_{\alpha\mathbb{R}}[F_{\mathfrak{w}}]$ on $\mathcal{T} + \alpha^2\mathcal{O}$ such that $\mathrm{I}_{\alpha R}[F_{\mathfrak{w}}] \in \alpha\mathbb{R}$.

Notion of integration are compatible :
$\mathrm{Pr}(\mathrm{I}_{\mathbb{C}}[F_{\mathfrak{w}}], \alpha\mathbb{R}) = \mathrm{I}_{\alpha\mathbb{R}}[F_{\mathfrak{w}}]$.
For K^{-1}, $\mathrm{Pr}(\mathrm{I}_{\mathbb{C}}[F_{\mathfrak{w}}], \alpha\mathbb{R})$ is harmonic everywhere and has constant value on all boundary vertices.

A second set of relations

Before, we used the t-holomorphicity relation around each white face. What happens around a vertex ?

- Neighbouring values $F^{\circ}_{\mathfrak{w}}(w)$ have the same projection on $\eta_b\mathbb{R}$.
- $F_{\mathfrak{w}}(w_1) - F_{\mathfrak{w}}(w_2) \in i\eta_{b_1}\mathbb{R}$
- The set of values $F^{\circ}_{\mathfrak{w}}(w)$ forms a polygon with fixed edge directions $i\eta_b\mathbb{R}$.



Linear relation on the values $\mathrm{Re}(F^{\circ}_{\mathfrak{w}}(w))$ around the face.

Second random walk interpretation

Proposition

The uniform measure is invariant for the random walk on $\mathcal{T} + \mathcal{O}$.

Proposition

If $F_{\mathfrak{w}}$ *is t-white holomorphic, then* $\mathrm{Re}(F^{\circ}_{\mathfrak{w}})$ *is harmonic for the backward random walk on* $\mathcal{T} - \bar{\mathcal{O}}$.

I have no conceptual insight on this fact. The proof comes from the computations.

Assumptions on the embedding

Now we consider a sequence $\mathcal{T}^\delta$ of t-embeddings. We make some non-degeneracy assumptions.

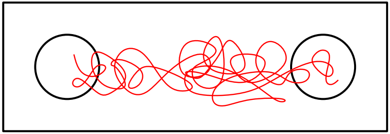
- There is a fixed bounded open set U such that for all δ, U is contained in the union of all interior triangles of $\mathcal{T}^{\#\delta}$.
- There exists C such that the length of all edges in $\mathcal{T}^{\#\delta}$ are bounded by δL.
- There exists $c \in (0,1)$ such that $\mathcal{O}(v) - \mathcal{O}(v') \leq (1-c)|v - v'| + \delta/c$.
- There exists $\beta > 0$ such that if one removes all edges of length more than δ^β, then no component has diameter more than δ.
- *There exists $\xi > 0$ such that all angles are in $[\xi, \pi - \xi]$.*

All constants must be independent of δ and uniform over compacts of U.

Understanding the forward walk

Since the forward walk is a martingale, we are in familiar grounds.

- From the angle condition we can prove that the variances of all projections are similar : $\exists c, \forall \alpha, \mathrm{Var}(\mathrm{Pr}(X_t, \alpha\mathbb{R})) \geq c Tr(\mathrm{Var}(X_t))$.
- Martingale and the variance condition gives a uniform crossing estimate



- Uniform crossing and planarity gives Harnack inequality and Hölder regularity of harmonic functions.

Understanding the backward walk

The backward walk looks a priori nasty. We want to connect its behaviour to the forward one.
It is not clear how to get a backward trajectory with some fixed starting point from the forward walk. *Excursions* however are symmetric.

- Using only uniform crossing, we prove that it stays true for forward random walk excursions.
- This gives the backward uniform crossing from the forward one.
- Harnack and Hölder continuity follow as before.

Remark : I don't know how to transfer a CLT from forward to backward.

Connecting the two

Actually having Harnack inequality on both a function and its primitive allows to bootstrap into a better result.

Proposition

For all open set V with $\bar{V} \subset U$, there exists a C (independent of δ) such that for all t-holomorphic function F^δ,

$$\sup_V |F^\delta| \leq C(\sup_U \mathrm{I}_{\mathbb{R}}[F^\delta] - \inf_U \mathrm{I}_{\mathbb{R}}[F^\delta]).$$

In other word, if a primitive is bounded then it is Lipschitz.

It is enough to control *only one* of the T-graph integrals.

Recap and conclusion

We want to do discrete complex analysis on t-embeddings/circle patterns.

- For a fixed w, $b \to \eta_w K^{-1}(w, b)$ is an example of t-holomorphic function.
- t-holomorphic functions are Hölder regular.
- t-holomorphic functions are bounded by their primitives.
- If $\mathcal{O}$ is small then any subsequential scaling limit of t-holomorphic function is (continuous) holomorphic. If $\mathcal{O}$ is not small, there is still an analogue.

Remains to do

- Introduce proper boundary conditions.
- Deduce height fluctuations from the theory.
- *Check the assumptions.*

Thank you for your attention.