On non-topological solutions for planar Liouville Systems of Toda-type

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$$\frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^{2N_i} e^{u_i^{(R)}} dx = \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^{2N_i} e^{u_i} dx. \tag{4}$$

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- In the case of $m = 1$, and $N = 0$ problem (2) reduces to:

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If $a_{ij} > 0$, $\det A \neq 0$ and $N_i = 0$ then (12) are necessary and sufficient condition for radial solvability of (5).

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In the settings of previous theorem a radial solution to (5) is unique (up to scaling (3)).

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If $a_{ij} > 0$ and $N_i > -1$ then (12) are necessary and sufficient condition for radial solvability of (5). Moreover, such a solution is unique (up to scaling (3)).

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Here A can be degenerate. The particular case when $det\, A \neq 0$ was treated independently by C.S.Lin and Zhang.

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Theorem (P-Tarantello)

Assume that v is a radial solution of (13).

$$\text{if } b > \frac{1}{N+1} \text{ then } \max\left\{\frac{4}{b}, 4(N+1) - \frac{4}{b}\right\} < \alpha < 4(N+1),$$

$$\text{if } 0 < b < \frac{1}{N+1} \text{ then } \max\left\{4(N+1), \frac{4}{b} - 4(N+1)\right\} < \alpha < \frac{4}{b}.$$

Moreover, in the later cases there exist the unique radial solution to (13).

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and positive definiteness reeds as:

$$a_{11} > 0, \quad a_{22} > 0, \quad \text{and} \quad a_{12}^2 < a_{11}a_{22}. \tag{16}$$

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$$u_1(x) = \psi(x) - \ln(a_{11}) \quad \text{and} \quad u_2(x) = \varphi(x) - \ln(a_{22}) \qquad (17)$$

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where

$$\tau_1 := -\frac{a_{12}}{a_{22}}, \quad \tau_2 := -\frac{a_{12}}{a_{11}} \quad \text{and} \quad \beta_1 = \frac{\beta}{a_{11}}, \quad \beta_2 = \frac{\alpha}{a_{22}}. \tag{19}$$

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$$\begin{cases} -\Delta u_1 = |x|^{2N_1} e^{u_1} - \tau_1 |x|^{2N_2} e^{u_2} & \text{in } \mathbb{R}^2 \\ -\Delta u_2 = |x|^{2N_2} e^{u_2} - \tau_2 |x|^{2N_1} e^{u_1} & \text{in } \mathbb{R}^2 \\ \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^{2N_1} e^{u_1} dx = \beta_1, \\ \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^{2N_2} e^{u_2} dx = \beta_2, \end{cases} \tag{18}$$

where

$$\tau_1 := -\frac{a_{12}}{a_{22}}, \quad \tau_2 := -\frac{a_{12}}{a_{11}} \quad \text{and} \quad \beta_1 = \frac{\beta}{a_{11}}, \quad \beta_2 = \frac{\alpha}{a_{22}}. \tag{19}$$

Moreover, in the case $a_{12} \neq 0$ (16) reeds as:

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Moreover, (6) and (7) together reed as:

$$\begin{cases} \beta_2 < \frac{2}{1-\tau_1\tau_2}\Big((N_2+1) + \tau_2(N_1+1) \\ +\sqrt{(N_2+1)^2 + 2\tau_2(N_2+1)(N_1+1) + \frac{\tau_2}{\tau_1}(N_1+1)^2}\Big), \\ \\ \beta_2 > \frac{2}{1-\tau_1\tau_2}\Big((N_2+1) + \tau_2(N_1+1) \\ +\left(\sqrt{\tau_1\tau_2}\right)\sqrt{(N_2+1)^2 + 2\tau_2(N_2+1)(N_1+1) + \frac{\tau_2}{\tau_1}(N_1+1)^2}\Big), \\ \\ \beta_1 = \left(2(N_1+1) + \tau_1\beta_2\right) \\ +\sqrt{\left(2(N_1+1) + \tau_1\beta_2\right)^2 - \frac{\tau_1}{\tau_2}\beta_2\left(\beta_2 - 4(N_2+1)\right)}. \end{cases} \quad (22)$$

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Moreover, similarly as it was done in the case $a_{12} > 0$, in the case $a_{12} < 0$ we also can find that the following condition

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is also one of the necessary conditions of radial solvability of problem (18).

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and if $\tau_1 = \frac{1}{2}$, $\tau_2 = 1$ then necessarily $\beta_1 = 8(N_1 + 1) + 4(N_2 + 1)$ and $\beta_2 = 8(N_1 + 1) + 8(N_2 + 1)$.

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• If $\tau = 1/2$ then it is well known Toda system, the radial solution exists if and only if $\beta_1 = \beta_2 = 4(N_1 + N_2 + 2)$ and they are completely classified (C.S.Lin-Wei-Ye).

Theorem (P-Tarantello)

For every $\tau \in (0,1) \setminus \{1/2\}$ the necessary and sufficient conditions on (β_1, β_2) for the existence of a radial solution to (24) are the following:

$$\begin{cases} \frac{1}{2}\beta_1^2 - 2(N_1+1)\beta_1 + \frac{1}{2}\beta_2^2 - 2(N_2+1)\beta_2 - \tau\beta_1\beta_2 = 0, \\ \underline{\beta}_1(\tau) < \beta_1 < \overline{\beta}_1(\tau) \\ \underline{\beta}_2(\tau) < \beta_2 < \overline{\beta}_2(\tau). \end{cases} \tag{25}$$

where $\underline{\beta}_1(\tau), \overline{\beta}_1(\tau), \underline{\beta}_2(\tau), \overline{\beta}_2(\tau)$ are given by some formulas.

Definitions of $\underline{\beta}_1(\tau), \overline{\beta}_1(\tau), \underline{\beta}_2(\tau), \overline{\beta}_2(\tau)$

- There exists unique $\delta_1 \in (0, 1/2)$ such that

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$$\underline{\beta}_1(\tau) = \begin{cases} 4(N_1+1) \quad \forall \tau \in (0,\sigma_1) \\ 2\tau\big(4(N_2+1)+8\tau(N_1+1)\big) \quad \forall \tau \in [\sigma_1, 1/2) \\ \big(4(N_1+1)+8\tau(N_2+1)\big) \quad \forall \tau \in [1/2, \delta_2) \\ \\ \frac{2\left((N_1+1)+\tau(N_2+1)+\tau\sqrt{(N_1+1)^2+(N_2+1)^2+2\tau(N_1+1)(N_2+1)}\right)}{1-\tau^2} \\ \ \forall \tau \geq \delta_2 \end{cases} \tag{30}$$

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$$\tilde{\beta}_1(\theta) := \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^{2N_1} e^{v_1^{(\theta)}} dx, \quad \tilde{\beta}_2(\theta) := \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^{2N_2} e^{v_2^{(\theta)}} dx.$$

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Then

$$\beta_1\neq 4(N_1+1)+8\tau_1(N_2+1) \quad \textit{and} \quad \beta_2\neq 4(N_2+1)+8\tau_2(N_1+1).$$

Thank You!