Abstract Homomorphisms of Algebraic Groups and Applications

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Outline

1. Introduction
   - Abstract homomorphisms: general philosophy
   - Work of Borel and Tits
   - Groups over commutative rings
2. Results and applications
   - Rigidity results over rings
   - Rigidity for some non-arithmetic groups
   - Applications to character varieties

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General philosophy

Given alg. groups G/K and G'/K', an abstract homomorphism

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can (often) be written (essentially) as $\varphi = \beta \circ \alpha$, where

- $\alpha\colon G(K) \to G_{K'}(K')$ is induced by a field homomorphism $\tilde{\alpha}\colon K \to K'$ ($G_{K'}$ is obtained from G by base change via $\tilde{\alpha}$);
- $\beta\colon G_{K'}(K') \to G'(K')$ is induced by a K'-defined morphism $G_{K'} \to G'$.

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(rigidity statement)

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$G^+ =$ *(normal) subgroup of* $G(K)$ *generated by* K*-points of unipotent radicals of* K*-defined parabolics.*

Then *any abstract homomorphism* $\varphi\colon G^+ \to G'(K')$ *with Zariski-dense image has a standard description.*

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No. B-T gave an example of $\varphi\colon G(K) \to G'(K)$ such that

- G is absolutely almost simple / infinite K;
- φ has Zariski-dense image;
- G' is not reductive.

Borel-Tits' example

CONSTRUCTION: **Let**

- G be an absolutely almost simple group / infinite k (e.g. $G = \mathrm{SL}_n$)
- K/k be a field extension with a nontrivial k-derivation $\delta \colon K \to K$ (e.g. $k(x)/k$, δ = differentiation)

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Define $\varphi \colon G(K) \to G'(K)$ by

$$G(K) \ni g \;\mapsto\; (g^{-1} \cdot \Delta(g)\,,\, g) \in G'(K),$$

where Δ is induced by δ.

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Then

- $\mathrm{Im}\,\varphi$ is Zariski-dense in G';
- unipotent radical of G' is $\mathfrak{g}$ (hence nontrivial).

Example (cont.)

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B-T **conjectured** that any abstract homomorphism can be obtained in (basically) this fashion.

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such that

$$\rho = \sigma \circ r_{B/K'} \circ F,$$

where

- $F\colon G(K) \to G_B(B)$ *is induced by* f;
- $r_{B/K'}\colon G_B(B) \to \mathbf{R}_{B/K'}(G_B)(K')$ – *canonical isomorphism;*
- $\sigma\colon \mathbf{R}_{B/K'}(G_B) \to G'$ *is a* K'-*morphism of algebraic groups.*

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- $\mathrm{char}\, K = \mathrm{char}\, K' = 0$

  G simply connected Chevalley group,

  G' has commutative unipotent radical

  (L. Lifschitz, A. Rapinchuk)

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- arithmetic groups having Congruence Subgroup Property (work of BASS, MILNOR, SERRE and others)
- lattices in higher rank Lie groups (MARGULIS' SUPERRIGIDITY THEOREM)

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- $G(R)^{+}$ – subgroup of $G(R)$ generated by R-points of root subgroups (elementary subgroup)

Notations and conventions (cont.)

- for a finite-dimensional commutative K-algebra B, $G(B)$ is an algebraic group;

  more precisely, there exists an algebraic K-group $\mathbf{R}_{B/K}(G)$ such that

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- Given an abstract representation $\rho\colon G(R)^+ \to GL_n(K)$, we set
  $$H = \overline{\rho(G(R)^+)} \quad \text{(Zariski-closure)}$$
  $$H^\circ = \text{ connected component of } H$$

Rigidity Theorem

Theorem 2. (I.R.) *Assume* (Φ, R) *is nice, and* R *is noetherian if* char $K > 0$. *Let* $\rho \colon G(R)^+ \to GL_n(K)$ *be a representation.*

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such that

$$\rho|_\Gamma = (\sigma \circ F)|_\Gamma$$

for a suitable finite-index subgroup $\Gamma \subset G(R)^+$, *where* $F \colon G(R)^+ \to G(B)^+$ *is induced by* f.

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- Let $B = K[\varepsilon_1] \times \cdots \times K[\varepsilon_s]$, with $\varepsilon_i^2 = 0$ for all i, and define

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Examples (cont.)

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- images of root subgroups of $G(R)^+$ can have (arbitrarily) large dimension.
- one can construct representations whose image has unipotent radical of prescribed nilpotence class.

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We have also proved analogous results for elementary groups of type A_n over noncommutative rings.

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  $\boldsymbol{\mu}(\boldsymbol{\alpha}(x, y), z) = \boldsymbol{\alpha}(\boldsymbol{\mu}(x, z), \boldsymbol{\mu}(y, z))$ ("distributivity").

Our algebraic rings will always be commutative and unital.

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Then $(A, \boldsymbol{\alpha})$ is a commutative algebraic group.

Define $f : R \to A$ by $t \mapsto \rho(e_{13}(t))$ and **note** that

$$\boldsymbol{\alpha}(f(t_1), f(t_2)) = f(t_1 + t_2) \quad \text{for all} \quad t_1, t_2 \in R.$$

Construction of algebraic ring for SL_3 (cont.)

To define multiplication operation $\mu : A \times A \to A$, we need

$$w_{12} = e_{12}(1)\, e_{21}(-1)\, e_{12}(1) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$w_{23} = e_{23}(1)\, e_{32}(-1)\, e_{23}(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

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We have

$$w_{12}^{-1}\, e_{13}(r)\, w_{12} = e_{23}(r) \quad , \quad w_{23}\, e_{13}(r)\, w_{23}^{-1} = e_{12}(r)$$

and

$$[e_{12}(r)\ ,\ e_{23}(s)] = e_{13}(rs)$$

Construction of algebraic ring for SL_3 (cont.)

Define a regular map $\boldsymbol{\mu} : A \times A \to H$ by

$$\boldsymbol{\mu}(a_1, a_2) = [\rho(w_{23})\, a_1\, \rho(w_{23})^{-1}\,,\; \rho(w_{12})^{-1} a_2 \rho(w_{12})\,].$$

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$$\boldsymbol{\mu}(A \times A) \subset A.$$

As R is a commutative ring and f has Zariski-dense image we conclude that

$(A\,,\,\boldsymbol{\alpha}\,,\,\boldsymbol{\mu})$ is a commutative algebraic ring with identity.

Structure of algebraic rings in characteristic 0

Notice that

Any finite-dimensional K-algebra A has a natural structure of an algebraic ring.

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Conversely:

Theorem. *Let A be an algebraic ring / K where* $\mathrm{char}\, K = 0$. *Then there exists a finite-dimensional K-algebra B and a finite ring C such that*

$$A = B \oplus C.$$

In particular, any connected algebraic ring / K is a finite-dimensional K-algebra.

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- Starting with a representation $\rho\colon G(R)^+ \to GL_n(K)$, we construct an algebraic ring A.
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To summarize:

- Starting with a representation $\rho\colon G(R)^+ \to GL_n(K)$, we construct an algebraic ring A.
- If $\operatorname{char} K = 0$, then $A = B \oplus C$.
- The finite-dimensional K-algebra B is the algebra that appears in Theorem 2.
- A nontrivial finite ring C necessitates the passage to a finite-index subgroup.

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EXAMPLE. Set $A = K \oplus K$ with the usual addition and the following multiplication

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Then A is an algebraic ring with identity element $(1, 0)$.

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Then A is an algebraic ring with identity element $(1, 0)$.

But A is not a K-algebra: consider

$$\varphi : A \to A, \quad a \mapsto \mu(a, (0, 1)).$$

Then $\varphi((x, y)) = (0, x^p)$, hence $d_{(0,0)} \varphi \equiv 0$.

If $A \simeq$ an algebra, **then** φ would be a nonzero linear map, hence its differential would be $\not\equiv 0$.

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The map

$$\psi : A' \to A, \quad (x, y) \mapsto (x, y^p)$$

is a *morphism of algebraic rings* **and** an *isomorphism* of *abstract* rings, **but** not an isomorphism of *algebraic* rings.

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In particular, we generalize a rigidity result of G. Seitz.

Outline

Rigidity over rings of integers

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Notations:

- Φ - reduced irreducible root system of rank $\geqslant 2$
- G - corresponding Chevalley-Demazure group scheme/ $\mathbb{Z}$
- R - commutative ring such that (Φ, R) is a *nice pair*
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Theorem 2 implies the following classical result:

Theorem 8. *Suppose $\mathcal{O}$ is a ring of S-integers in a number field L. Then* any *representation $\rho: G(\mathcal{O})^+ \to GL_n(K)$ has a standard description.*

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This general strategy can be applied to rings with "few" derivations to analyze reps of some *non-arithmetic* groups.

Derivations and standard descriptions

For a ring homomorphism $g\colon R \to K$, let $\mathrm{Der}^{g}(R,K)$ be the K-vector space of maps $\delta\colon R \to K$ such that

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Theorem 9. (I.R.) *Suppose* $\dim_K \mathrm{Der}^g(R, K) \leqslant 1$ *for all homomorphisms* $g\colon R \to K$. *Then any representation* $\rho\colon G(R)^+ \to GL_n(K)$ *has a standard description.*

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Corollary. *If $\mathcal{O}$ is a ring of integers in a number field, then any representation* $\rho\colon SL_m(\mathcal{O}[X]) \to GL_n(K)$ $(m \geqslant 3)$ *has a standard description.*

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with $B \simeq K[\varepsilon_1] \times \cdots \times K[\varepsilon_r]$, $\varepsilon_i^{d_i} = 0$ for $d_i \geqslant 1$, and C *finite*;
- For $\widetilde{A} = K[\varepsilon]$, $\varepsilon^d = 0$ for $d \geqslant 1$, any central extension of algebraic groups over K of the form
$$1 \to W \to E \to G(\widetilde{A}) \to 1,$$
with $W = \mathbb{G}_a^\ell$ a vector group, splits. (Observed by Gabber.)

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In general:

If R a comm. k-algebra and $g\colon R \to K$ a ring hom., consider

$\mathrm{Der}^g_k(R, K) =$ set of derivations $\delta\colon R \to K$ such that $\delta|_k = 0$.

Rigidity over coordinate rings of affine curves

Theorem 10. (I.R.) *Suppose* $\dim_K \mathrm{Der}_k^g(R, K) \leqslant 1$ *for all homomorphisms* $g \colon R \to K$. *Then* any *representation* $\rho \colon G(R)^+ \to GL_n(K)$ *such that* $\rho|_{G(k)^+}$ *is completely reducible has a standard description.*

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Corollary. *Suppose* C *is a smooth affine algebraic curve over a number field* k*, with coordinate ring* $R = k[C]$*. Then* any *representation* $\rho: G(R)^+ \to GL_n(K)$ *has a standard description.*

Outline

1. Introduction
   - Abstract homomorphisms: general philosophy
   - Work of Borel and Tits
   - Groups over commutative rings

2. Results and applications
   - Rigidity results over rings
   - Rigidity for some non-arithmetic groups
   - Applications to character varieties

Representation and character varieties

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Let

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One can define

- $R_n(\Gamma)$ = variety of representations $\rho : \Gamma \to GL_n(K)$ (n^{th} representation variety)
- $X_n(\Gamma)$ = (categorical) quotient of $R_n(\Gamma)$ by $GL_n(K)$ (n^{th} character variety)

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So, varieties $R_n(\Gamma)$ and $X_n(\Gamma)$ are defined.

Assume now that

R is a finitely generated commutative ring, and

(Φ, R) is a nice pair.

Linear bound on the dimension

Theorem 5. (I.R.) *There exists a constant* $c = c(R)$ *(depending only on R) such that* $\varkappa_\Gamma(n) := \dim X_n(\Gamma)$ *satisfies*

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Remarks.

- Constant c is related to dimension of space of derivations of R.
- If R is the ring of S-integers in a number field (e.g. $\mathbb{Z}$), then $c = 0$, **hence** Γ is SS-rigid.

Elements of the proof

Bound dimension of

tangent space to $X_n(\Gamma)$ at $[\rho]$

by dimension of $H^1(\Gamma\,,\,\mathrm{Ad}_{GL_n} \circ \rho)$.

(based on ideas going back to A. Weil)

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One then uses standard descriptions of representations of Γ with non-reductive image (Theorem 2) to relate this cohomology group to a space of derivations of R.

A conjecture

Essentially all known examples of discrete linear groups having Kazhdan's property (T) are of the form $\Gamma = G(R)^+$.

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Conjecture. *Let Γ be a discrete linear group having Kazhdan's property (T). Then there exists a constant $c = c(\Gamma)$ such that*

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- For $\Gamma = F_d$, the free group on $d > 1$ generators,
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- **Thus**, conjecture predicts that rate of growth of $\varkappa_\Gamma(n)$ is minimum possible if Γ is Kazhdan.

- For any $f \colon \mathbb{N} \to \mathbb{N}$ such that $f(n)/n$ is non-decreasing and $f(n) \leqslant n(n-1)/2$, there exists a f.g. group Γ_f such that $\varkappa_{\Gamma_f}(n) = f(n)$ for all $n \geqslant 3$ (M. Kassabov).

Realizing affine varieties as character varieties

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Note that $X_n(\Gamma)$ is an affine variety defined over $\mathbb{Q}$.

Realizing affine varieties as character varieties

Question. *What affine varieties can be realized as* $X_n(\Gamma)$ *for some finitely generated group* Γ *and some* $n \geqslant 1$*?*

Same question with Γ having some special properties.

Note that $X_n(\Gamma)$ is an affine variety defined over $\mathbb{Q}$.

Are there any *other* restrictions?

Realizing affine varieties as character varieties (cont.)

Theorem 6. (Kapovich-Millson, 1998) *For any affine variety S defined over $\mathbb{Q}$, there is an Artin group Γ such that a Zariski-open subset U of S is biregular isomorphic to a Zariski-open subset of $X(\Gamma, PO(3))$.*

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Theorem 7. (I.R.) *Let S be an affine algebraic variety defined over $\mathbb{Q}$. There exist a finitely generated group Γ having Kazhdan's property (T) and an integer $m \geqslant 1$ such that there is a biregular isomorphism of complex algebraic varieties*

$$\varphi \colon S(\mathbb{C}) \to X_m(\Gamma) \setminus \{[\rho_0]\}$$

(where ρ_0 is the trivial representation).

Idea of the proof

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$$R \to R\left[\frac{1}{N}\right]$$

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- Set $\Gamma = G(R)^+$, $m = 2n$.

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