WOA: Women in Operator Algebras

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1 Overview

This report summarizes the organization and scientific progress made at the first workshop for women in operator algebras at the Banff International Research Station in Banff, Canada. The main purposes of the workshop were

1. for women in operator algebras to conduct cutting-edge collaborative research, and
2. to build a network of women working in the field to drive future research collaborations and to alleviate the commonly reported feeling of isolation.

We think that we have achieved both of these aims by using the format from the BIRS workshops "Women in Numbers" where the focus is on research in groups on current problems.

Our workshop had 8 groups, each led by 1 or 2 researchers. Prior to the workshop, the leaders designed a broad outline of a research project, and provided background reading and references for their group. On the first day of the workshop each group briefly outlined the research project they would be working on, and on the last day each group reported on progress made and future directions for the work. See Section 3 below for a report from each group. Several of the groups are planning to meet again this year or are preparing articles for publication.

The workshop had 37 participants from 15 countries (Australia, Brazil, Canada, China, France, Germany, India, Israel, Italy, Korea, Denmark, Netherlands, New Zealand, USA, UK). Unfortunately, 5 woman had to cancel at the very last moment due to visas not being issued, caring responsibilities or illness.

To avoid inviting only researchers we already knew well, we initially invited 28 participants, and then filled the remaining places via an application process. The participants initially invited included the project leaders and researchers whose expertise would enable progress on the projects. The call for applications was very broad via social media; email lists for women in mathematics, women in operator algebras and related fields; e-mail lists for major conferences of interest to operator algebraists. We requested a CV, ranked preferences of 2–3 projects of interest, and a short statement about how their expertise fit with the projects and how they would benefit from participating.

2 Outcomes of the workshop

The immediate outcomes of the workshop include several new collaborations, a new network of women in operator algebras supported by a website[1] hosted by the Association for Women in Mathematics (AWM) and articles in preparation. Several groups reported that they are planning to meet again this year and one group reported that they are preparing a journal article for publication (see the reports from the groups in Section 3 below). The feedback from the participants via an AWM survey and testimonials was overwhelmingly positive (see Section 4).

3 Scientific progress made: reports from the research groups

The study of operator algebras is a very active branch of functional analysis dealing with problems that are intrinsically infinite-dimensional. Such problems arise in quantum mechanics where, for example, a famous theorem of von Neumann says that the only solutions of Heisenberg's commutation relations are families of operators on an infinite-dimensional Hilbert space. Operator algebras have a rich and remarkably rigid structure, and there is a powerful general theory which makes this precise and applicable. Over the past few decades, operator algebras have influenced diverse areas of mathematics, including number theory, harmonic analysis, knot theory, dynamical systems and ergodic theory.

During the workshop, our groups worked on current research problems in the field. Each group's work is reported below.

3.1 Cartan subalgebras of twisted groupoid C^*-algebras

Group members

Anna Duwenig (University of Victoria), Elizabeth Gillaspy (group co-leader, University of Montana), Rachael Norton (Northwestern University), Sarah Reznikoff (group co-leader, Kansas State University), Sarah Wright (Fitchburg State University).

Research synopsis and progress

Let A be a C^*-algebra and let $B \subseteq A$ be a Cartan subalgebra. That is, B is a maximal abelian sub-C^*-algebra of A that contains an approximate unit for A, such that there exists a faithful conditional expectation from A onto B, and the normalizer of B generates A as a C^*-algebra. In 2008, building on earlier work of Kumjian [27], Renault proved [46] that every Cartan pair (A, B) is of the form $(C_r^*(G; \Sigma), C_0(G^{(0)}))$, where $C_r^*(G; \Sigma)$ denotes the reduced C^*-algebra of a twisted, topologically principal, Hausdorff étale groupoid $(G; \Sigma)$, and $G^{(0)}$ denotes the unit space of G. (A groupoid G is a generalization of a group, in which every element has an inverse but multiplication is not globally defined. Thus, each $g \in G$ has two associated units, namely, its source $s(g)$ and its range $r(g)$. Two elements $g, h \in G$ can be multiplied if and only if the source of g equals the range of h. When the only elements in G with $s(g) = r(g)$ are the units themselves, we say G is *principal*; we say G is *topologically principal* if G comes equipped with a topology such that the set $\{u \in G^{(0)} : s(g) = r(g) = u \Rightarrow g = u\}$ is dense in $G^{(0)}$.)

However, even if G is not topologically principal, the twisted groupoid C^*-algebra $C_r^*(G; \Sigma)$ may have Cartan subalgebras. This is the case for the rotation algebras A_θ, for example, which can be realized as a twisted group C^*-algebra, $A_\theta \cong C_r^*(\mathbb{Z}^2; c_\theta)$. For any group G, we have $G^{(0)} = \{e\}$, so if G is nontrivial then G cannot be topologically principal. Despite this, other descriptions of the rotation algebras enable one to check that $C_r^*(\mathbb{Z}) \cong C(\mathbb{T})$ is a Cartan subalgebra in A_θ.

The goal of this project is to identify Cartan subalgebras of twisted groupoid C^*-algebras when the groupoid in question is not topologically principal. In the untwisted case, a similar question was investigated by Brown et al. in [13]. Inspired by the Cartan pair $(A_\theta, C(\mathbb{T}))$, we have identified sufficient conditions on a subgroupoid S of a twisted groupoid $(G; \Sigma)$ that ensure $C_r^*(S; \Sigma|_S)$ is a Cartan subalgebra of $C^*(G; \Sigma)$. When restricted to the case of a trivial twist, our theorem recovers [13, Corollary 4.4].

[1] https://awmadvance.org/research-networks/woa/

Benefits of the WOA format

The ten-minute presentations given on the first day were extremely beneficial to our group; we recruited a new group member on the strength of that presentation.

Another of our group members was able to join the project thanks to the pre-conference application process. Obtaining her group assignment three months in advance of the workshop gave this team member enough time to familiarize herself with groupoids and Cartan subalgebras, which were new topics for her. This consequently enabled her to contribute to the research during the week at BIRS.

Future plans

Although we proved our main theorem during the week at BIRS, there are still many open questions which we plan to explore, such as the uniqueness of the subgroupoid S and/or the Cartan subalgebra that it generates. Several of these questions are also still open in the untwisted case. We are planning to meet at Northwestern University for a week in May 2019 to continue working on this project, and we will also apply to MSRI's Summer Research Program for Women in the hopes of spending two weeks in July 2019 working on this project at Berkeley.

3.2 Index theory and K-theory with applications to arithmetic groups

Group members

Sara Azzali (University of Potsdam), Sarah Browne (The Pennsylvania State University), Maria Paula Gomez Aparicio (group co-leader, Université Paris-Sud 11), Lauren Ruth (Vanderbilt University), Hang Wang (group co-leader, East China Normal University).

Format

Since we had formed a group before the BIRS workshop, on arrival we all introduced ourselves and discussed our connections to the potential research avenues on connections between K-theory, the Baum–Connes conjecture and representation theory. From these discussions we decided on a format which involved us all giving talks connected to the topic (we list the titles and abstracts below). On the first day we presented our plans to the others at the conference. This generated a lot of interest. In particular, Jacqui Ramagge gave us a guest talk on connected joint work by herself, G. Robertson and T. Steger.

Sara Azzali, Discrete groups, counterexamples to the Baum–Connes conjecture, and a localised assembly map

We describe the counterexamples to the Baum–Connes conjecture with coefficients, given by Higson, Lafforgue and Skandalis in [22]. We follow Puschnigg's Bourbaki seminar [42] and focus on the case of discrete groups Γ. The counterexamples are based on the different behaviour of the left and right hand sides of the Baum–Connes map with respect to exact sequences of Γ-algebras: the left hand side is exact in the middle, whereas the right hand side can fail exactness. The discrete group Γ in the counterexample is the so called Gromov "Monster group", in whose Cayley graph one can suitably embed an expander graph.

In the second part of the talk, we report on joint work with P. Antonini and G. Skandalis where we use KK-theory with coefficients in $\mathbb{R}$ to study discrete group actions on C^*-algebras and to give a localised form of the Baum–Connes conjecture [3]. We start with the definition of (equivariant) KK-theory with coefficients in $\mathbb{R}$, which is constructed by means of an inductive limit over II_1-factors [2]. We see how the standard group trace defines a class $[\tau] \in KK^{\Gamma}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ which is shown to be an idempotent. The image of $[\tau]$ acting on $KK^{\Gamma}_{\mathbb{R}}(A, B)$ by exterior product is called the "τ-part" of $KK^{\Gamma}_{\mathbb{R}}(A, B)$. A Baum–Connes type morphism μ_τ is naturally defined between the τ-parts of the usual left and right hand sides of the Baum–Connes map (where on the right hand side the τ acts via descent). One can show that the τ-form of the Baum–Connes conjecture is weaker[2] than the classical one, but the injectivity of μ_τ still implies the strong Novikov conjecture.

[2](more precisely, it is verified by a Γ-algebra A if the classical Baum–Connes is verified by $A \otimes N$ for every II_1-factor N with trivial Γ-action)

Sarah Browne, E-theory and the Baum-Connes conjecture

We introduced the notion of E-theory and gave properties and understood the relations with both K-theory and KK-theory. After we gave a definition of equivariant E-theory in the papers of Guentner-Higson-Trout [20] and Guentner-Higson [23] and talked about the technicality of composition of elements. There after we described the Baum-Connes conjecture in terms of E-theory and in particular with coefficients to relate to crossed product C^*-algebras. The talk finished with a connection to the speakers' current joint work which may be useful for future avenues.

Maria Paula Gomez Aparicio, Property (T) as an obstruction to prove the Baum-Connes conjecture

The Baum-Connes conjecture gives a way of computing the K-theory of the reduced C*-algebra of a locally compact group. This C*-algebra encodes the topology of the temperate dual of the group and its K-theory is a topological invariant of this topological space. This conjecture, and some generalizations, are still open for some groups having property (T) (e.g $SL_3(\mathbb{Z})$), a rigidity property on groups representations. In this workshop I explained how property (T) appears as an element in the maximal C*-algebra of the group and how it prevents the use of the Dirac-dual Dirac method to succeed. I also explained why a stronger version of property (T) is an obstruction to all the methods that have been used so far to prove the conjecture and that a direction that is still open concerns applying the ideas of Bost, who defined a version of Oka principle in Noncommutative Geometry. Indeed, for higher rank Lie groups, there is a statement that says that proving the Baum-Connnes conjecture is equivalent to prove a Bost's kind of Oka principle for some group algebras. References included [11], [10], [26], [30], [31].

Lauren Ruth, Von Neumann dimension and lattices in algebraic groups

First, we give background on discrete series representations and lattices in algebraic groups. Then we explain the coupling constant, or von Neumann dimension, and how it measures the size of the commutant of a finite factor. Importantly, representations of II_1 factors are classified up to unitary equivalence by their von Neumann dimension, which assumes a continuum of values. In the main part of the talk, we explain a theorem that has its roots in Atiyah's work on L^2-index in [8], which was used by Atiyah and Schmid to realize discrete series representations in [9]; another proof is given in [19]. Let G be a simple algebraic group, with trivial center and no compact factors, having square-integrable irreducible unitary representations; let $(\pi, \mathcal{H})$ be such a representation of G; and let Γ be a lattice in G. The theorem states that the restriction of π to Γ extends to a representation of the II_1 factor $R\Gamma$ on $\mathcal{H}$, and this representation of $R\Gamma$ has von Neumann dimension $\mathrm{d}_p i \cdot \mathrm{vol}(G/\Gamma)$, where d_π is the formal dimension of $(\pi, \mathcal{H})$. We conclude by discussing two results from our dissertation, [47]: calculating the formula in a $\mathfrak{p}$-adic setting by dealing carefully with Haar measure, and obtaining representations of II_1 factors on spaces of automorphic functions.

Hang Wang, Elements in the Baum-Connes assembly map

We introduce for a locally compact second countable group, the universal example of proper actions, the equivariant K-homology and the precise definition the Baum-Connes assembly map. We describe in details the examples of finite groups, free groups, fundamental group of Riemann surfaces, where the Baum-Connes conjecture is true. Using the fact that $SL(2, \mathbb{Z})$ is the $\mathbb{Z}/2$-amalgamated product of $\mathbb{Z}/4$ and $\mathbb{Z}/6$ we are able to label the precise map of the Baum-Connes isomorphism for $SL(2, \mathbb{Z})$, and the generators all come from finite dimensional representations. For $SL(3, \mathbb{Z})$, the difficulty is that we no longer have a nice amalgamated product structure as in $SL(2, \mathbb{Z})$. Therefore, infinite dimensional representations of the group have to be investigated in order to understand the generators of K-homology and K-theory associated to those representations. The references we used here are Valette's "Introduction to Baum-Connes conjecture" Chapters 5, 6 and Natsumi's paper in year 1985 calculating K-theory of the group C^*-algebras of $SL(2, \mathbb{Z})$.

Jacqui Ramagge, Property (RD) for $\tilde{A}_2$-groups, including cocompact lattices in $SL_3(Q_p)$

We proved property (RD) for $\tilde{A}_2$-groups using a geometric argument to prove a Haagerup inequality. That is, we showed that the the operator norm of a group element under the left regular representation on $\ell^2(\Gamma)$ is polynomially bounded by the two-norm of the element, where the polynomial is a function of the length of the element with respect to a length function on the group. We began by giving a geometric reinterpretation of the proof of result of Haagerup on free groups from [21]. By adding one extra idea, we then generalised this to a result on $\tilde{A}_2$-groups as in [45]. By the results of Lafforgue in [29], this proves the Baum-Connes Conjecture for $\tilde{A}_2$-groups.

Outcomes

From all of the discussions and talks for the first half of the week, we then came up with some questions to tackle and ways to attack these together. The project now has a clear structure and we hope to meet again in Shanghai in June 2019.

3.3 Semigroup C^*-algebras and simplicity

Group members

Zahra Afsar (group co-leader, University of Sydney), Nadia S. Larsen (group co-leader, University of Oslo), Carla Farsi (University of Colorado, Boulder), Judith Packer (University of Colorado, Boulder).

Description of project and outcomes

The project took aim to investigate simplicity of the boundary quotient C^*-algebra $\mathcal{Q}(S)$ associated to a right LCM semigroup with identity (right LCM monoid) S. Simplicity of $\mathcal{Q}(S)$ is proved in [51, Theorem 4.12] under additional assumptions imposed so as to use the characterization of simplicity of étale Hausdorff groupoid C^*-algebras from [12]. We have that $\mathcal{Q}(S)$ is constructed as the tight groupoid C^*-algebra of an inverse semigroup, which should correspond to a "reduction groupoid" of the left inverse hull $I_l(S)$ whose C^*-algebra $C^*(I_l(S))$ models the universal semigroup C^*-algebra of S, see for example, [34, Theorem 5.17].

We started investigating the proof of simplicity from [51, Theorem 4.12], and in particular the construction of the tight groupoid from S. Then we moved on to understand how the conditions on S securing the Hausdorff property can be relaxed if we instead invoke the recent simplicity result for non-Hausdorff groupoids from [15], see in particular their Theorem 4.10. One reason why it would be interesting to explore this more general, non-Hausdorff universe, is that for arbitrary right LCM monoids without right cancellativity one does not expect their tight groupoids to be Hausdorff. For the class of right LCM semigroups arising from self-similar actions, there is a very interesting simplicity result in [15, Theorem 5.21] which identifies ω-faithful as condition for simplicity of the reduced groupoid C^*-algebra. This should admit useful generalizations.

Our efforts during the workshop concentrated on understanding the various ingredients that go into the construction of $\mathcal{Q}(S)$, and the new subtle points that go into the proof of simplicity from [15]. We have agreed on a plan to move forward with our project. We are grateful to Lisa Orloff Clark who joined our group one morning and gave a short, expert lecture on her recent work in [15].

3.4 Quantum majorization in infinite-dimensional Hilbert Space

Group members

Priyanga Ganesan (Texas A & M University), Sasmita Patnaik (Indian Institute of Technology), Sarah Plosker (project leader, Brandon University), Emily Redelmeier.

Project description

The research project related to an open problem in quantum information theory (QIT). In QIT, a quantum system is described by a Hilbert space $\mathcal{H}$. The predual of $\mathcal{B}(\mathcal{H})$, the set of all bounded linear operators acting on $\mathcal{H}$, is the ideal of trace class operators $\mathcal{T}(\mathcal{H})$, i.e., $\mathcal{T}(\mathcal{H}) = \mathcal{B}(\mathcal{H})_*$. Note that if $\mathcal{H}$ is finite dimensional, then $\mathcal{T}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$. The study of quantum states and quantum channels is central in QIT. Quantum states, also called density operators, are positive (semi-definite) trace-one operators $\rho \in \mathcal{T}(\mathcal{H})$. We will focus on bipartite states: let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two Hilbert spaces and $\rho^{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a state. For two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, a *quantum channel* $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a completely positive, trace preserving (CPTP) linear map.

Entanglement is one of the key resources that sets QIT apart from its classical counterpart. It is therefore important to quantify this resource, and particular attention is paid to when it is maximal. It should be noted that maximally entangled states do not exist in infinite dimensions as the Schmidt coefficients are an infinite set of nonnegative numbers summing to one, which has no minimal element with respect to majorization, whereas maximally entangled states are minimal elements with respect to majorization in finite dimensions.

Majorization has become an important tool in QIT since Nielsen's theorem (1999) which states that a pure state $|\psi_1\rangle$ shared by two parties can be transformed by LOCC into another state $|\psi_2\rangle$ (that is, $|\psi_1\rangle$ is at least as entangled as $|\psi_2\rangle$) if and only if $\lambda_{\psi_1} \prec \lambda_{\psi_2}$, where λ_{ψ_1} and λ_{ψ_2} are the Schmidt coefficient vectors of $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively, and $\prec$ denotes majorization. This result has become a springboard for various generalizations giving rise to related comparison relations in various contexts.

Many results in QIT are given in the context of bounded operators acting on finite-dimensional Hilbert spaces, and as such these results reduce to results about matrices. The context of infinite-dimensional Hilbert space is of interest to e.g. theoretical physicists, so if a result can be generalized to infinite dimensions it is of interest to do so. Solutions to such open problems are often of independent interest from a pure math standpoint. There are questions of closure and convergence that need to be addressed, issues related to the existence of certain objects in infinite dimensions, and gaps in the literature. In particular, generalizing the various versions of the majorization relation to infinite-dimensional Hilbert space is more than merely incremental in nature. The techniques used during the week relied heavily on operator theory, including comparisons of various norms, and the understanding of small but important details about tensor products of trace-class operators.

Unfortunately, due to a variety of factors (visa issues, family issues, etc.) I. Beltita, P. Ganesan, L. Ismert, and S. Srivastava could not attend. Genesan was able to meet via video chat for a few hours each day, so she was able to be involved remotely. Originally seven group members, the group wound up being three present at BIRS, plus an additional member (Ganesan) via Skype. This setup had its advantages and disadvantages. By the end of the week, we had identified a number of major issues that need to be addressed before attempting to solve the main open problem, and we developed a clear picture of what approach to use.

3.5 Twisted Steinberg algebras

Group members

Becky Armstrong (University of Sydney), Lisa Orloff Clark (group leader, Victoria University of Wellington), Kristen Courtney (University of Münster) , Ying-Fen Lin (Queen's University Belfast), Kathryn McCormick (University of Minnesota) and Jacqui Rammage (University of Sydney).

Description of project and outcomes

Jean Renault introduced the notion of a twisted groupoid C^*-algebra in his 1980 thesis and these C^*-algebras have been used in a variety of contexts over the years. Non-twisted groupoid C^*-algebras are themselves a broad class of algebras that includes a number of interesting subclasses, like graph and higher-rank graph C^*-algebras. The class of twisted groupoid C^*-algebras is even broader and there are a few results in particular highlighting this.

Results of an Huef, Kumjian and Sims [24, Theorem 5.2] say that every Fell algebra is isomorphic to a twisted groupoid C^*-algebra. More generally, Renault shows in [46, Theorem 5.9]: If a C^*-algebra contains a Cartan subalgebra, then it is isomorphic to a twisted groupoid C^*-algebra. This result, along with recent progress in the classification program, has placed twisted groupoid C^*-algebras at the forefront of C^*-algebraic research.

In this project, we considered ample groupoids, which are étale groupoids that have a basis of compact open sets. In this case, sitting inside of the groupoid C^*-algebra is a dense subalgebra called the Steinberg algebra [52]. As an analogy, the Steinberg algebra sits inside the groupoid C^*-algebra like the algebra generated by a Cuntz-Krieger family (no closure) sits inside the graph C^*-algebra. Steinberg algebras are gaining popularity as they 1. give rise to a lot of interesting purely algebraic examples and 2. give insight into groupoid C^*-algebras in surprising ways.

Up until now, twisted Steinberg algebras have not been considered. Our first task was to define twisted Steinberg algebras. We took into account the various notions of twisted groupoid C^*-algebras in the literature, and formulated a unified treatment in the purely algebraic setting. From there, we began considering the an Huef-Kumjian-Sims result and Renault's result and are making progressing on proving a purely algebraic version.

Our group will reconvene for 2 weeks in March 2019 at the University of Sydney to continue work on this project.

3.6 Quantum principal bundles and their C*-algebras

Group members

Francesca Arici (group leader, Max Planck Institute for Mathematics in the Sciences), Erin Grisenauer (Eckerd College), Chiara Pagani (group leader, Università del Piemonte Orientale).

Noncommutative fiber bundles and spaces

This is project lies at the interface of two important research directions in operator algebra: the theory of C^*-algebras, and Connes' approach to noncommutative geometry (NCG), including the algebraic approach to quantum groups.

The focus of this project is the study of noncommutative spaces and principal bundles over them, starting from the analysis of relevant examples in the literature, with the aim of giving a C*-algebraic description of such objects.

In the purely *algebraic* setting, a principal bundle over a noncommutative space B consists of an algebra extension $B \subseteq A$ of the algebra B which is H-Galois with respect to a certain Hopf algebra H. The base space B is the subalgebra of (co)invariant elements for the coaction of H on A.

It is less clear how to describe a principal bundle in the *C*- algebraic* context. Deepening our knowledge about bundles in the analytic framework is one of the goals of this research project.

Some partial results in this direction were obtained in [7, 6], where the authors described quantum principal *circle* bundles in terms of Cuntz–Pimsner algebras [41] of self-Morita equivalence bimodules. Prototypical examples of this construction are the inclusions $C(\mathbb{CP}^n_q) \subseteq C(S^{2n+1}_q)$, as well as more general algebra extensions coming from $U(1)$-actions, such as those involving weighted lens and projective spaces [14].

Inspired by these results, we look for a C*- algebraic characterisation of quantum principal bundles with non-Abelian structure group, inspired by [38]. Current work in progress in a parallel project by the first author suggests that the setting of subproduct systems [49, 57] will help describing Hopf–Galois extensions at the C*- algebraic level. A relevant class of fundamental examples to be studied are deformations of the $SU(2)$)-Hopf bundle.

A C^*-algebraic version of the symplectic instanton bundle

The q-deformed instanton bundle $\mathcal{O}(S^4_q) \subseteq \mathcal{O}(S^7_q)$ of [32] is an important example of a Hopf–Galois extension which is genuinely noncommutative, in that the structure group is Woronowicz's quantum $SU(2)$ and the total space is a quantum homogeneous spaces of symplectic quantum groups. The total space of this bundle is the symplectic seven-dimensional sphere S^7_q, which is obtained as a quantum homogeneous space for symplectic quantum groups.

During the first days of the WOA week, we have recalled the classical construction of projections that described the modules of sections of the associated vector bundles coming from the representations of $SU(2)$, in order to address the analogous computation for the symplectic sphere S^7 and the corepresentations of the quantum group $SU_q(2)$. These corepresentations are labelled by integers. The projection $\mathfrak{p}_{(1)}$ for the module of sections of the associated bundle coming from the fundamental corepresentation $\mathcal{E}_1$ was constructed in [32].

We have performed lengthy explicit computations and were able to construct the projection $\mathfrak{p}_{(2)}$ for the module $\mathcal{E}_2$, i.e., the module of sections of the associated bundle coming from the weight two corepresentation. In order to do so, we were lead to introduce a suitable ten-dimensional vector subspace of $\mathbb{C}^4 \otimes \mathbb{C}^4$ that can be thought of as a quantum analogue of the second symmetric power of $\mathbb{C}^4$, and that we have denoted by $Sym_q^2(\mathbb{C}^4)$.

Future plans and open questions

1. We plan to construct higher q-symmetric powers of $\mathbb{C}^4$ as maximal standard subproduct systems of Hilbert spaces with prescribed fibres $\mathbb{C}$, $\mathbb{C}^4$, $Sym_q^2(\mathbb{C}^4)$ in the sense of [49, Sec. 6.1]:

$$Sym_q^n(\mathbb{C}^4) = \bigcap_{i+j=n} Sym_q^i(\mathbb{C}^4) \otimes Sym_q^j(\mathbb{C}^4).$$

   To any such product system, one can associate a Cuntz–Pimsner like algebra, as described in [57]. We expect to be able to interpret the resulting algebra as the C*- algebra of the noncommutative symplectic seven-sphere, much as the C*- algebra of the standard symmetric product system over $\mathbb{C}^4$ is the commutative C*- algebra $C(S^7)$.

2. Motivated by the study of $SU(2)$-gauge theories, a parallel goal is that of inductively constructing all projections $\mathfrak{p}_{(n)}$ for $n \geq 2$, possibly giving a unified description, using the higher symmetric tensor powers defined in the previous point. Since the projection $\mathfrak{p}_{(n)}$, is determined by the quantum analogues of the totally symmetric n-th tensors products of ψ_1, ψ_2. We expect powers of q to appear in their expression.

One should observe that, in principle, one could also construct the modules $\mathcal{E}_n$ as maximal standard subproduct systems of Hilbert C*- modules over $C(S_q^4)$ with prescribed fibres $C(S_q^4)$, $\mathcal{E}_1$, $\mathcal{E}_2$. This, however, does not immediately yield the projections, therefore we plan to adopt the more explicit approach described in the two previous points.

Acknowledgments

We would like to thank the organisers of the WOA workshop for this great opportunity.

3.7 Nuclearity of C^*-algebras of quasi-lattice ordered groups

Group members

Astrid an Huef (group leader, Victoria University of Wellington), Camila F. Sehnem, Brita Nucinkis (Royal Holloway, University of London), Dilian Yang (University of Windsor).

Description of project and outcomes

A C^*-algebra A is *nuclear* if for every C^*-algebra B there is a unique C^*-norm on the algebraic tensor product $A \odot B$. Nuclearity is an important hypothesis of the classification program for C^*-algebras, and so it is important to be able to decide if a C^*-algebra is nuclear. For example, the C^*-algebra of a group is nuclear if and only if G is amenable.

The question of when the C^*-algebra of a semigroup is nuclear also has deep connections with notions of amenability. In [33, Theorem 6.1] Li studied a subsemigroup P of a discrete group G. He showed, under some hypotheses, that the full C^*-algebra $C^*(P)$ of P is nuclear if and only if, whenever a C^*-algebra A admits an action of G, the reduced and full crossed products of A by the semigroup P are isomorphic. Li's theorem applies, for example, to the quasi-lattice ordered groups (G, P) introduced by Nica in [39]. The project was to find sufficient and checkable conditions for the Toeplitz algebra $C^*(P)$ of a quasi-lattice ordered group to be nuclear.

The existence of generalised length functions, called "controlled maps", from G to an amenable group K is sufficient to ensure that (G, P) is amenable [28, 16, 25]. Given the deep connections between amenability and nuclearity, it seemed reasonable to explore this circle of ideas for nuclearity as well. Indeed, [55] used ideas from [33, Theorem 6.1] to show that a C^*-algebra of a doubly quasi-lattice ordered groups admitting such a controlled map is nuclear.

The Toeplitz algebra is also the C^*-algebra of a Fell bundle extended from P. Properties of controlled maps into $(\mathbb{Z}, \mathbb{N})$ were implicitly used in [48] to prove that Fell bundles extended from free semigroups and from Baumslag–Solitar semigroups are amenable. So it is natural to ask how we might deduce nuclearity of Toeplitz algebras from the amenability of those Fell bundles.

By [50, Example 2.2] every (G, P) gives a category of paths $\Lambda = \Lambda(G, P)$. Spielberg associated a groupoid $\mathcal{G}_\Lambda$ to a category of paths in such a way that C^*-algebra of $\mathcal{G}_\Lambda$ is isomorphic to $C^*(P)$ [50, Theorem 6.3]. So it is possible that nuclearity could be deduced from the results in [50, §9]. To this end, we wanted to know if a "controlled map" of [25, Definition 3.1] satisfies the hypotheses of [50, Theorem 9.8].

During the week at Banff we explored the above strategies and made significant progress. We currently have one paper in preparation and ideas for another.

3.8 Generalized Crossed Products

Group members

Maria Stella Adamo (University of Catania), Dawn Archey (group co-leader, University of Detroit Mercy), Magdalena Georgescu (Ben Gurion University), Ja A Jeong (Seoul National University), (Maria Grazia Viola (Lakehead University), Karen Strung (group co-leader, Radboud University).

Description of project and outcomes

The crossed products C^*-algebras by group actions have long been a source of interesting and elegant examples in the theory of C^*-algebras. One of the most well-studied cases is the case of the crossed product of $C(X)$, for X a compact metric space, by an action of the integers (which is, necessarily, determined by a single automorphism). One reason that these crossed products have been so valuable as examples is that while they are are in some ways complicated, they also have a lot of concrete structure.

There are some interesting generalizations of this construction, one of which is to take a crossed product by a Hilbert A-A-bimodule [1]. This is a generalization in the sense that every crossed product by an action of $\mathbb{Z}$ has a natural $\mathbb{Z}$-grading which allows one to express the crossed product using bimodules instead. In the broadest possible terms, our project is to study structural properties of these generalized crossed products. Of particular interest are those generalized crossed products which are simple, as this allows us to make links to the Elliott classification program. Groupoid C^*-algebras arising from minimal principal étale groupoids provide a different approach to generalizing crossed products of commutative C^*-algebras by the integers. Indeed, the orbit equivalence relation of a minimal dynamical system gives rise to such a C^*-algebra, which is isomorphic to the crossed product.

When studying simple crossed products by group actions, a technique that is frequently used is to find a suitable C^*-subalgebra of the crossed product which is "large enough" to retain most of the structural information of the crossed product while at the same time having a more tractable, recognizable structure. Putnam was the first to use this technique in his work on crossed products arising from Cantor minimal systems, by examining the construction of what we now call *orbit-breaking algebras* [43]. These have been used successfully in many places to understand the structure of their containing crossed products (see, for example, [37, 54, 56, 35, 36, 53, 18]) and have also been of interest in their own right, as they can also be viewed as C^*-algebras of minimal principal étale groupoids [17, 44].

Definition. Let X be an infinite compact metric space and $\varphi : X \to X$ a minimal homeomorphism. Put $A := C(X) \rtimes_\varphi \mathbb{Z}$. Let $Y \subset X$ be a closed nonempty subset meeting every φ orbit at most once. The *orbit-breaking algebra* at Y is given by

$$A_Y := \mathrm{C}^*(C(X), uC_0(X \setminus Y)) \subset C(X) \rtimes_\varphi \mathbb{Z},$$

where u is the unitary implementing φ, that is, $u \in C(X) \rtimes_\varphi \mathbb{Z}$ is the unitary satisfying $ufu^* = f \circ \varphi^{-1}$ for every $f \in C(X)$.

In 2011, Phillips introduced a generalization of orbit-breaking algebras which he called *large subalgebras* [40]. In this more general setting, many properties have been shown to pass between the large subalgebra and its containing C^*-algebra [40, 5, 4]. Although there are cases where this more general setting has proven useful, there is so far a dearth of examples of large subalgebras and their containing algebras that do not come from orbit breaking subalgebra in a crossed product of a commutative C^*-algebra. The goal of this project is to find other examples of large subalgebras, with an aim to exploit the preservation of known structural properties. Our starting point is to look at C^*-algebras which are very similar to crossed products by group actions, namely the two generalizations mentioned above.

Our time at WOA was spent sharing knowledge about Hilbert A-A-bimodules, groupoid C^*-algebras, labeled graph C^*-algebras, and large subalgebras until a candidate for a setting where we might apply the large subalgebra framework emerged. By the end of the week we had identified a (type of) subalgebra of bimodule crossed products which is similar, in spirit, to an orbit breaking algebras of a crossed product $C(X) \rtimes_\varphi \mathbb{Z}$, where the orbit is broken at a single point $x \in X$, the prototypical large subalgebra. We were able to confirm that our candidate is simple if the crossed product is simple, which is known to be a necessary condition for a large subalgebra of a simple algebra.

The next step in our project is to prove that our candidate subalgebras truly are large in the bimodule crossed product. We also hope to find the corresponding large subalgebra in minimal principal étale groupoid C^*-algebras. With these examples in hand, we will then look to prove results on structural properties of generalized crossed products using the properties of the large subalgebras we identified and known results about properties which pass from a large subalgebra to the containing algebra. We are planning to meet again in early spring in Germany or the Netherlands. Exact details are pending confirmation by the host institutions. We have also applied for a two week stay at MSRI during summer of 2019.

4 Feedback

The Association for Women in Mathematics ADVANCE evaluation team surveyed the WOA participants to provide feedback for the organisers. Of the 37 workshop participants, 31 responded to the survey, yielding an excellent response rate of 84 percent. Of the 31 survey respondents, most (87%) had earned a doctorate degree, and the vast majority (97%) were employed in academia. Graduate students and post-doctorates (39%) were balanced with academics in tenure-track positions (45%), below. This workshop was a new experience for most; almost all (94%) of the participants were newcomers to the Research Conference Collaboration Workshop model. Overall, the survey showed that the workshop was very successful: all participants would attend this conference again. Below are some of the open ended comments from the survey and testimonials supplied to BIRS:

- This was a phenomenal workshop. It was unusually effective in generating meaningful mathematical conversations and results. It also provided a great opportunity to bring together researchers with diverse specialties and to introduce researchers to related fields and questions.

- As far as research goes, I am kind of isolated at my current institution. This workshop was very beneficial for me. It helped me make connections with several people and really make progress on a research project. I am very grateful to the organizers, AWM, and other funding sources for this opportunity.

- It was a great workshop and one of the most friendly conferences/workshops I have been to.

- The workshop was fantastic. The collaborative and supportive atmosphere was totally different than any other workshop I've attended, in the most wonderful way. I think we all felt comfortable, respected and valued as mathematicians. [?] Many of us felt simultaneously that the week was too short, and that a two-week workshop wouldn't be practicable. Maybe a 9-day workshop, including weekends? Or an optional first week that had more of a mini-course structure, followed by the workshop? I also think it's important to support/fund mixed gender collaborations, in addition to the WOA collaboration network.

- The workshop allowed me to have the confidence to ask questions and make the most of a conference unlike any other conference. Being a women only conference it gave me the opportunity to communicate mathematics without having to prove myself. It gave me also a new foresight on job prospects, and actually made me rethink my capabilities as a researcher. I can be a successful researcher! I now have new collaborators with whom I feel comfortable working with.

- This workshop was absolutely remarkable. It provided me with an opportunity to meet and talk maths with outstanding researchers from around the world. During the week, I joined a new collaboration with researchers that I would have never normally had an opportunity to work with. The composition of our group and the structure of the workshop enabled us to prove several preliminary results, which we will develop into a paper in the coming months. Most importantly, this workshop brought together many women in a field with strikingly poor representation of women. The connections formed here, both mathematical and personal, will positively impact the diversity and, consequently, the mathematical developments in our field.
- The meeting was a unique opportunity to meet colleagues that I would not meet at regular workshops and offered me the chance to start a new research collaboration.
- Participating in this BIRS workshop has broadened my knowledge in the research field which I study, as well as new fresh insight of some research questions which I am interested. It was quite an unique experience that in the workshop not only I could have new contacts, but also we started a new collaboration on a research project. It is really good that we could just focus on our research problems during the whole week and achieved some results. On the top of all scientific aspects, I feel very much supported in our discipline through the workshop and the colleagues whom I got to know, which is very valuable for me.

5 Thanks

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