Pablo A. Ferrari, Buenos Aires

Soliton decomposition of the Box Ball System in $\mathbb{Z}$

with

Chi Nguyen, Leonardo Rolla, Minmin Wang (arXiv:1806.02798)

Davide Gabrielli (In preparation)

Banff, August 2018

Box at each $x \in \mathbb{Z}$.

Ball configuration $\eta \in \{0,1\}^{\mathbb{Z}}$

$$\eta(x) = 0 \leftrightarrow \text{empty box}, \qquad \eta(x) = 1 \leftrightarrow \text{ball at } x$$

Carrier visits boxes from left to right.

Carrier picks balls from occupied boxes.

Carrier deposits one ball, if carried, at empty boxes.

0	0	1	0	1	1	0	0	0	1	1	1	0	1	0	0	0	0	0	η
0	0	1	0	1	2	1	0	0	1	2	3	2	3	2	1	0	0	0	carrier
0	0	0	1	0	0	1	1	0	0	0	0	1	0	1	1	1	0	0	$T\eta$

$T\eta$: configuration after the carrier visited all boxes.

Ball-Box-System by Takahashi-Satsuma (1990)

Motivation: Korteweg & de Vries equation

$$\dot{u} = u''' + u\,u'$$

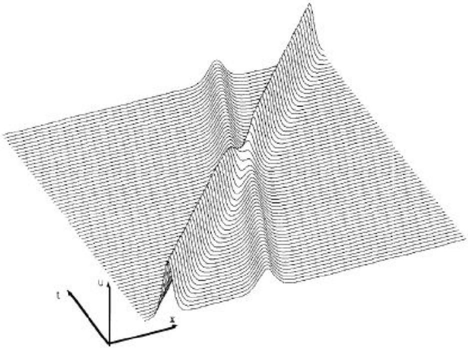

Figure from Rendiconti di Matematica, Serie VII, 11, p.351-376, 1991

Interacting soliton solutions for KdV.

Summary

0) Soliton identification and conservation.

1) Asymptotic speed of solitons.

2) *Soliton decomposition* of ball configurations.

3) Evolution is a *hierarchical translation* of soliton components.

4) Measures with independent soliton components are invariant for T.

5) Explicit soliton decomposition for iid Bernoulli, Ising models and other ball distributions.

Soliton: a solitary wave that propagates with little loss of energy and retains its shape and speed after colliding with another such wave

Conserved solitons. Motivation.

isolated *k-soliton:* set of k successive ones followed by k zeroes.

Isolated k-solitons travel at speed k and conserve shape and distances:

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k-solitons and distances are conserved after interacting with m-solitons:

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Conserved solitons. Motivation.

isolated k-soliton: set of k successive ones followed by k zeroes.

Isolated k-solitons travel at speed k and conserve shape and distances:

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k-solitons and distances are conserved after interacting with m-solitons:

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Conserved solitons. Motivation.

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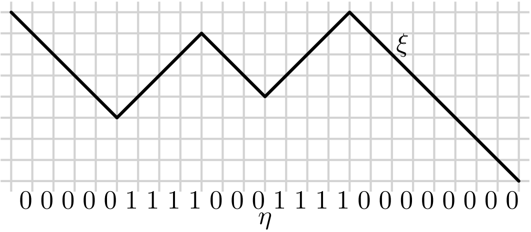
Walk representation



$$\xi(z) - \xi(z-1) = 2\eta(z) - 1$$

Records: $\{z : \xi(z) < \xi(y)$ for all $y < z\}$.

Excursion: configuration between two successive **records**

Infinite volume dynamics

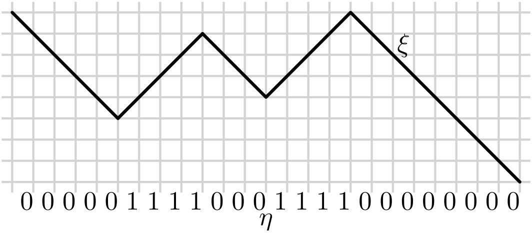
$$T\eta(x) := (1 - \eta(x))\mathbf{1}\{x \text{ is not a record}\}$$

$\mathcal{X}_\lambda :=$ configurations with density of particles $\lambda \in [0, 1]$

If $\lambda < 1/2$ and $\eta \in \mathcal{X}_\lambda$ then $T\eta \in \mathcal{X}_\lambda$.

Takahashi-Satsuma soliton identification

Call *runs* the segments induced by broken lines in the **walk**



Explore runs from left to right. If a run has length $k \leq$ length of the next run, then its k boxes and the first k boxes of the next run form a k-soliton. Remove these sites and start again exploring from the left.

Takahashi-Satsuma soliton identification

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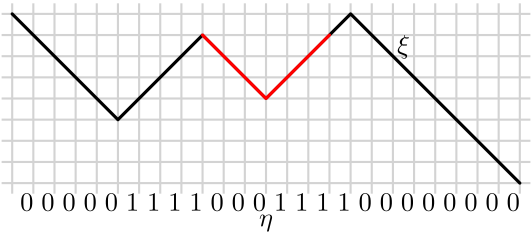

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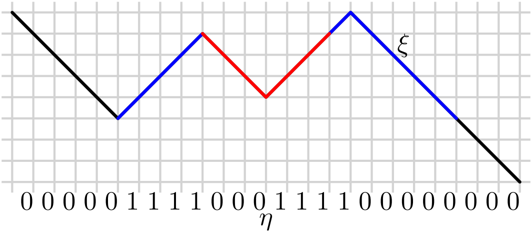

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Takahashi-Satsuma soliton identification

head of k-soliton γ:= position of ones $h(\gamma) = \{h_1, \ldots, h_k\}$,

tail of k-soliton γ:= zeroes $t(\gamma) = \{t_1, \ldots, t_k\}$.

Infinite configurations:

Apply TS algorithm to each *excursion*, pretending it is isolated.

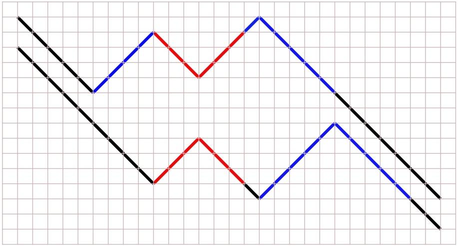
k-soliton conservation under T

Proposition. *For any $\eta \in \mathcal{X}$:*

$$\eta \text{ has } k\text{-soliton } \gamma \text{ with tail } t(\gamma) = a$$

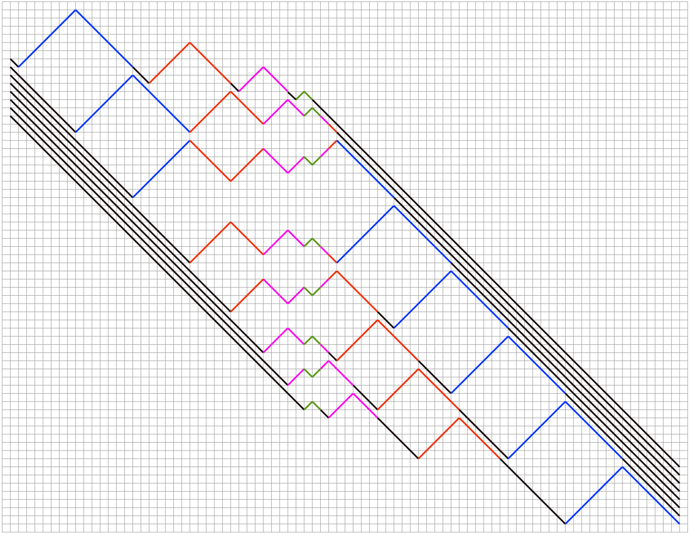
if and only if

$$T\eta \text{ has } k\text{-soliton } \gamma' \text{ with head } h(\gamma') = a.$$



We can follow solitons along time!

k-soliton conservation under T



Asymptotic speed of solitons

Theorem (FNRW). *Let μ be a shift-ergodic T-invariant measure.*
$\rho_k :=$ *mean number of k-solitons per excursion.*

$x(\gamma^t) :=$ *position of k-soliton γ at time t.*

Then, there exists $v = (v_k)_{k\geq 1}$ deterministic such that

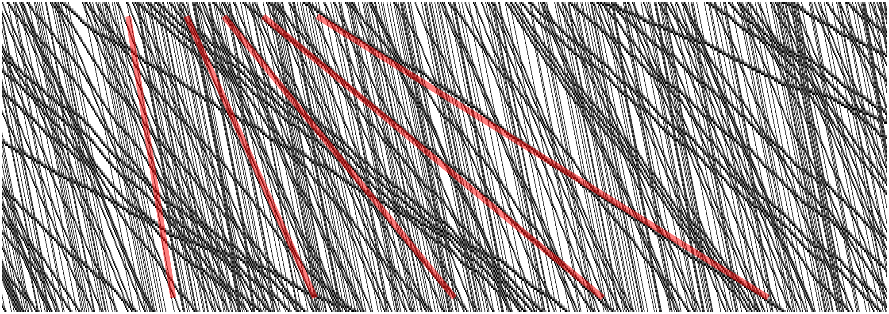
$$\lim_{t\to\infty} \frac{x(\gamma^t)}{t} = v_k, \quad \mu\text{-a.s.}$$

v is a solution of

$$v_k = k + \sum_{m<k} 2m\rho_m(v_k - v_m) - \sum_{m>k} 2k\rho_m(v_m - v_k).$$

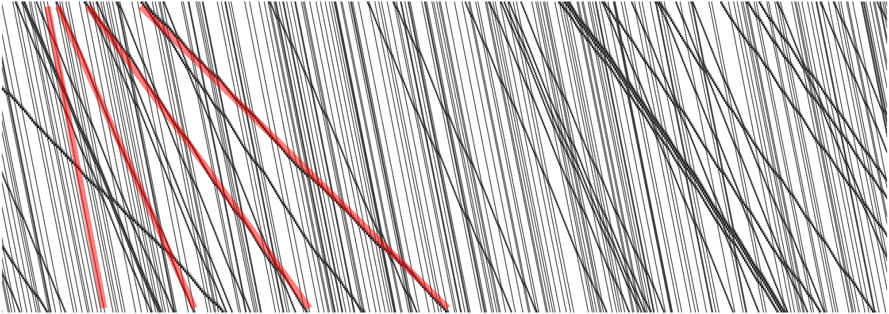
and can be computed explicitely in function of $(\rho_k)_{k\geq 1}$.

Asymptotic speed of solitons



Simulation for an iid configuration with density 0.25.
Deterministic red straight lines have slopes computed by the theorem.
2000 boxes $\times$ 140 T iterations, going downwards
(stretched vertically by a factor of 5)

Asymptotic speed of solitons



Simulation for an iid configuration with density 0.15.
Deterministic red straight lines have slopes computed by the theorem.
2000 boxes $\times$ 140 T iterations, going downwards
(stretched vertically by a factor of 5)

Detail

Asymptotic speed of solitons

Motivation of

$$v_k = k + \sum_{m<k} 2m\rho_m(v_k - v_m) - \sum_{m>k} 2k\rho_m(v_m - v_k).$$

Isolated k-solitons have speed k

When a k-soliton encounters an m-soliton:

- it advances $2m$ extra units if $m < k$ or
- it is delayed by 2 time steps if $m > k$.

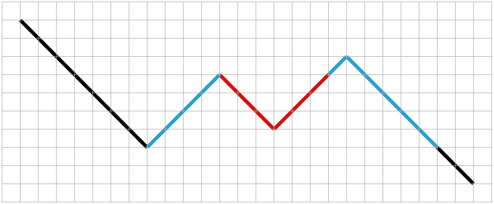
$\rho_m|v_k - v_m|$ is the frequency of such encounters as seen from a k-soliton.

Slots

Let η configuration with finite excursions.

Will decompose η in soliton components.

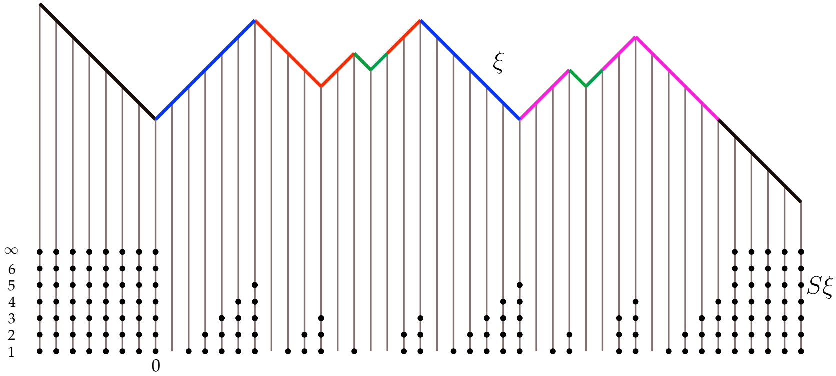
Motivation: Insert 3-soliton in 3-slot of 5-soliton:



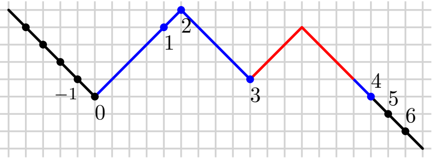
k-slots := Places where k-solitons can be inserted.

k-slots := records plus $\{h_\ell(\gamma), t_\ell(\gamma)\}$, $\ell > m$, m-solitons γ with $m > k$.

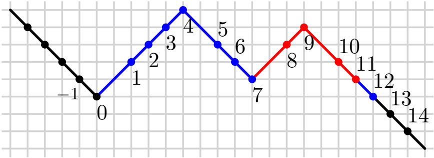
Slot configuration



Enumerating the k-slots



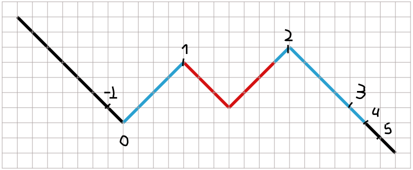
enumerating 3-slots



enumerating 1-slots

Soliton components

Insert 3-soliton in 3-slot number 1:

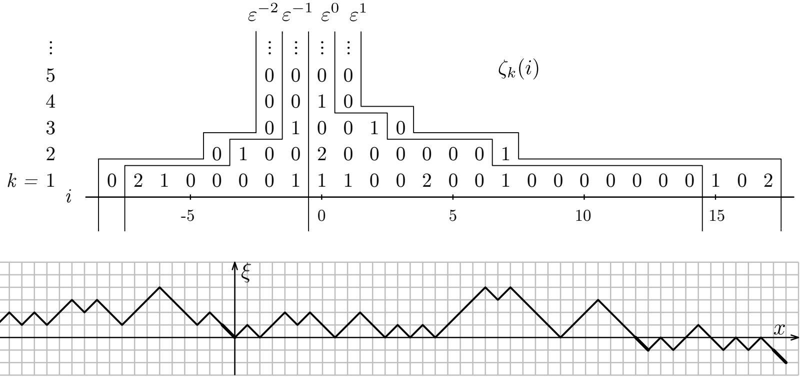


$M_3\eta(1) = 1$ means that 3-component at coordinate 1 has 1 soliton.

$M_k\eta(i)$:=number of k-solitons inserted in k-slot number i

Soliton decomposition: $M\eta = (M_k\eta)_{k\geq 1}$

The map $\eta \mapsto M\eta$ (or $\xi \mapsto \zeta$).



Below: Records -2 to 2 in boldface and the excursions between them.

Above: the parts of the field ζ corresponding with excursions $\varepsilon^{-2}, \varepsilon^{-1}, \varepsilon^{0}, \varepsilon^{1}$.

Solitons!

Theorem (FNRW).

k-soliton component of $\hat{T}^t\eta$ is a shift of the k-soliton component of η:

$$M_k \hat{T}^t \eta = \theta^{o_k^t(\eta,0)+kt} M_k \eta$$

$$\hat{T}^t\eta = T^t\eta \text{ as seen from Record 0}$$

$$\theta^x = \text{translation by } x$$

$$o_k^t(\eta,0) := \sum_{m>k} 2(m-k) J_m^t(\eta)$$

$$J_m^t(\eta) := \text{Flow of } m\text{-solitons thru Record 0}$$

Evolution of components

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Evolution of components

Ball configuration

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Components

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Evolution of components

Ball configuration

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Components

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Evolution of components

Ball configuration

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Components

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Evolution of components

Ball configuration

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Components

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Evolution of components

Ball configuration

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Components

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Evolution of components

Ball configuration

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Components

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Evolution of components

Ball configuration

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Components

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Evolution of components

Ball configuration

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Components

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Evolution of components

Ball configuration

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Components

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Independent-component invariant measures

Theorem [FNRW]

$\zeta_k \in \mathbb{N}^{\mathbb{Z}}$ independent with shift-stationary law such that

$$\sum_k k E\zeta_k(0) < \infty$$

Let $\zeta = (\zeta_k)_{k\geq 1}$ and $\hat{\mu} :=$ law of $M^{-1}\zeta$.

Then $\hat{\mu}$ is $\hat{T}$-invariant and

$\mu := \mathrm{Palm}^{-1}(\hat{\mu})$ is shift-stationary and T-invariant.

k-components of iid Bernoulli are independent iid geometrics.

With Davide Gabrielli

Let $\lambda \in (0, \frac{1}{2})$ and $\alpha_k := (\lambda(1-\lambda))^k$

Let $q_1 = \alpha_1$ and for $k \geq 2$,

$$q_k := \frac{\alpha_k}{\prod_{j=1}^{k-1}(1-q_j)^{2(k-j)}}\,.$$

Theorem (FG). *If $(\eta(x) : x \in \mathbb{Z})$ iid Bernoulli(λ) conditioned to have a Record at the origin, then*

$(M_k\eta(s) : s \in \mathbb{Z})$ *iid Geometric*$(1-q_k)$ *and*

$(M_k\eta : k \geq 1)$ *are independent.*

Other measures with independent geometric k-components.

Let $\alpha_k \geq 0$ such that $\sum_{k\geq 0} \alpha_k < \infty$.

Let ε be an excursion between Record 0 and Record 1 and

$n_k(\varepsilon) :=$ number of k-solitons of ε.

$$\text{weight} \quad w_\alpha(\varepsilon) := \prod_{k=1}^{\infty} \alpha_k^{n_k(\varepsilon)} \tag{1}$$

induces a measure

$$\nu_\alpha(\varepsilon) = \frac{w(\varepsilon)}{Z_\alpha} \tag{2}$$

Concatenate independent excursions to obtain a measure $\hat{\mu}_\alpha$ on $\widehat{\mathcal{X}}$.

$\mu := \mathrm{Palm}^{-1}(\mu)$ has independent components geometric with parameters q_i

Theorem (FG).

Let $\hat{\mu}_\alpha :=$ *independent excursions with law* ν_α *with Record 0 at the origin.*

Let $(\eta(x) : x \in \mathbb{Z}) \sim \hat{\mu}_\alpha$. *Then*

$(M_k\eta(s) : s \in \mathbb{Z})$ *iid Geometric*$(1 - q_k)$ *and*

$(M_k\eta : k \geq 1)$ *are independent.*

Special cases

- $\alpha_k = [\lambda(1-\lambda)]^k$, $\lambda < \frac{1}{2}$. Product measure with density λ
- $\alpha_k = e^{2J}e^{kh}$. Ising measure with pair interaction J and external field $h < 0$.

When the k-components of invariant measures are independent?

Let $\hat{\mu}$ on $\mathcal{X}^o$ be invariant for $\hat{T}$ and record-mixing(?). Then,

$$\hat{\mu}M = \bigotimes_{k\geq 1} \hat{\mu}M_k.$$

That is, if η has law $\hat{\mu}$,

$(M_k\eta : k \geq 1)$ is a family of independent configurations.