# A cocycle Perron-Frobenius theorem for random dynamical systems on Banach spaces

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# Outline

Motivation: Markov Chains

Generalized Perron-Frobenius Theorem (for Cocycles)

Paired Tent Maps Example

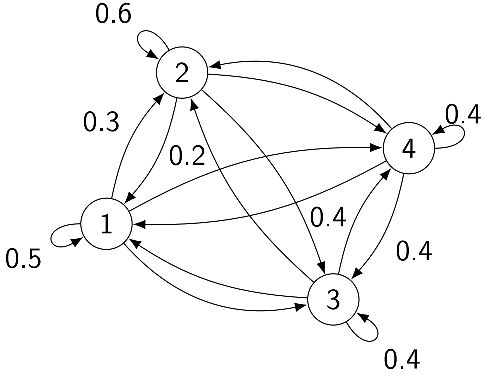
# Classical Perron-Frobenius Theorem

## Theorem (Perron 1908, Frobenius 1912)

*Let $P \in M_d(\mathbb{R}_{\geq 0})$ be such that there exists $n \geq 1$ with $(P^n)_{ij} > 0$ for all $i, j$ ($P$ is primitive). Then:*

1. *the spectral radius $\rho(P)$ of $P$ is a simple eigenvalue of $P$, no other eigenvalues of modulus $\rho(P)$;*
2. *the eigenvector $v$ corresponding to $\rho(P)$ is positive (that is, has all positive entries);*
3. *if $w$ is the left-eigenvector for $P$ corresponding to $\rho(P)$ (with $w \cdot v = 1$), then $\rho(P)^{-n} P^n x \underset{n \to \infty}{\longrightarrow} (w \cdot x) v$ for all $x \in \mathbb{R}^d$.*

# Markov Chain Example




$$P = \begin{bmatrix} 0.5 & 0.2 & 0.1 & 0.1 \\ 0.3 & 0.6 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{bmatrix}$$

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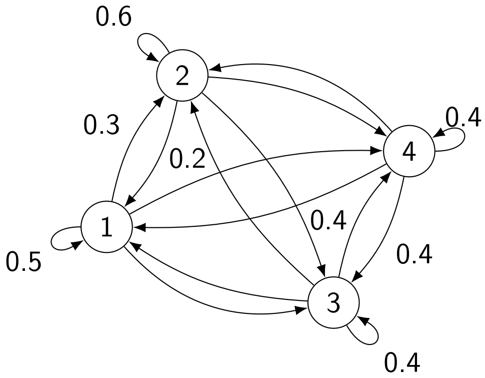


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$$\sigma(P) = \left\{ 1, \frac{3}{5}, \frac{3}{10}, 0 \right\}$$

$$v = \begin{bmatrix} 3/14 \\ 2/7 \\ 1/4 \\ 1/4 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

If $w \cdot x = 0$, then

$$\|P^n x\|_1 = O\left( \left( \frac{3}{5} \right)^n \right).$$

# Definitions

$(\Omega, \mu, \sigma)$ an invertible, ergodic probability-preserving transformation: "base dynamics".

*Cocycle:* $A_\omega \in M_d(\mathbb{R})$ or $\mathcal{B}(X)$, $A_\omega^{(n)} = A_{\sigma^{n-1}(\omega)} \cdots A_{\sigma(\omega)} A_\omega$.

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*Lyapunov exponents:* exponential growth rates for the cocycle.

$$\limsup_{n\to\infty} \frac{1}{n} \log \left\| A_\omega^{(n)} x \right\| = \lambda(x, \omega)$$

(Multiplicative Ergodic Theorem: actually discrete! Like eigenvalues.)

# Definitions

Cone: $\mathcal{C} \subset \mathbb{R}^d$ or $X$, closed, convex, $\mathcal{C} \cap -\mathcal{C} = \{0\}$.
Generating: $\mathcal{C} - \mathcal{C} = X$.
Partial order: $x \preceq y$ if and only if $y - x \in \mathcal{C}$.
$D$-adapted: $-y \preceq x \preceq y$ implies $\|x\| \leq D \|y\|$ (a.k.a. "normal").

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*Hilbert projective metric:* For a cone $\mathcal{C}$ and $v, w \in \mathcal{C}$, define:

$$\alpha(v, w) = \sup \{\lambda > 0 \ : \ \lambda v \preceq w\},$$
$$\beta(v, w) = \inf \{\mu > 0 \ : \ w \leq \mu v\},$$
$$\theta(v, w) = \log \left( \frac{\beta(v, w)}{\alpha(v, w)} \right).$$

# History

- M. Krein, M. Rutman, 1948: Compact linear operators preserving a cone
- Ga. Birkhoff, 1957: Operator on vector lattice preserving a cone
- I. Evstigneev, 1974: Cocycles of positive matrices
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- J. Buzzi, 1999: Cone technique for cocycle of dynamical P-F operators
- I. Evstigneev and S. Pirogov, 2009: Cocycles of non-linear positive operators on $\mathbb{R}^d$
- J. Mierczynski and W. Shen, 2013: Cocycles of positive linear operators

# Main Theorem

## Matrix Cocycle Version

Cocycle $A_\omega^{(n)} \in M_d(\mathbb{R}^d)$ over base dynamics $(\Omega, \mu, \sigma)$.
$\int_\Omega \log^+ \|A_\omega\|_{\mathrm{op}} \; d\mu(\omega) < \infty$. Cone $\mathcal{C} = \mathbb{R}^d_{\geq 0}$.

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Suppose that there is $k_P \in \mathbb{Z}_{\geq 1}$, $G_P \subset \Omega$ with $\mu(G_P) > 0$, and $D_P \in \mathbb{R}_{>0}$ such that for all $\omega \in G_P$, $\mathrm{diam}_\theta \left( A_\omega^{k_P}(\mathcal{C}) \right) \leq D_P$. Then:

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1. there is $v(\omega) \in \mathcal{C}$ with $\|v(\omega)\| = 1$ and $\phi(\omega) > 0$ such that $A_\omega v(\omega) = \phi(\omega) v(\sigma(\omega))$;
2. $\int_\Omega \log(\phi) \ d\mu = \lambda_1$ (the largest Lyapunov exponent for $A_\omega^{(n)}$) and the top Oseledets space is one-dimensional;
3. $\lambda_2 \leq \lambda_1 - \dfrac{\mu(G_P)}{k_P} \log \left( \tanh \left( \dfrac{D_P}{4} \right)^{-1} \right) < \lambda_1$.

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Moreover, $k_P$, $G_P$, and $D_P$ exist if and only if $n_P(\omega) := \inf \left\{ k \geq 1 \;:\; \mathrm{diam}_\theta \left( A_\omega^{(k)}(\mathcal{C}) \right) < \infty \right\}$ is finite on a set of positive measure.

# Main Theorem

## Cocycle of Linear Operators Version

Cocycle $A_\omega^{(n)} \in \mathcal{B}(X)$ over base dynamics $(\Omega, \mu, \sigma)$.
$\int_\Omega \log^+ \|A_\omega\|_{\mathrm{op}} \, d\mu(\omega) < \infty$, $A_\omega$ "nice". Cone $\mathcal{C} \subset X$, $D$-adapted.

Suppose that there is $k_P \in \mathbb{Z}_{\geq 1}$, $G_P \subset \Omega$ with $\mu(G_P) > 0$, and $D_P \in \mathbb{R}_{>0}$ such that for all $\omega \in G_P$, $\mathrm{diam}_\theta \left(A_\omega^{k_P}(\mathcal{C})\right) \leq D_P$. Then:

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# Definition and Background

A *paired tent map* is a map $T_{\epsilon_1,\epsilon_2} : [-1,1] \to [-1,1]$, with $\epsilon_1, \epsilon_2 \in [0,1]$, that looks like:

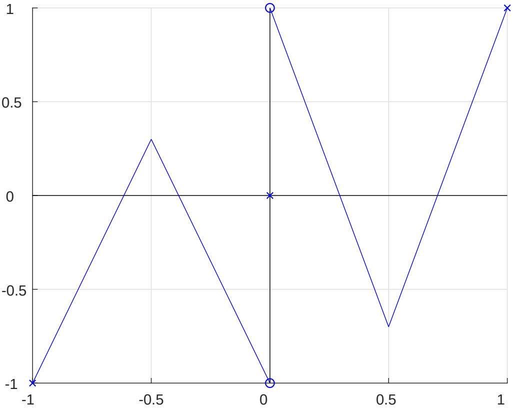


Figure: $T_{\epsilon_1,\epsilon_2}$, with parameters $\epsilon_1 = 0.3$ and $\epsilon_2 = 0.7$.

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C. Gonzalez Tokman, B. Hunt, and P. Wright (2011) studied invariant densities for $C^2$ perturbations of maps like $T_{0,0}$ that "leak" between $[-1, 0]$ and $[0, 1]$.

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What are the more in-depth spectral properties; what about cocycles?

# Application of Generalized P-F Theorem to Paired Tent Maps

## Theorem

*Base dynamics* $(\Omega, \mu, \sigma)$, $\epsilon_1, \epsilon_2 : \Omega \to [0, 1]$ *both not* 0*, countable range. Consider the cocycle of P-F operators* $P_\omega^{(n)}$ *associated to* $T_\omega := T_{\epsilon_1(\omega), \epsilon_2(\omega)}$.

*Then there is an explicitly computable* $C = C(\epsilon_1, \epsilon_2) > 0$ *with* $\lambda_2 \leq -C < 0 = \lambda_1$*, where* $\lambda_1$ *and* $\lambda_2$ *are the largest and second largest Lyapunov exponents for* $P_\omega^{(n)}$.

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### Proposition (New Lasota-Yorke Inequality)

*Inequality of the form* $\operatorname{Var}(P_\omega(f)) \leq a_1 \operatorname{Var}(f) + a_2 \|f\|_1$ *that can hold uniformly in* $\omega$.

# Asymptotic Properties of $\lambda_2(\kappa)$

Theorem

*Consider* $T_{\omega,\kappa} = T_{\kappa\epsilon_1(\omega),\kappa\epsilon_2(\omega)}$. *For* $\epsilon_1, \epsilon_2$ *both not* 0, *countable range, there is* $c > 0$ *such that* $\lambda_2(\kappa) \lesssim -c\kappa$ *for sufficiently small* $\kappa$.

# Asymptotic Properties of $\lambda_2(\kappa)$

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### Proposition

*There is a decreasing sequence $\{\kappa_j\} \subset (0,1]$ with $\kappa_j \underset{j\to\infty}{\longrightarrow} 0$ such that the maps $T_{\kappa_j,\kappa_j}$ are Markov. Set $P_j = P_{T_{\kappa_j,\kappa_j}}$; the cocycles of P-F operators $P_\omega^{(n)} = P_j^n$ have $\lambda_2(j) \sim -2\kappa_j$.*

# The End

Thank you!