Value-Distribution of Cubic Hecke L-Functions

Alia Hamieh
(Joint work with Amir Akbary)

University of Northern British Columbia

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Some Distribution Theorems for the Riemann Zeta Function

Bohr-Jessen Theorem

Theorem (1932)

Let E be a fixed rectangle in the complex plane whose sides are parallel to the real and imaginary axes, and let $\sigma > \frac{1}{2}$ be a fixed real number. Then the limit

$$\lim_{T\to\infty} \frac{1}{2T} \mathrm{meas}\left(\{-T \le t \le T;\ \log\zeta(\sigma+it) \in E\}\right)$$

exists.

Selberg Theorem

Theorem (1949, unpublished)

For $E \subset \mathbb{C}$, *we have*

$$\lim_{T\to\infty} \frac{1}{2T}\text{meas}\left(\left\{-T \le t \le T;\ \frac{\log\zeta(1/2+it)}{\sqrt{\frac{1}{2}\log\log T}} \in E\right\}\right) = \frac{1}{2\pi}\iint_E e^{-\frac{1}{2}(x^2+y^2)}dxdy.$$

Distribution Theorems for Dirichlet and Hecke L-functions

The Case of the Fundamental Discriminant

If d is a fundamental discriminant, we set

$$L_d(s) = L(s, (d/.)) = \sum_{n=1}^{\infty} \frac{\left(\frac{d}{n}\right)}{n^s},$$

where $\left(\frac{d}{n}\right)$ is the Kronecker symbol.

Chowla-Erdos Theorem

Theorem (1951)

If $\sigma > 3/4$, we have

$$\lim_{x\to\infty} \frac{\#\{0 < d \leq x;\ d \equiv 0,1 \pmod 4 \text{ and } L_d(\sigma) \leq z\}}{x/2} = G(z)$$

exists. Furthermore $G(0) = 0$, $G(\infty) = 1$, and $G(z)$, the distribution function, is a continuous and strictly increasing function of z.

Elliott Theorem

Theorem (1970)

There is a distribution function $F(z)$ such that

$$\frac{\#\{0 < -d \leq x;\ d \equiv 0, 1 \pmod 4 \text{ and } L_d(1) < e^z\}}{x/2} = F(z) + O\left(\sqrt{\frac{\log\log x}{\log x}}\right)$$

holds uniformly for all real z, and real $x \geq 9$. $F(z)$ has a probability density, may be differentiated any number of times, and has the characteristic function

$$\phi_F(y) = \prod_p \left(\frac{1}{p} + \frac{1}{2}\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p}\right)^{-iy} + \frac{1}{2}\left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p}\right)^{-iy}\right)$$

which belongs to the Lebesgue class $L(-\infty, \infty)$.

Granville-Soundararajan Theorem

In 2003, Granville and Soundararajan investigated the distribution of values of $L_d(1)$ as d varies over all fundamental discriminants with $|d| \leq x$. They followed the approach of probabilistic random models.

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A weaker version of their results implies that the proportion of fundamental discriminants d with $|d| \leq x$ such that $L_d(1) \geq e^{\gamma}\tau$ decays doubly exponentially in $\tau = \log\log x$ (i.e. is between $\exp(-B\frac{e^{\tau}}{\tau})$ and $\exp(-A\frac{e^{\tau}}{\tau})$ for some absolute constants $0 < A < B$) and similarly for the low extreme values (i.e. $L_d(1) \leq \frac{\zeta(2)}{e^{\gamma}\tau}$).

Random Euler products

The idea of Elliott and then Granville-Soundararajan is to compare the distribution of the values $L_d(1)$ with the distribution of $L(1,X) = \prod_p \left(1 - X(p)p^{-1}\right)^{-1}$ where the $X(p)$'s are independent random variables given by:

$$X(p) = \begin{cases} 0 & \text{with probability } 1/(p+1); \\ 1 & \text{with probability } p/2(p+1); \\ -1 & \text{with probability } p/2(p+1). \end{cases}$$

Then

$$E\left[(L(1,X;x)^z\right] = \prod_{p \le x} E\left[\left(1 - X(p)p^{-1}\right)^{-z}\right]$$

$$= \prod_{p \le x} \left(\frac{1}{p+1} + \frac{1}{2}\left(1 - \frac{1}{p+1}\right)\left(1 - \frac{1}{p}\right)^{-z} + \frac{1}{2}\left(1 - \frac{1}{p+1}\right)\left(1 + \frac{1}{p}\right)^{-z} \right).$$

Ihara-Matsumoto's Work

Let k be $\mathbb{Q}$ or an imaginary quadratic field, and let $\mathfrak{f} \subset \mathfrak{O}_k$ be an ideal.
Consider characters χ of $H_\mathfrak{f} = I_\mathfrak{f}/P_\mathfrak{f}$.
Consider $\mathcal{L}(s, \chi)$ where $\mathcal{L}$ is either $\frac{L'}{L}(s, \chi)$ or $\log L(s, \chi)$.

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Theorem (2011)

Let $\sigma := \Re(s) \geq 1/2 + \epsilon$ be fixed, and let $|dw| = (dxdy)/2\pi$. Assume the GRH. Then there exists a density function $\mathcal{M}_\sigma(w)$ such that

$$\lim_{\substack{\mathrm{N}(\mathfrak{f})\to\infty \\ \mathfrak{f} \text{ prime}}} \frac{1}{|\widehat{H}'_{\mathfrak{f}}|} \#\{\chi \in \widehat{H}'_{\mathfrak{f}} :\ \mathcal{L}(s,\chi_{\mathfrak{f}}) \in S\} = \int_S \mathcal{M}_\sigma(w)\,|dw|,$$

if $S \subset \mathbb{C}$ is either compact or complement of a compact set.

Ihara-Matsumoto $\mathcal{M}$-Function

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$$\tilde{\mathcal{M}}_\sigma(z) = \sum_{\mathfrak{a} \subset \mathfrak{O}_k} \lambda_z(\mathfrak{a}) \lambda_{\overline{z}}(\mathfrak{a}) \mathrm{N}(\mathfrak{a})^{-2\sigma}.$$

Mourtada-Murty Theorem

Theorem (2015)

Let $\sigma \geq 1/2 + \epsilon$, and assume GRH. Let $\mathcal{F}(Y)$ denote the set of the fundamental discriminants in the interval $[-Y, Y]$ and let $N(Y) = \#\mathcal{F}(Y)$. Then, there exists a probability density function M_σ, such that

$$\lim_{Y\to\infty} \frac{1}{N(Y)} \#\{d \in \mathcal{F}(Y);\ \left(L'_d/L_d\right)(\sigma) \leq z\} = \int_{-\infty}^{z} M_\sigma(t)dt.$$

Moreover, the characteristic function $\varphi_{F_\sigma}(y)$ of the asymptotic distribution function $F_\sigma(z) = \int_{-\infty}^{z} M_\sigma(t)dt$ is given by

$$\varphi_{F_\sigma}(y) = \prod_p \left(\frac{1}{p+1} + \frac{p}{2(p+1)} \exp\left(-\frac{iy\log p}{p^\sigma - 1}\right) + \frac{p}{2(p+1)} \exp\left(\frac{iy\log p}{p^\sigma + 1}\right)\right).$$

A Value-Distribution Result for Cubic Hecke L-functions

The Setup

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- The $\langle c \rangle$-ray class group of k is denoted by $H(\langle c \rangle) = I(\langle c \rangle)/P(\langle c \rangle)$.
- For $\chi_c \in \hat{H}_{\langle c \rangle}$ and $\Re(s) > 1$, let

$$L(s, \chi_c) = \sum_{\mathfrak{a}} \frac{\chi_c(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s},$$

  be the Hecke L-function associated with the Hecke character χ_c. For a non-trivial χ_c, this L-function has an analytic continuation to the whole complex plane.

The family $\mathcal{C}$

Consider the set

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Useful Estimate:

$$N^*(Y) = \sum_{c \in \mathcal{C}} \exp\left(-\frac{\mathrm{N}(c)}{Y}\right) \sim \frac{3\mathrm{res}_{s=1}\zeta_k(s)}{4|H_{\langle 9 \rangle}|\zeta_k(2)} Y.$$

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One can show that, for $c \in \mathcal{C}$, the cubic residue symbol $\chi_c(.) = (\frac{\cdot}{c})_3$ is an ideal class (Hecke) character.

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- For $\Re(s) > 1$, we have

$$L_c(s) = \frac{3^{2s}}{(3^s - 1)^2} \sum_{\substack{a,b \\ a \equiv 1 \pmod{\langle 3 \rangle} \\ b \equiv 1 \pmod{\langle 3 \rangle}}} \frac{\left(\frac{c}{a}\right)_3 \left(\frac{\bar{c}}{b}\right)_3}{\mathrm{N}(ab)^s}.$$

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- We set

$$\mathcal{L}_c(s) = \begin{cases} \log L_c(s) & \text{(Case 1)}, \\ (L_c'/L_c)(s) & \text{(Case 2)}. \end{cases}$$

The function $L_c(s)$

Another look at $L_c(s)$:
The function $L_c(s)$ is the quotient of the Dedekind zeta functions associated with the extension $k(c^{1/3})/k$. In other words,

$$L_c(s) = \frac{\zeta_{k(c^{1/3})}(s)}{\zeta_k(s)}.$$

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The classical $L_d(s)$:
The quotient of the Dedekind zeta functions associated with the extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ is

$$L_d(s) = \frac{\zeta_{\mathbb{Q}(\sqrt{d})}(s)}{\zeta(s)}.$$

The Problem

Let z be a real number, and let $\sigma > \frac{1}{2}$ be fixed. Does $\mathcal{L}_c(\sigma)$ possess an asymptotic distribution function F_σ? We are interested in studying the asymptotic behaviour as $Y \to \infty$ of

$$\frac{1}{\mathcal{N}(Y)} \#\{c \in \mathcal{C} : \mathrm{N}(c) \leq Y \ \text{ and } \ \mathcal{L}_c(\sigma) \leq z\}.$$

The Main Result

Theorem (Akabry - H.)

Let $\sigma \geq 1/2 + \epsilon$. Let $\mathcal{N}(Y)$ be the the number of elements $c \in \mathcal{C}$ with norm not exceeding Y. There exists a smooth density function M_σ such that

$$\lim_{Y \to \infty} \frac{1}{\mathcal{N}(Y)} \# \{c \in \mathcal{C} : \mathrm{N}(c) \leq Y \text{ and } \mathcal{L}_c(\sigma) \leq z\} = \int_{-\infty}^{z} M_\sigma(t)\, dt.$$

The asymptotic distribution function $F_\sigma(z) = \int_{-\infty}^{z} M_\sigma(t)\, dt$ can be constructed as an infinite convolution over prime ideals $\mathfrak{p}$ of k,

$$F_\sigma(z) = *_{\mathfrak{p}}\, F_{\sigma,\mathfrak{p}}(z),$$

The Main Result

Theorem (Continuation)

Moreover, the density function M_σ can be constructed as the inverse Fourier transform of the characteristic function $\varphi_{F_\sigma}(y)$, which in (Case 1) is given by

$$\varphi_{F_\sigma}(y) = \exp\left(-2iy\log(1-3^{-\sigma})\right) \prod_{\mathfrak{p}\nmid\langle 3\rangle}\left(\frac{1}{\mathrm{N}(\mathfrak{p})+1} + \frac{1}{3}\frac{\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})+1}\sum_{j=0}^{2}\exp\left(-2iy\log\left|1-\frac{\zeta_3^j}{\mathrm{N}(\mathfrak{p})^\sigma}\right|\right)\right)$$

and in (Case 2) is given by

$$\varphi_{F_\sigma}(y) = \exp\left(-2iy\frac{\log 3}{3^\sigma-1}\right) \prod_{\mathfrak{p}\nmid\langle 3\rangle}\left(\frac{1}{\mathrm{N}(\mathfrak{p})+1} + \frac{1}{3}\frac{\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})+1}\sum_{j=0}^{2}\exp\left(-2iy\Re\left(\frac{\zeta_3^j\log\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^\sigma-\zeta_3^j}\right)\right)\right).$$

The case $\sigma = 1$

Theorem (Brauer-Siegel)

If K ranges over a sequence of number fields Galois over $\mathbb{Q}$, degree N_K and absolute value of discriminant $|D_K|$, such that $N_K/\log|D_K|$ tends to 0, then we have

$$\log(h_K R_K) \sim \log|D_K|^{1/2},$$

where h_K is the class number of K, and R_K is the regulator of K.

The case $\sigma = 1$

- By the class number formula we know that

$$L_c(1) = \frac{(2\pi)^2\sqrt{3}h_c R_c}{\sqrt{|D_c|}},$$

  where h_c, R_c, and $D_c = (-3)^5(\mathrm{N}(c))^2$ are respectively the class number, the regulator, and the discriminant of the cubic extension $K_c = k(c^{1/3})$.

- By the Brauer-Siegel theorem

$$\log(h_c R_c) \sim \log|D_c|^{1/2},$$

  as $\mathrm{N}(c) \to \infty$.

The case $\sigma = 1$

Corollary

Let $E(c) = \log(h_c R_c) - \log|D_c|^{1/2}$. *Then*

$$\lim_{Y\to\infty} \frac{1}{\mathcal{N}(Y)} \#\{c \in \mathcal{C} : \mathrm{N}(c) \le Y \ \textit{ and } \ E(c) \le z\} = \int_{-\infty}^{\bar{z}} M_1(t)\, dt,$$

where $\bar{z} = z + \log(4\sqrt{3}\pi^2)$ *and* $M_1(t)$ *is the smooth function described in the main result (Case 1) for* $\sigma = 1$.

The case $\sigma = 1$

- The Euler-Kronecker constant of a number field K is defined by
$$\gamma_K = \lim_{s \to 1} \left(\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s-1} \right).$$
- We have
$$\frac{L_c'(1)}{L_c(1)} = \gamma_{K_c} - \gamma_k.$$

The case $\sigma = 1$

Corollary

There exists a smooth function $M_1(t)$ (as described in the main result (Case 2) for $\sigma = 1$) such that

$$\lim_{Y\to\infty} \frac{1}{\mathcal{N}(Y)} \# \{c \in \mathcal{C} : \mathrm{N}(c) \leq Y \ \text{ and } \ \gamma_{K_c} \leq z\} = \int_{-\infty}^{\bar{z}} M_1(t)\, dt,$$

where $\bar{z} = z - \gamma_k$.

The Key Lemma

Lemma

Let f be a real arithmetic function. Suppose that

$$\lim_{N\to\infty} \frac{\displaystyle\sum_{n=1}^{\infty} e^{iyf(n)} e^{-n/N}}{\displaystyle\sum_{n=1}^{\infty} e^{-n/N}} = \widetilde{M}(y),$$

which is continuous at 0. Then f possesses a distribution function F. In this case, $\widetilde{M}$ is the characteristic function of F.

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which is continuous at 0. Then f possesses a distribution function F. In this case, $\widetilde{M}$ is the characteristic function of F. Moreover, if

$$\left|\widetilde{M}(y)\right| \leq \exp\left(-\eta\,|y|^{\gamma}\right),$$

for some $\eta, \gamma > 0$, then $F(z) = \int_{-\infty}^{z} M(t)dt$ for a smooth function M, where

$$M(z) = (1/2\pi) \int_{\mathbb{R}} \exp\left(-izy\right) \widetilde{M}(y) dy.$$

The Steps of The Proof

- **Step One:** Establishing $\lim_{Y\to\infty} \frac{1}{\mathcal{N}^*(Y)} \sum_{c\in\mathcal{C}}^{\star} \exp\left(iy\mathcal{L}_c(\sigma)\right) \exp(-\mathrm{N}(c)/Y) = \widetilde{M}_\sigma(y)$,
  - The method is based on the previous works of Luo and Ihara-Matsumoto.
  - A version of Polya-Vinogradov inequality and a version of large sieve inequality both due to Heath-Brown are two main ingredients.
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- **Step Two:** Finding a product formula for $\widetilde{M}_\sigma(y)$.
- **Step Three:** Proving that $\widetilde{M}_\sigma(y)$ has exponential decay following a method devised by Wintner and employed by Mourtada and Murty.

Step 1

Proposition

Fix $\sigma \geq 1/2 + \epsilon$ *and* $y \in \mathbb{R}$. *Then*

$$\lim_{Y \to \infty} \frac{1}{\mathcal{N}^*(Y)} \sum_{c \in \mathcal{C}}^{\star} \exp\left(iy\mathcal{L}_c(\sigma)\right) \exp(-\mathrm{N}(c)/Y) = \widetilde{M}_\sigma(y),$$

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where $\star$ *means that the sum is over* c *such that* $L_c(\sigma) \neq 0$. *The function* $\widetilde{M}_\sigma(y)$ *is given by*

$$\sum_{r_1, r_2 \geq 0} \frac{\lambda_y(\langle 1 - \zeta_3 \rangle^{r_1}) \lambda_y(\langle 1 - \zeta_3 \rangle^{r_2})}{3^{(r_1 + r_2)\sigma}}$$

$$\times \sum_{\substack{\gcd(\mathfrak{a}\mathfrak{b}\mathfrak{m}, \langle 3 \rangle) = 1 \\ \gcd(\mathfrak{a}, \mathfrak{b}) = 1}} \frac{\lambda_y(\mathfrak{a}^3\mathfrak{m}) \lambda_y(\mathfrak{b}^3\mathfrak{m})}{\mathrm{N}(\mathfrak{a}^3\mathfrak{b}^3\mathfrak{m}^2)^\sigma \prod_{\substack{\mathfrak{p} | \mathfrak{a}\mathfrak{b}\mathfrak{m} \\ \mathfrak{p} \text{ prime}}} \left(1 + \mathrm{N}(\mathfrak{p})^{-1}\right)}.$$

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$$\exp\left(iy \frac{L'}{L}(s, \chi_c)\right) = \prod_{\mathfrak{p}} \exp\left(-\frac{iy \log(\mathrm{N}(\mathfrak{p}))\, \chi_c(\mathfrak{p}) \mathrm{N}(\mathfrak{p})^{-s}}{1 - \chi_c(\mathfrak{p}) \mathrm{N}(\mathfrak{p})^{-s}}\right)$$

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$$\text{with } \; H_0(u) = 1 \; \text{ and } \; H_r(u) = \frac{1}{r!}\prod_{n=1}^{r}(u+n-1)\,.$$

In (Case 2),

$$\lambda_y(\mathfrak{p}^{\alpha_\mathfrak{p}}) = G_{\alpha_\mathfrak{p}}\left(-\frac{iy}{2}\log \mathrm{N}(\mathfrak{p})\right),$$

$$\text{with } \; G_0(u) = 1 \; \text{ and } \; G_r(u) = \sum_{n=1}^{r}\frac{1}{n!}\binom{r-1}{n-1}u^n.$$

The Arithmetic Function λ_y

We define λ_y to be an arithmetic multiplicative function on $\mathfrak{O}_k$:
In (Case 1),

$$\lambda_y(\mathfrak{p}^{\alpha_\mathfrak{p}}) = H_{\alpha_\mathfrak{p}}\left(\frac{iy}{2}\right),$$

$$\text{with } \; H_0(u) = 1 \; \text{ and } \; H_r(u) = \frac{1}{r!}\prod_{n=1}^{r}(u+n-1)\,.$$

In (Case 2),

$$\lambda_y(\mathfrak{p}^{\alpha_\mathfrak{p}}) = G_{\alpha_\mathfrak{p}}\left(-\frac{iy}{2}\log \mathrm{N}(\mathfrak{p})\right),$$

$$\text{with } \; G_0(u) = 1 \; \text{ and } \; G_r(u) = \sum_{n=1}^{r}\frac{1}{n!}\binom{r-1}{n-1}u^n.$$

Moreover, for any $\epsilon > 0$ and all $|y| \leq R$, we have

$$\lambda_y(\mathfrak{a}) \ll_{\epsilon,R} \mathrm{N}(\mathfrak{a})^{\epsilon}$$

More on Step 1

- For $\sigma > 1$ we have

$$\exp\left(iy\mathcal{L}_c(\sigma)\right) = \sum_{\mathfrak{a},\mathfrak{b}\subset\mathfrak{O}_k} \frac{\lambda_y(\mathfrak{a})\lambda_y(\mathfrak{b})\chi_c(\mathfrak{a}\mathfrak{b}^2)}{\mathrm{N}(\mathfrak{a}\mathfrak{b})^\sigma}.$$

More on Step 1

- For $\sigma > 1$ we have

$$\exp\left(iy\mathcal{L}_c(\sigma)\right) = \sum_{\mathfrak{a},\mathfrak{b}\subset\mathfrak{O}_k} \frac{\lambda_y(\mathfrak{a})\lambda_y(\mathfrak{b})\chi_c(\mathfrak{a}\mathfrak{b}^2)}{\mathrm{N}(\mathfrak{a}\mathfrak{b})^\sigma}.$$

- For $\frac{1}{2} < \sigma \leq 1$ and $c \in \mathcal{Z}^{\mathrm{c}}$ we have

$$\exp\left(iy\mathcal{L}_c(\sigma)\right) = \sum_{\mathfrak{a},\mathfrak{b}\subset\mathfrak{O}_k} \frac{\lambda_y(\mathfrak{a})\lambda_y(\mathfrak{b})\chi_c(\mathfrak{a}\mathfrak{b}^2)}{\mathrm{N}(\mathfrak{a})^\sigma\mathrm{N}(\mathfrak{b})^\sigma} \exp\left(-\frac{\mathrm{N}(\mathfrak{a}\mathfrak{b})}{X}\right) - \frac{1}{2\pi i}\int_L \exp\left(iy\mathcal{L}_c(\sigma+u)\right)\Gamma(u)X^u du,$$

for an appropriate contour L.

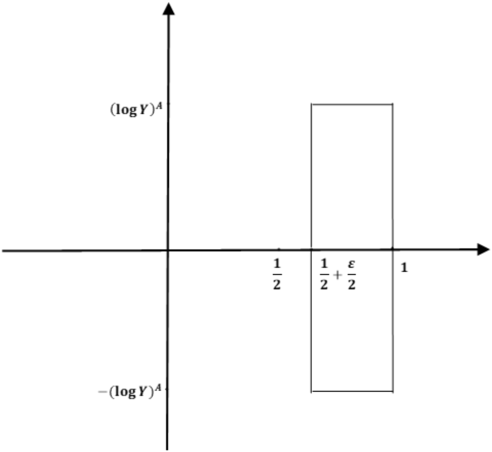

Figure: The rectangle $R_{Y,\epsilon,A}$

An element $c \in \mathcal{Z}^c$, if $L(s, \chi_c) \neq 0$ in $R_{Y,\epsilon,A}$. Otherwise, $c \in \mathcal{Z}$.

We prove that

$$\begin{aligned}
&\sum_{c\in\mathcal{C}}^{\star} \exp\left(iy\mathcal{L}_c(\sigma)\right)\exp(-\mathrm{N}(c)/Y)\\
&= \sum_{c\in\mathcal{Z}^{\mathrm{c}}} \exp\left(iy\mathcal{L}_c(\sigma)\right)\exp(-\mathrm{N}(c)/Y) + O(Y^{\delta})\\
&= (I) - (II) + (III) + O(Y^{\delta}),
\end{aligned}$$

with

$$(I) = \sum_{c\in\mathcal{C}} \left(\sum_{\mathfrak{a},\mathfrak{b}\subset\mathfrak{O}_k} \frac{\lambda_y(\mathfrak{a})\lambda_y(\mathfrak{b})\chi_c(\mathfrak{a}\mathfrak{b}^2)}{\mathrm{N}(\mathfrak{a})^{\sigma}\mathrm{N}(\mathfrak{b})^{\sigma}} \exp\left(-\frac{\mathrm{N}(\mathfrak{a}\mathfrak{b})}{X}\right)\right) \exp\left(-\frac{\mathrm{N}(c)}{Y}\right).$$

Step 2

Proposition

In (Case 1) we have

$$\widetilde{M}_\sigma(y) = \exp\left(-2iy\log(1-3^{-\sigma})\right)$$
$$\times \prod_{\mathfrak{p}\nmid\langle 3\rangle}\left(\frac{1}{\mathrm{N}(\mathfrak{p})+1}+\frac{1}{3}\left(\frac{\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})+1}\right)\sum_{j=0}^{2}\exp\left(-2iy\log\left|1-\frac{\zeta_3^j}{\mathrm{N}(\mathfrak{p})^\sigma}\right|\right)\right).$$

In (Case 2) we have

$$\widetilde{M}_\sigma(y) = \exp\left(-2iy\frac{\log 3}{3^\sigma-1}\right)$$
$$\times \prod_{\mathfrak{p}\nmid\langle 3\rangle}\left(\frac{1}{\mathrm{N}(\mathfrak{p})+1}+\frac{1}{3}\left(\frac{\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})+1}\right)\sum_{j=0}^{2}\exp\left(-2iy\Re\left(\frac{\zeta_3^j\log\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^\sigma-\zeta_3^j}\right)\right)\right).$$

Step 3

Proposition

Let $\delta > 0$ be given, and fix $\sigma > \frac{1}{2}$. For sufficiently large values of y, we have

$$\left|\widetilde{M}_\sigma(y)\right| \le \exp\left(-C|y|^{\frac{1}{\sigma}-\delta}\right),$$

where C is a positive constant that depends only on σ and δ.

More on Step 3

Recall in (Case 2)

$$\widetilde{M}_\sigma(y) = \exp\left(-2iy\frac{\log 3}{3^\sigma - 1}\right)\prod_{\mathfrak{p}\nmid\langle 3\rangle}\widetilde{M}_{\sigma,\mathfrak{p}}(y),$$

where

$$\widetilde{M}_{\sigma,\mathfrak{p}}(y) = \frac{1}{\mathrm{N}(\mathfrak{p})+1} + \frac{1}{3}\left(\frac{\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})+1}\right)\sum_{j=0}^{2}\exp\left(-2iy\log\mathrm{N}(\mathfrak{p})\Re\left(\frac{\zeta_3^j}{\mathrm{N}(\mathfrak{p})^\sigma - \zeta_3^j}\right)\right)$$

We prove that

$$\left|\widetilde{M}_{\sigma,\mathfrak{p}}(y)\right| \le \frac{1}{\mathrm{N}(\mathfrak{p})+1} + 0.3256\left(\frac{\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})+1}\right) \le 0.8256.$$

for all $\mathfrak{p}$ with

$$\frac{2y\log 2y}{2.36\sigma} \le \mathrm{N}(\mathfrak{p})^\sigma \le \frac{2y\log 2y}{1.8\sigma}$$

The number of prime ideals satisfying this inequality is

$$\Pi_\sigma(y) \gg_\sigma y^{\frac{1}{\sigma}}.$$

The number of prime ideals satisfying this inequality is

$$\Pi_\sigma(y) \gg_\sigma y^{\frac{1}{\sigma}}.$$

It follows that

$$\left|\widetilde{M}_\sigma(y)\right| = \prod_{\mathfrak{p}} \left|\widetilde{M}_{\sigma,\mathfrak{p}}(y)\right| \leq 0.8256^{\Pi_\sigma(y)} \leq \exp\left(-Cy^{\frac{1}{\sigma}}\right),$$

where C is a positive constant depending only on σ.

Thank you for listening!