Asymptotic properties of weighted recursive and preferential attachment trees

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Delphin Sénizergues

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University of British Columbia

Introduction

- Start with $(w_n)_{n\geq 1}$ a sequence of weights.

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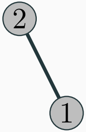

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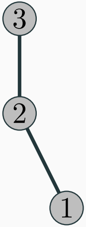

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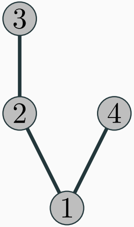

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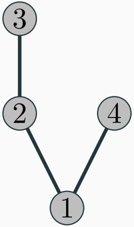

$$\forall k \in \{1, \dots n\}, \quad \mathbb{P}\left((n+1) \to (k) \;\middle|\; \mathsf{T}_n \right) = \frac{w_k}{\sum_{i=1}^{n} w_i}.$$

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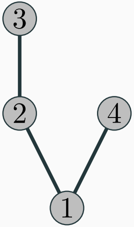

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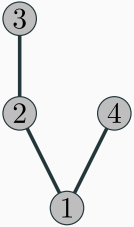

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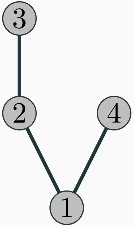

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- We can also use *random* sequences of weights.

Goals and motivations

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- One motivation is the connection with another model of growing trees: the preferential attachment trees.
- Also, connection to the monkey walk (Mailler-Uribe 2018+).

Preferential attachment trees with additive fitnesses

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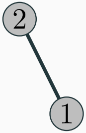

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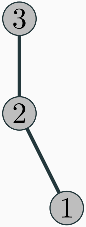

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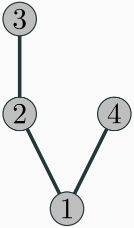

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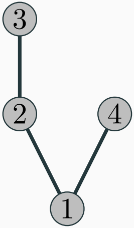

$$\forall k \in \{1,\dots n\}, \quad \mathbb{P}\left(\textcircled{n+1} \to \textcircled{k} \;\middle|\; \mathsf{P}_n \right) = \frac{a_k + \deg_n^+(\textcircled{k})}{\sum_{i=1}^n a_i + \sum_{i=1}^n \deg_n^+(\textcircled{i})}.$$

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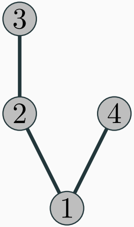

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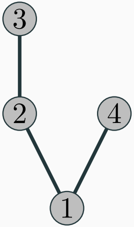

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The representation theorem

Theorem (S. 2019+)
Preferential attachment trees are weighted recursive trees i.e. for any sequence of fitnesses $\mathsf{a} = (a_n)_{n\geq 1}$*, there exists a random sequence* $(\mathsf{w}_n^{\mathsf{a}})_{n\geq 1}$ *such that the distributions* $\mathsf{PA}((a_n)_{n\geq 1})$ *and* $\mathsf{WRT}((\mathsf{w}_n^{\mathsf{a}})_{n\geq 1})$ *coincide.*

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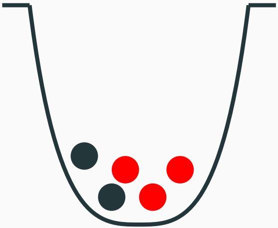
- Results for weighted recursive trees automatically apply to preferential attachment trees!

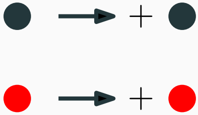
Definition of the random sequence $(\mathrm{w}^{\mathrm{a}}_n)_{n\geq 1}$

For any sequence $\mathrm{a} = (a_n)_{n\geq 1}$ of fitnesses, the random sequence $(\mathrm{w}^{\mathrm{a}}_n)_{n\geq 1}$ is defined as

$$\mathrm{w}^{\mathrm{a}}_1 = \mathrm{W}^{\mathrm{a}}_1 = 1 \qquad \text{and} \qquad \forall n \geq 2, \quad \mathrm{W}^{\mathrm{a}}_n = \prod_{k=1}^{n-1} \beta_k^{-1},$$

where the $(\beta_k)_{k\geq 1}$ are independent with respective distribution $\mathrm{Beta}(A_k + k, a_{k+1})$, with $A_k = \sum_{i=1}^{k} a_i$.





- At time 0, the urn contains a black balls and b red balls.
- At each time $n \geq 1$, a ball is drawn from the urn and returned to the urn together with a new ball of the same colour.
- For all $n \geq 1$, we let $X_n := \mathbb{1}_{\{\text{the ball drawn at time } n \text{ is black}\}}$.

Theorem
Almost surely

$$\frac{\#\{\text{black balls at time } n\}}{\#\{\text{balls at time } n\}} = \frac{a + \sum_{i=1}^{n} X_i}{a+b+n} \underset{n\to\infty}{\longrightarrow} \beta,$$

where β *is a random variable with distribution* $\mathrm{Beta}(a, b)$.

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where β *is a random variable with distribution* $\mathrm{Beta}(a, b)$*.*
Furthermore, conditionally on β*, the sequence* $(X_n)_{n\geq 1}$ *is a sequence of i.i.d. Bernoulli random variables with parameter* β*.*

Convergence results for weighted recursive trees

Proposition (S. 2019+)
If $W_n \sim C \cdot n^{\gamma}$ *as* $n \to \infty$ *for* $0 < \gamma < 1$, *then we almost surely have*

$$n^{-(1-\gamma)} \cdot \left(\deg_n^+(\text{①}), \deg_n^+(\text{②}), \dots\right) \underset{n\to\infty}{\to} \frac{1}{C(1-\gamma)} \cdot (w_1, w_2, \dots),$$

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- Under some additional condition on $(w_n)_{n\geq 1}$, the last convergence can be proved to hold in ℓ^p for $p > \frac{1}{1-\gamma}$.

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If $W_n \sim C \cdot n^{\gamma}$ as $n \to \infty$ for $0 < \gamma < 1$, then we almost surely have

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- Under some additional condition on $(w_n)_{n\geq 1}$, the last convergence can be proved to hold in ℓ^p for $p > \frac{1}{1-\gamma}$.
- Thanks to the representation theorem, this also holds for preferential attachment trees.

For any $k \geq 1$ we can write:

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$$\deg_n^+(\text{(k)}) \underset{n\to\infty}{\sim} \sum_{i=k+1}^{n} \mathbb{P}\left(\text{(i)} \to \text{(k)}\right) \underset{n\to\infty}{\sim} \sum_{i=k+1}^{n} \frac{w_k}{W_{i-1}} \underset{n\to\infty}{\sim} w_k \cdot \sum_{i=k+1}^{n} \frac{1}{C \cdot i^{\gamma}}$$

$$\underset{n\to\infty}{\sim} w_k \cdot \frac{n^{1-\gamma}}{C(1-\gamma)}.$$

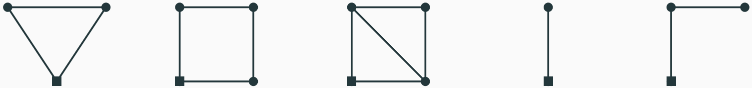
If $(\mathrm{P}_n)_{n\geq 1}$ is a sequence of trees with distribution $\mathrm{PA}((a_n)_{n\geq 1})$ with $A_n \simeq c \cdot n$ as $n \to \infty$ then we have the almost sure convergence in some ℓ^p space,

$$n^{-\frac{1}{c+1}} \cdot (\deg_n^+(①), \deg_n^+(②), \dots) \underset{n\to\infty}{\to} (\mathrm{m}_1, \mathrm{m}_2, \dots),$$

where $(\mathrm{m}_n)_{n\geq 1}$ is a constant times the sequence $(\mathrm{w}_n^{\mathrm{a}})_{n\geq 1}$.

Scaling limits for generalisation of Rémy's algorithm

A generalised version of Rémy's algorithm



G_1 G_2 G_3 G_4 G_5

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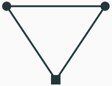
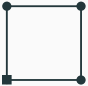
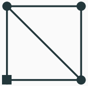


G_1



G_2



G_3

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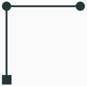

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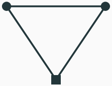

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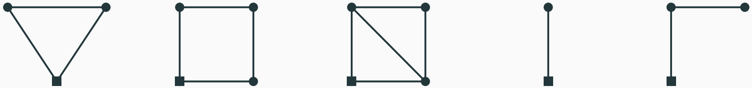

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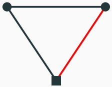

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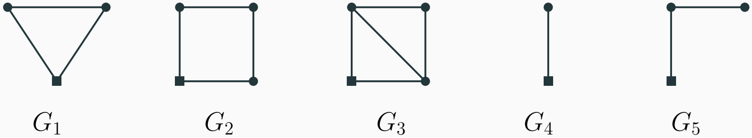

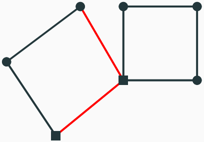

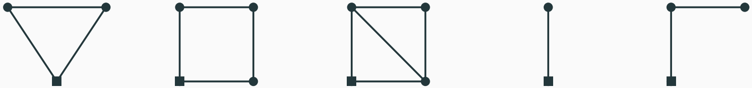

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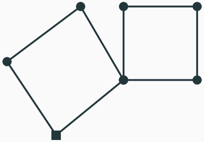

H_2

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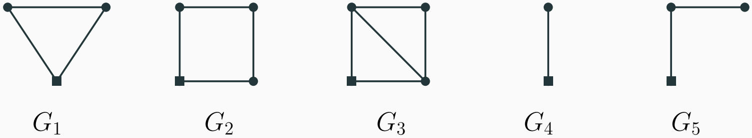

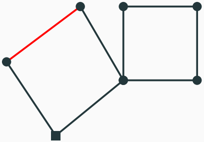

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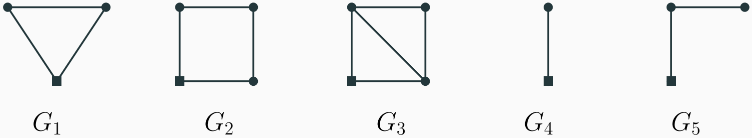
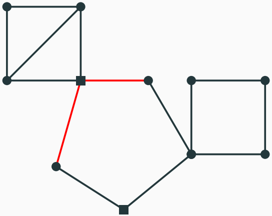
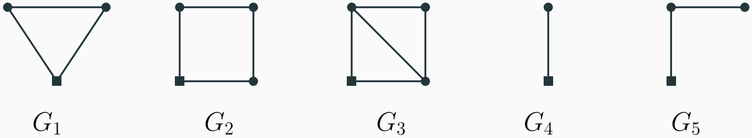
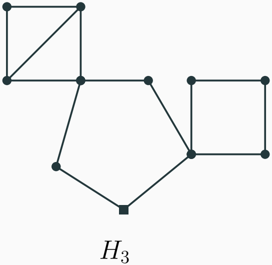
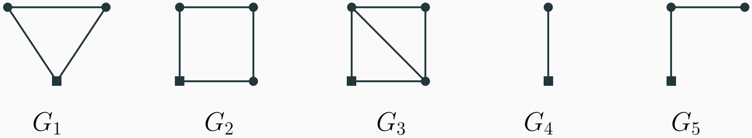





G_1
G_2
G_3
G_4
G_5


H_3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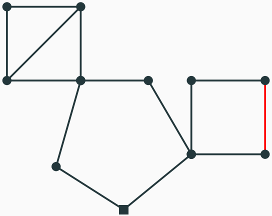

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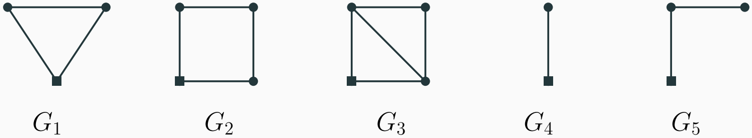

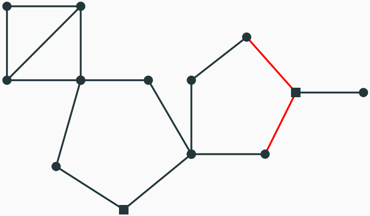

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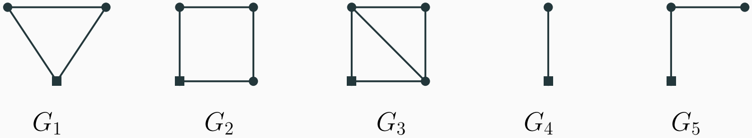

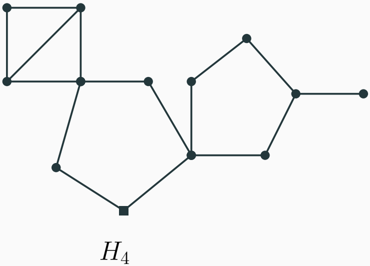

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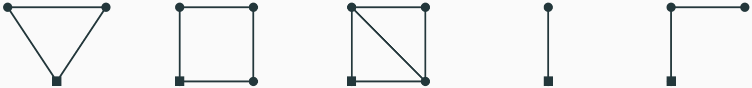
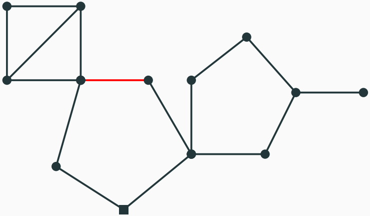


G_1 G_2 G_3 G_4 G_5



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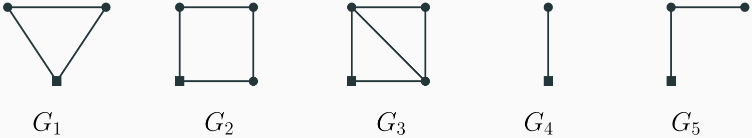

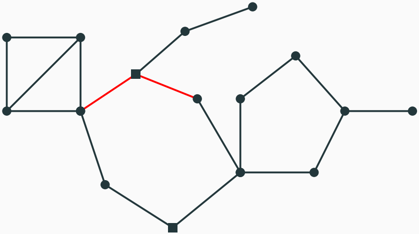

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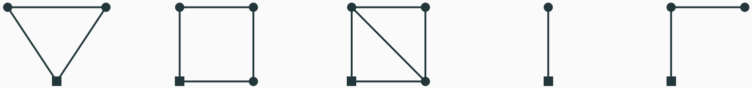
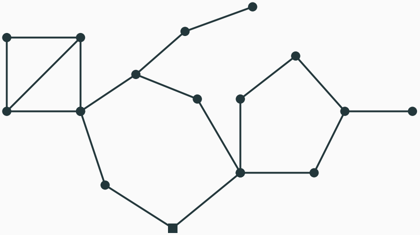


G_1 G_2 G_3 G_4 G_5



H_5

Questions about the model

- In the original algorithm, the diameter of H_n grows as $n^{1/2}$, and we have an almost sure convergence in a metric space sense

$$(H_n, n^{-1/2} \cdot \mathrm{d}_{gr}) \underset{n\to\infty}{\to} \mathcal{T},$$

where $\mathcal{T}$ is Aldous' CRT.

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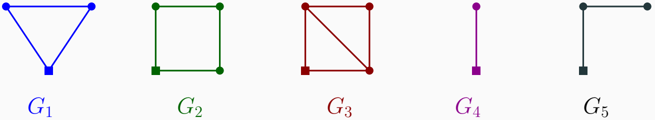
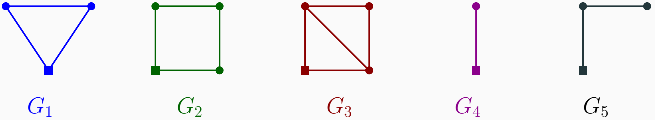
- Do we get a scaling limit?
- If yes, what does it look like?


G_1
G_2
G_3
G_4
G_5


H_1
T_1

A generalised version of Rémy's algorithm



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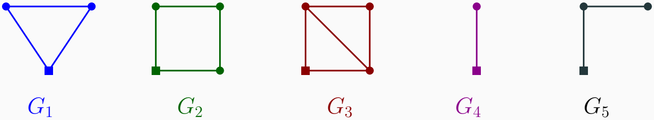

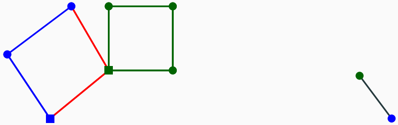

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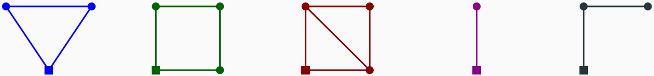
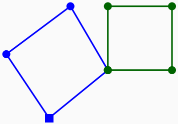
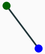


G_1 G_2 G_3 G_4 G_5

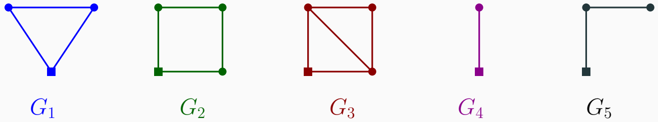


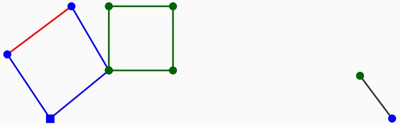
H_2



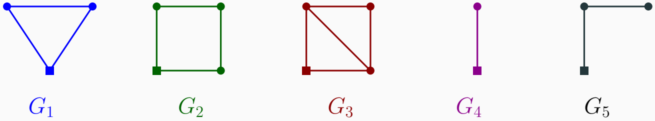
T_2

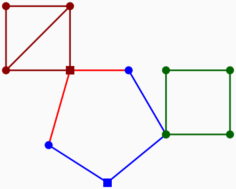
A generalised version of Rémy's algorithm

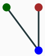


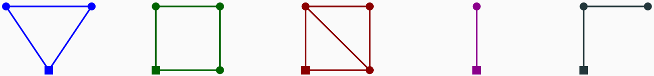


A generalised version of Rémy's algorithm

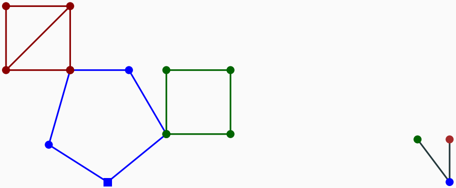






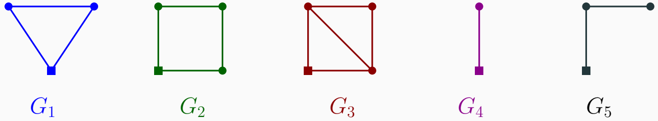


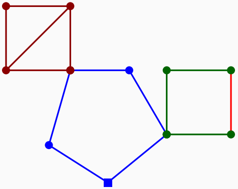
G_1 G_2 G_3 G_4 G_5

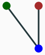


H_3 T_3

A generalised version of Rémy's algorithm







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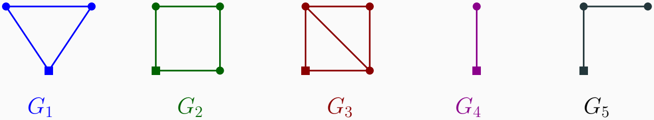

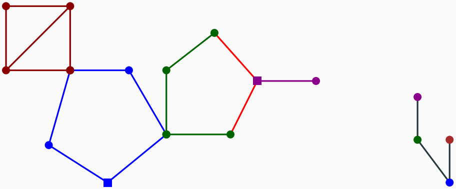

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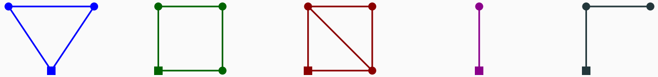
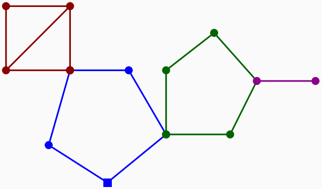


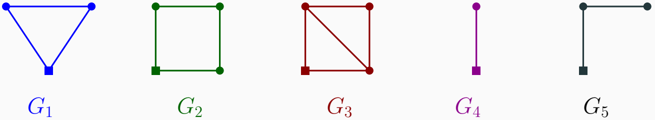
G_1 G_2 G_3 G_4 G_5

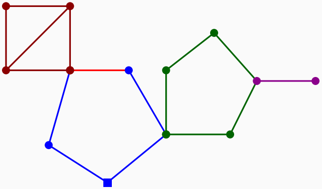


H_4

T_4

A generalised version of Rémy's algorithm





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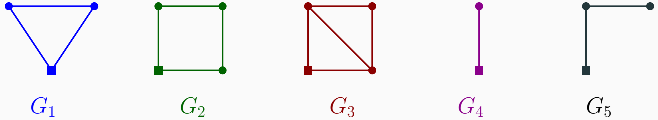

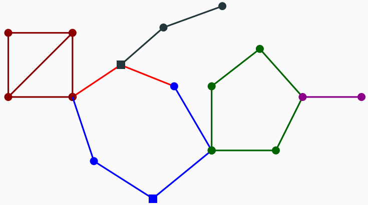

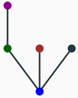

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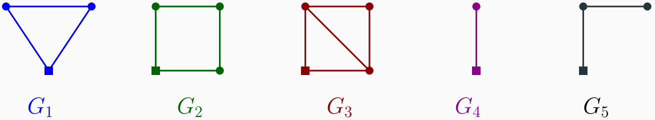
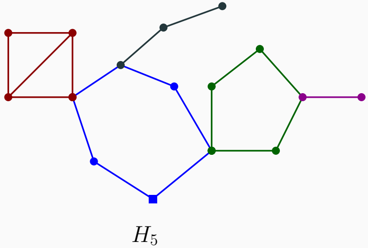
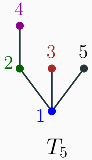
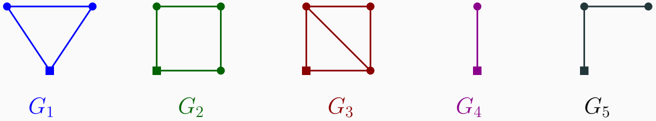


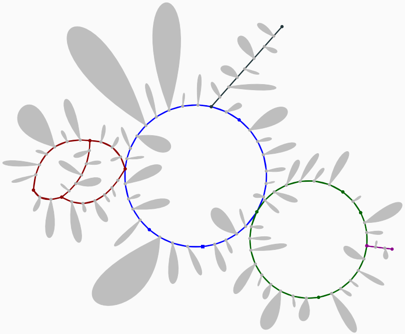






large n

distances $\times n^{-1/(c+1)}$



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Theorem (S. 2019+)
Suppose that $\sum_{i=1}^{n} a_i = c \cdot n + O\left(n^{1-\epsilon}\right)$ *for some constant* $c > 0$ *and* $a_n \leq n^{\frac{1}{c+1}-\epsilon+o(1)}$. *Then we have the following convergence*

$$\left(H_n, n^{-\frac{1}{c+1}} \cdot \mathrm{d_{gr}}, \mu_{n,\mathrm{unif}}\right) \underset{n\to\infty}{\longrightarrow} (\mathcal{H}, \mathsf{d}, \mu),$$

almost surely in Gromov-Hausdorff-Prokhorov topology.

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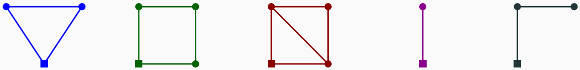
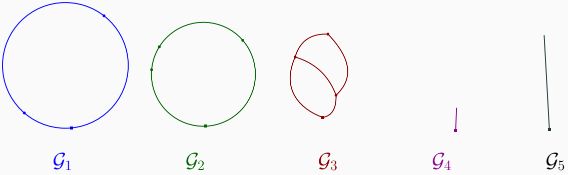
- Controlling the degrees in the tree $\to$ controlling distances in H_n.
- Description of the tree as a WRT $\to$ iterative gluing construction.

Iterative gluing construction



G_1 G_2 G_3 G_4 G_5

Iterative gluing construction



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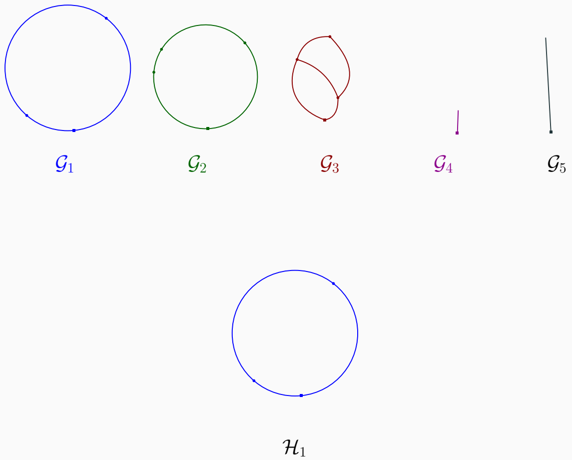

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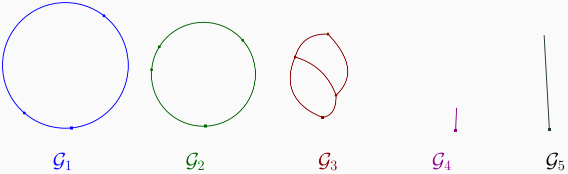

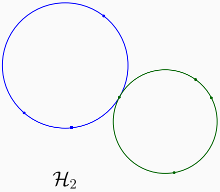

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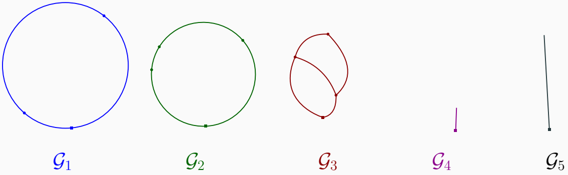

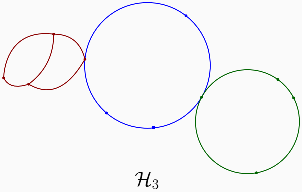

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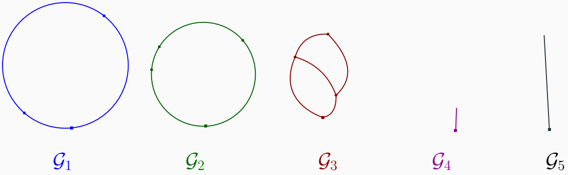

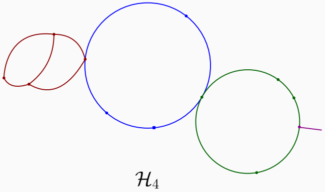

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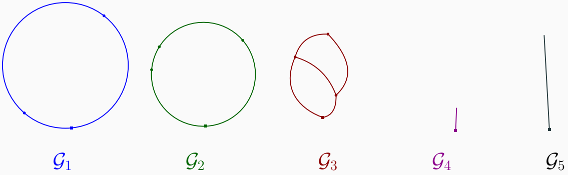
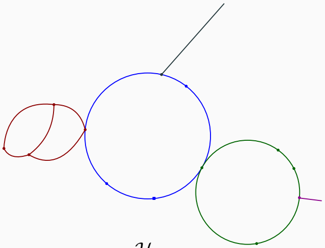
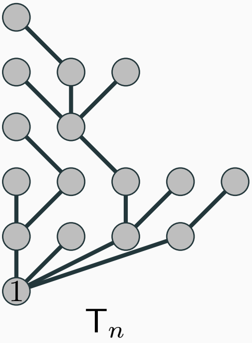
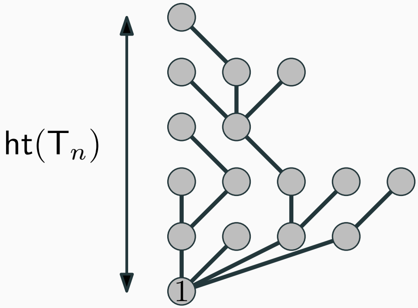


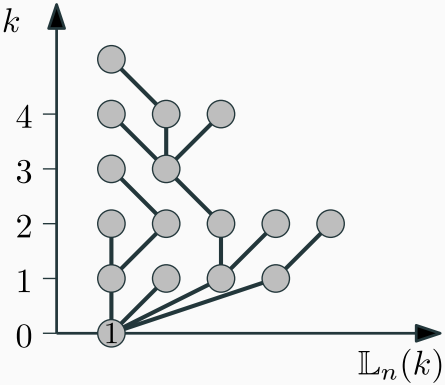


$\mathcal{H}_5$

Thank you for your attention!

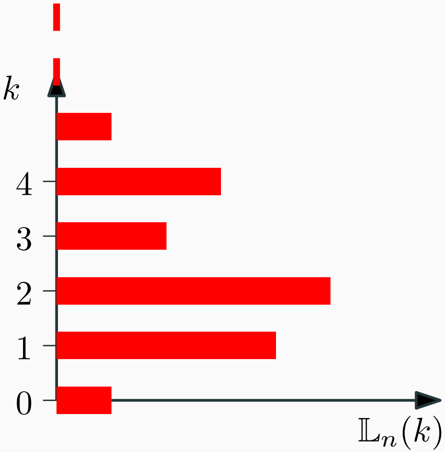

1
T_n


ht(T_n)
1



The profile of the tree T_n is the function $\mathbb{L}_n : \mathbb{N} \to \mathbb{N}$ defined as

$$\forall k \geq 0, \quad \mathbb{L}_n(k) = \#\{\text{vertices at height } k \text{ in the tree } \mathsf{T}_n\}.$$



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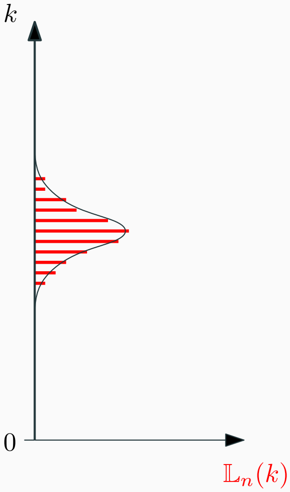
The profile is almost surely asymptotically Gaussian



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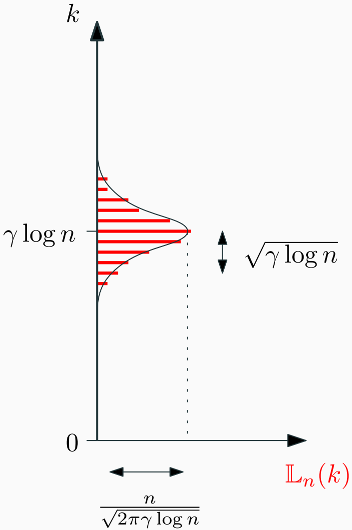
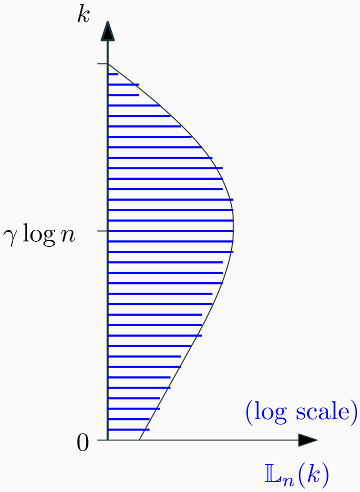


The log of the profile converges to a function



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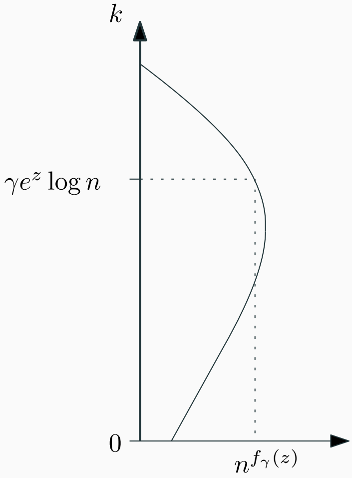

- $f_\gamma(z) := 1 + \gamma(e^z - 1 - ze^z)$.

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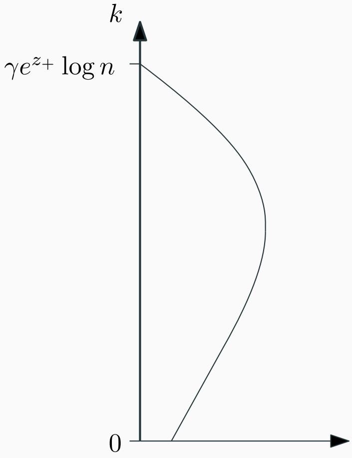

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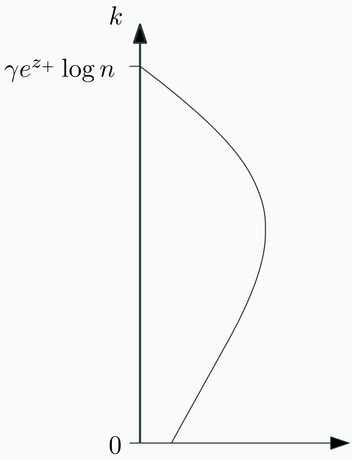
- z_+ is the unique positive zero of f_γ.



- $f_\gamma(z) := 1 + \gamma(e^z - 1 - ze^z)$.
- z_+ is the unique positive zero of f_γ.
- Almost surely $\mathrm{ht}(\mathrm{T}_n) \underset{n\to\infty}{\sim} \gamma \cdot e^{z_+} \log n$.

Theorem (S. 2019+)
Under the assumption $W_n \simeq \mathrm{cst} \cdot n^\gamma$ (+ additional condition) we almost surely have

- $\dfrac{\mathrm{ht}(\mathrm{T}_n)}{\log n} \underset{n\to\infty}{\longrightarrow} \gamma \cdot e^{z_+}.$
- $\mathbb{L}_n(k) \underset{n\to\infty}{=} \dfrac{n}{\sqrt{2\pi\gamma \log n}} \exp\left\{-\dfrac{1}{2} \cdot \left(\dfrac{k-\gamma\log n}{\sqrt{\gamma \log n}}\right)^2\right\} + O\left(\dfrac{n}{\log n}\right),$
- *for all* $z \in (z_-, z_+)$, $\quad \mathbb{L}_n\left(\lfloor \gamma e^z \log n \rfloor\right) = n^{f_\gamma(z)+o(1)}.$