# The geometry of symplectic quasi-Hitchin representations.

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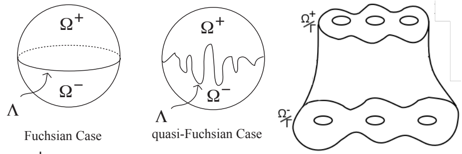
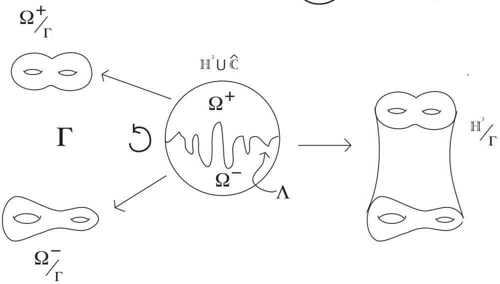
# Fuchsian and quasi-Fuchsian theory

Every quasi-Fuchsian $\rho\colon \pi_1(\Sigma) \longrightarrow SL(2,\mathbb{C}) \cong Sp(2,\mathbb{C})$ acts prop. disc. on $\mathbb{H}^3$ and on $\Omega_\rho := \mathbb{CP}^1 \setminus \Lambda_\rho \subset \partial\mathbb{H}^3 = \mathbb{CP}^1 \cong Lag(\mathbb{C}^2)$.

- $M_\rho = \mathbb{H}^3/\rho \cong \Sigma \times \mathbb{R}$;
- $\Omega_\rho/\rho \cong \Sigma^+ \sqcup \Sigma^-$.




**Question**

*What are 'quasi-Fuchsian' representations in* $Sp(4,\mathbb{C})$*?*

**Question**

*What is the topology of the quotient* $\Omega_\rho/\rho$ *for* $\Omega_\rho \subset Lag(\mathbb{C}^4)$*?*

# Symplectic spaces and group

- Symplectic space $(V_{\mathbb{K}}, \omega_{\mathbb{K}})$ (with $\mathbb{K} = \mathbb{R}, \mathbb{C}$).
- Symplectic group $\mathrm{Sp}(V_{\mathbb{K}}, \omega_{\mathbb{K}})$.
- $L \subset V_{\mathbb{K}}$ *isotropic* if $L \subset L^{\perp_{\omega_{\mathbb{K}}}}$ and *Lagrangian* if $L = L^{\perp_{\omega_{\mathbb{K}}}}$.

## Example

- $V_{\mathbb{C}} = \mathbb{C}^4 = \mathbb{C}^{(3)}[X, Y]$ and $\omega_{\mathbb{C}}$ defined by $\omega_{\mathbb{C}}(X^3, Y^3) = 1$ and $\omega_{\mathbb{C}}(X^2Y, XY^2) = -\frac{1}{3}$.
- $\mathrm{Sp}(V_{\mathbb{C}}, \omega_{\mathbb{C}}) \cong \mathrm{Sp}(4, \mathbb{C})$.

# Symplectic Anosov representations

## Definition (Labourie)

$\rho\colon \pi_1(\Sigma) \longrightarrow \mathrm{Sp}(2n,\mathbb{K})$ is *$Q_1$–Anosov* if $\exists$ continuous $\rho$–equivariant $\xi^1_\rho\colon \partial_\infty(\pi_1(\Sigma)) \longrightarrow \mathbb{P}(\mathbb{K}^{2n})$ s.t.

1. $\xi^1_\rho$ is *dynamics preserving* ($\forall\, \gamma \in \pi_1(\Sigma)$, $\xi^1_\rho(\gamma^\pm) = x^\pm_{\rho(\gamma)}$);
2. $\xi^1_\rho$ *transverse* ($\forall\, t \neq s$, $\xi^1_\rho(t)$ and $\xi^1_\rho(s)$ are transverse);
3. $+$ contraction/expansion properties.

## Example

- *Hitchin* reps: conn. comp. of $\mathfrak{X}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbb{R}))$ containing *Fuchsian* reps $\pi_1(\Sigma) \xrightarrow{\text{d. f.}} \mathrm{SL}(2,\mathbb{R}) \xrightarrow{\text{irred.}} \mathrm{Sp}(2n,\mathbb{R})$.
- *$Q_1$–quasi-Hitchin* reps: deformation of *Hitchin* reps $\pi_1(\Sigma) \xrightarrow{\text{d. f.}} \mathrm{SL}(2,\mathbb{R}) \xrightarrow{\text{irred.}} \mathrm{Sp}(2n,\mathbb{R}) \longrightarrow \mathrm{Sp}(2n,\mathbb{C})$ inside $Q_1$–Anosov reps $\mathfrak{X}_{Q_1}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbb{C}))$.
- deform. of $\mathrm{SL}(2,\mathbb{R})$ embeddings; *Maximal* reps; *positive* reps; ...

# Properties of (symplectic) Anosov representations

## Theorem (Labourie, Guichard-Wienhard)

*Given* $\rho\colon \pi_1(\Sigma) \longrightarrow \mathrm{Sp}(2n,\mathbb{K})$ $Q_1$–Anosov, *then*

- $\rho$ *is discrete and faithful.*
- $\rho$ *acts proximally on* $\mathbb{P}(\mathbb{K}^{2n})$ *(that is,* $\forall\gamma \in \pi_1(\Sigma)$, $\exists x_\gamma^+, x_\gamma^- \in \mathbb{P}(\mathbb{K}^{2n})$ *s.t.* $\forall y \in \mathbb{P}(\mathbb{K}^{2n})$ *transverse to* $x_\gamma^\mp$ $\rho(\gamma^{\pm n})y \longrightarrow x_\gamma^\pm$*).*
- *the orbit map* $\pi_1(\Sigma) \longrightarrow \mathrm{Sp}(2n,\mathbb{K})/K$ *is a quasi-isometrix embedding wrt the word distance and the Riemannian distance, resp.*

*In addition,* $\mathfrak{X}_{Q_1}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbb{K}))$ *is* open *and* $\mathrm{Mod}(\Sigma)$ *acts* prop. disc. *on* $\mathfrak{X}_{Q_1}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbb{K}))$.

# Domain of discontinuity

Given $\rho$ $Q_1$–Anosov, we have $\xi^1_\rho\colon \partial_\infty(\pi_1(\Sigma)) \longrightarrow \mathbb{CP}^{2n-1}$.

$\forall \ell \in \mathbb{CP}^{2n-1}$ let $K_\ell = \{W \in \mathrm{Lag}(\mathbb{C}^{2n}) | \ell \subset W\} \subset \mathrm{Lag}(\mathbb{C}^{2n})$.

Define

$$K_{\xi^1_\rho} = \bigcup_{t\in\partial_\infty(\pi_1(\Sigma))} K_{\xi^1_\rho(t)} \text{ and } \Omega_{\xi^1_\rho} = \mathrm{Lag}(\mathbb{C}^{2n}) \setminus K_{\xi^1_\rho}$$

## Theorem (Guichard–Wienhard)

*$\Omega_{\xi^1_\rho}$ is a cocompact domain of discontinuity for the action of $\rho$ on $\mathrm{Lag}(\mathbb{C}^{2n})$ (that is, $\Omega_{\xi^1_\rho}$ is open and $\rho$ acts on it freely, properly discontinuosly and cocompactly).*

## Question

*What is the topology of $\Omega_{\xi^1_\rho}/\rho$?*

# Main theorem

## Conjecture (Dumas-Sanders)

*Given $\rho\colon \pi_1(\Sigma) \longrightarrow G$ $B$–quasi-Hitchin, then $\Omega_{\xi^i_\rho}/\rho$ is a fiber bundle over a surface with fiber a compact Poincaré duality space.*

Dumas - Sanders prove the conjecture for $G = \mathrm{SL}(3,\mathbb{C})$.

## Theorem (Alessandrini - M. - Wienhard)

*Given $\rho\colon \pi_1(\Sigma) \longrightarrow \mathrm{Sp}(4,\mathbb{C})$ $Q_1$–quasi-Hitchin, then $\Omega_{\xi^1_\rho}/\rho$ is a fiber bundle over a surface with fiber $F$ and structure group $\mathrm{SO}(2)$ and Euler class $2g-2$. In addition, the fiber $F$ is homeomorphic to a quotient of $(\mathbb{S}^2 \times \mathbb{S}^2)/A_4$.*

The cont. projection $\Omega_{\xi^1_\rho} \longrightarrow \mathbb{H}^2$ (which descends to $\Omega_{\xi^1_\rho}/\rho \longrightarrow \Sigma$) comes from the study of the space $\mathrm{Lag}(\mathbb{C}^4)$ and its $\mathrm{SL}(2,\mathbb{C})$–orbits.

# $\mathrm{SL}(2,\mathbb{C})$–orbits

First, we study the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathrm{Lag}(\mathbb{C}^4)$.

Recall that $\mathbb{C}^4 = \mathbb{C}^{(3)}[X,Y]$ and $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathbb{C}^4$ by acting on the roots of the polynomials.

Consider the Veronese embeddings:

- $\xi^1\colon\ \mathbb{RP}^1 \longrightarrow \mathbb{CP}^3$
  $[a:b] \mapsto (bX-aY)^3$
- $\xi^2\colon\ \mathbb{RP}^1 \longrightarrow \mathrm{Lag}(\mathbb{C}^4)$
  $[a:b] \mapsto \langle (bX-aY)^3, (bX-aY)^2X\rangle$.

which can be extended to

- $\xi^1_{\mathbb{C}}\colon\ \mathbb{CP}^1 \longrightarrow \mathbb{CP}^3$;
- $\xi^2_{\mathbb{C}}\colon\ \mathbb{CP}^1 \longrightarrow \mathrm{Lag}(\mathbb{C}^4)$.

Recall that $\forall \ell \in \mathbb{CP}^3$, $K_\ell = \{W \in \mathrm{Lag}(\mathbb{C}^4) | \ell \subset W\} \subset \mathrm{Lag}(\mathbb{C}^4)$.

**Question**

*What are the $\mathrm{SL}(2,\mathbb{C})$–orbits in $\mathrm{Lag}(\mathbb{C}^4)$?*

# $\mathrm{SL}(2,\mathbb{C})$–orbits of $\mathrm{Lag}(\mathbb{C}^4)$

## Proposition

*There are* 3 $\mathrm{SL}(2,\mathbb{C})$*–orbits in* $\mathrm{Lag}(\mathbb{C}^4)$*:*

- $\mathrm{Lag}(\mathbb{C}^4) \setminus K_{\xi^1_{\mathbb{C}}} \cong \mathfrak{T} = \{$*ideal regular hyp. tetrahedra*$\}$ *(open orbit).*
- $K_{\xi^1_{\mathbb{C}}} \setminus \xi^2_{\mathbb{C}}(\mathbb{CP}^1)$*;*
- $\xi^2_{\mathbb{C}}(\mathbb{CP}^1)$ *(closed orbit).*

Recall that an ideal hyperbolic tetrahedra is regular $\iff$ it has max volume $\iff$ the cross-ratio of its vertices is $\frac{1-i\sqrt{3}}{2}$.

Note that $K_{\xi^1_{\mathbb{C}}}$ corresponds to “degenerate” ideal regular tetrahedra.

# Sketch of the proof

## $K_{\xi^1_{\mathbb{C}}} \cong \mathbb{CP}^1 \times \mathbb{CP}^1$.

$K_{\xi^1_{\mathbb{C}}} = \cup_{t\in\mathbb{CP}^1} K_{\xi^1_{\mathbb{C}}(t)} = \{W \in \mathrm{Lag}(\mathbb{C}^4) \mid \exists\, p = (X - z_0 Y)^3 \in W\}$
$= \{W \in \mathrm{Lag}(\mathbb{C}^4) \mid \forall p \in W, p(X,Y) = (X - z_0 Y)q(X,Y)\}$.

So $F\colon \mathbb{CP}^1 \times \mathbb{CP}^1 \xrightarrow{\cong} K_{\xi^1_{\mathbb{C}}}$ by $F([a:b],[c:d]) =$

$$\begin{cases} <(bX-aY)^3,(bX-aY)^2X>=\xi^2_{\mathbb{C}}([a:b]) & \text{if } [a:b]=[c:d] \\ <(bX-aY)^3,(dX-cY)^2(bX-aY)> & \text{if } [a:b]\neq[c:d] \end{cases}$$

□

## Remark

$K_{\xi^1} \cong \mathbb{RP}^1 \times \mathbb{CP}^1$.

# Sketch of the proof (summary)

$\mathrm{Lag}(\mathbb{C}^4) \setminus K_{\xi^1_{\mathbb{C}}} \cong \mathfrak{T}$.

- $\forall W \in \mathrm{Lag}(\mathbb{C}^4) \setminus K_{\xi^1_{\mathbb{C}}}$, $\exists p \in W$ with a double root. Up to $\mathrm{SL}(2,\mathbb{C})$, we can suppose $p = X^2 Y$.
- We study all the Lagrangians containing $p = X^2 Y$.
- $\forall W \in \mathrm{Lag}(\mathbb{C}^4) \setminus K_{\xi^1_{\mathbb{C}}}$ $\exists 4$ $p \in W$ with double roots and these 4 roots form a regular ideal hyperbolic tetrahedra.

□

# Sketch of the proof (details)

## $\mathrm{Lag}(\mathbb{C}^4) \setminus K_{\xi^1_{\mathbb{C}}} \cong \mathfrak{T}$.

- $\forall W \in \mathrm{Lag}(\mathbb{C}^4)$, $\exists p \in W$ s.t. $p(X,Y) = (X - z_0 Y)^2 (X - z_1 Y)$, with $z_i \in \mathbb{CP}^1$.
- Let $z_0 = 0$ and $z_1 = \infty$, so $p = X^2 Y$.
  $L_{X^2Y} = \{W \in \mathrm{Lag}(\mathbb{C}^4) \mid X^2 Y \in W\}$
  $= \{W = \langle X^2 Y, X^3 + \frac{b}{a} Y^3 \rangle \mid \frac{b}{a} \in \mathbb{CP}^1\}$.

  By using the action of $\mathrm{SL}(2,\mathbb{C})$, we can assume $\frac{b}{a} = 1$ and study $W = \langle X^2 Y, X^3 + Y^3 \rangle$. So

$$K_{\xi^1_{\mathbb{C}}} = \mathrm{SL}(2,\mathbb{C}) \cdot \langle X^2 Y, X^3 + Y^3 \rangle.$$

□

# Sketch of the proof (details)

$\mathrm{Lag}(\mathbb{C}^4) \setminus K_{\xi^1_{\mathbb{C}}} \cong \mathfrak{T}$.

- In $W = [\langle X^2Y, X^3 + Y^3\rangle]$ $\exists$ 4 'special' polynomials with double roots. Their associate double and single roots are:
  - (i) $z_0 = 0$ and $w_0 = \infty$;
  - (ii) $z_1 = \sqrt[3]{2}$ and $w_1 = -\frac{1}{\sqrt[3]{4}}$;
  - (iii) $z_2 = \frac{-1-i\sqrt{3}}{\sqrt[3]{4}}$ and $w_2 = \frac{1+i\sqrt{3}}{2\sqrt[3]{4}}$;
  - (iv) $z_3 = \frac{-1+i\sqrt{3}}{\sqrt[3]{4}}$ and $w_3 = \frac{1-i\sqrt{3}}{2\sqrt[3]{4}}$.

  (Proof: use the notion of discriminant.)
- The cross ratio is $[z_0 : z_1 : z_2 : z_3] = [w_0 : w_1 : w_2 : w_3] = \frac{1-i\sqrt{3}}{2}$.

□

# Main theorem

By sending a tetrahedra into its barycenter (or its degenerations), we define the cont. projections

$$\mathrm{Lag}(\mathbb{C}^4) \longrightarrow \mathbb{H}^3 \cup \mathbb{CP}^1 \text{ and } \Omega_{\xi^1} \longrightarrow \mathbb{H}^2,$$

where

$$\Omega_{\xi^1} = \mathcal{T} \cup \left((\mathbb{CP}^1 \setminus \mathbb{RP}^1) \times \mathbb{RP}^1\right) \longrightarrow \mathbb{H}^3 \cup (\mathbb{CP}^1 \setminus \mathbb{RP}^1) \longrightarrow \mathbb{H}^2$$

## Theorem

*Given* $\rho\colon \pi_1(\Sigma) \longrightarrow \mathrm{Sp}(4,\mathbb{C})$ *$Q_1$–Anosov, then* $\Omega_{\xi^1_\rho}/\rho$ *is a fiber bundle over a surface with fiber $F$ and structure group* $\mathrm{SO}(2)$ *and Euler class* $2g-2$.

## Question

*What can we say about the fiber $F$?*

# What can we say about $F$?

## Theorem

*The fiber $F$ is homeomorphic to a quotient of $(\mathbb{S}^2 \times \mathbb{S}^2)/A_4$.*

Let's describe first $\mathbb{S}^2 \times \mathbb{S}^2$ via mapping cylinders:

- Let $M_p = \mathbb{T}^{\leq 1}(\mathbb{S}^2) = \mathbb{T}^1(\mathbb{S}^2) \times [0,1]/(\mathbb{T}^1(\mathbb{S}^2) \times \{0\} \sim \mathbb{S}^2)$ via the projection $p\colon\ \mathbb{T}^1(\mathbb{S}^2) \longrightarrow \mathbb{S}^2$.
- Then $M_p \sqcup_{id} M_p/\sim\ \cong \mathbb{S}^2 \times \mathbb{S}^2$.

$\mathbb{T}^1(\mathbb{S}^2)/A_4 \cong \mathbb{T}^{1,orb}(\mathbb{S}^2(2,3,3))$ (reg. tetrahedra with fixed barycenter).

If we do the same construction replacing $\mathbb{T}^1(\mathbb{S}^2)$ with $\mathbb{T}^{1,orb}(\mathbb{S}^2(2,3,3))$, we obtain $X = (\mathbb{S}^2 \times \mathbb{S}^2)/A_4$. The fiber $F$ is a quotient of $X$.

## Proposition

*The fiber $F$ is not an orbifold and it has 4 singular points: two are cones over $\mathrm{L}(3,1)\#\mathbb{RP}^3$, and two are cones over $\mathrm{L}(3,1)$.*

# $\mathrm{Sp}(4,\mathbb{R})$–orbits

## Theorem (Wolf J.)

*There are 6 $\mathrm{Sp}(4,\mathbb{R})$–orbits in $\mathrm{Lag}(\mathbb{C}^4)$:*

$$\mathcal{R}_i = \{W \in \mathrm{Lag}(\mathbb{C}^4) \mid \dim(W \cap \overline{W}) = i\}.$$

*Then*

- $\mathcal{R}_0 = \mathcal{H}_{2,0} \bigcup \mathcal{H}_{1,1} \bigcup \mathcal{H}_{0,2}$ *where* $\mathcal{H}_{i,j} \cong X_{i,j} = \mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(i,j)$ *(open).*
- $\mathcal{R}_1$ *fibers over* $\mathbb{P}(\mathbb{R}^4)$ *with fiber isomorphic to* $X_{0,1} \bigcup X_{1,0}$.
- $\mathcal{R}_2 \cong \mathrm{Lag}(\mathbb{R}^4)$ *(closed).*

# Sketch of the proof

## Proof.

1. $\mathcal{R}_0$:
   - $\omega_{\mathbb{C}}$ defines a non-degenerate $\mathrm{Sp}(4,\mathbb{R})$–invariant Hermitian form $h$:

$$h(v,w) := i\omega_{\mathbb{C}}(\overline{v},w);$$

   - $\mathcal{R}_0 = \mathcal{H}_{2,0} \bigcup \mathcal{H}_{1,1} \bigcup \mathcal{H}_{0,2}$ w/

$$\mathcal{H}_{p,q} = \{W \in \mathcal{R}_0 \mid h|_{W\times W} \text{ has signature } (p,q)\}.$$

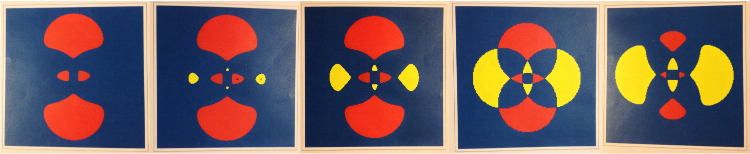
2. $\mathcal{R}_1$:
   - $\forall W \in \mathcal{R}_1$, then $Z = W \cap \overline{W}$ is the complexification of $Z' \in \mathbb{P}(\mathbb{R}^4)$ and this gives $p\colon \mathcal{R}_1 \longrightarrow \mathbb{P}(\mathbb{R}^4)$;
   - Let $M = Z^{\perp_{\omega_{\mathbb{C}}}}/Z$ is a 2–dim. sympl. space. Any $W \in p^{-1}(Z')$ is uniquely determined by $Y \in \mathrm{Lag}(M)$ s.t. $Y \cap \overline{Y} = \{0\}$.
3. $\mathcal{R}_2$: any $W \in \mathcal{R}_2$ is the complexification of $W' \in \mathrm{Lag}(\mathbb{R}^4)$.

□

# Ralationship with $\Omega_{\xi^1}$

## Question

*What is the relationship between F and the* $\mathrm{Sp}(4,\mathbb{R})$*–orbits?*



# Open questions

- What can we say about the d.o.d. $\Omega \subset \mathcal{F}(\mathbb{C}^4)$ for a Fuchsian representation $\rho\colon \pi_1(\Sigma) \longrightarrow \mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{Sp}(4,\mathbb{C})$?
- What is the connection with Dumas-Sanders' work? What can we say for $G = \mathrm{Sp}(2n,\mathbb{C})$ or other cases?
- What can we say about the quotient of the symmetric space?
- What can we say about limit of these representations? Can you combine punctured Fuchsian groups in order to understand 'geometrically finite' groups? Are there 'geometrically infinite' groups?
- Can we use other methods to find a fibration?

# End

# Cartan decomposition and contraction properties

Let $\mathfrak{a} = \{\mathrm{diag}(\lambda_1,\ldots,\lambda_n,-\lambda_n,\ldots,-\lambda_1) \mid \lambda_i \in \mathbb{R}\} \subset \mathfrak{sp}$.
Decompose $\mathrm{Sp}(2n,\mathbb{K}) = K\,\exp(\mathfrak{a})\,K$. [Problem: not unique!]

Given $\mathfrak{a}^+ = \{\mathrm{diag}(\lambda_1,\ldots,\lambda_n,-\lambda_n,\ldots,-\lambda_1 \mid \lambda_1 \geq \lambda_2 \ldots \lambda_n \geq 0\}$, then
$\mathrm{Sp}(2n,\mathbb{K}) = K\,\exp(\mathfrak{a}^+)\,K$ is unique:
$\forall g \in \mathrm{Sp}(2n,\mathbb{K}), \exists!\, a_g \in \mathfrak{a}^+$ s.t. $k_1 \exp(a_g) k_2$, where

## Definition

$\mu\colon\ \mathrm{Sp}(2n,\mathbb{K}) \longrightarrow \mathfrak{a}^+$ defined by $g \mapsto a_g$ is called the *Cartan projection* of $\mathrm{Sp}(2n,\mathbb{K})$.

Let $\alpha_i := \epsilon_i - \epsilon_{i+1} \in \mathfrak{a}^*$ and $\alpha_n := 2\epsilon_n \in \mathfrak{a}^*$, where
$\epsilon_i\left(\mathrm{diag}(\lambda_1,\ldots,\lambda_n,-\lambda_n,\ldots,-\lambda_1)\right) = \lambda_i$.
$\rho$ satisfies the contraction property if: $\forall$ diverging $\gamma_n \in \pi_1(\Sigma)$,
$\lim_{n\longrightarrow\infty} \alpha_i\left(\mu(\rho(\gamma_n))\right) = \infty$.