AH extensions and estimates for an analogue of the Bartnik mass

Stephen McCormick
Uppsala University

Joint work with:
Armando Cabrera Pacheco (Tübingen)
& Carla Cederbaum (Tübingen)

May 16, 2018

Banff – AH Manifolds, 2018

arXiv:1802.03331

TABLE OF CONTENTS

INTRODUCTION AND BACKGROUND

In this talk, we consider only time-symmetric initial data slices.

Some notation:

Σ is diffeomorphic to the two-sphere and throughout g will denote a metric on Σ.

We will consider 3-manifolds M (usually asymptotically hyperbolic) equipped with metrics γ.

The mean curvature H of a closed surface in an asymptotically hyperbolic manifold will always be with respect to the normal pointing towards infinity in the manifold.

Bartnik data refers to a triple (Σ, g, H), where H is a non-negative function (that we will take to be constant usually).

There are well known definitions for the total mass of an asymptotically flat, or asymptotically hyperbolic manifold.

There is no energy density of the gravitational field in general relativity

Can we determine the total mass or energy contained in a bounded domain? This is the problem of *quasi-local mass*.

The Hawking mass is a well-known example of such a definition

$$\mathfrak{m}_H(\Sigma, g, H) = \frac{1}{2}\left(\frac{|\Sigma|}{4\pi}\right)^{1/2}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2 \, d\mu\right), \qquad (1)$$

and its asymptotically hyperbolic counterpart

$$\mathfrak{m}_H^{AH}(\Sigma, g, H) = \frac{1}{2}\left(\frac{|\Sigma|}{4\pi}\right)^{1/2}\left(1 - \frac{1}{16\pi}\int_\Sigma (H^2 - 4) \, d\mu\right) \qquad (2)$$

Bartnik's quasi-local mass functional is usually given by

$$\mathfrak{m}_B(\Sigma, g, H) := \inf\{\mathfrak{m}_{ADM}(M,\gamma) : (M,\gamma) \text{ is an 'admissible extension'.}\},$$

where an admissible extension (M, γ) is an asymptotically flat manifold with non-negative scalar curvature, boundary isometric to (Σ, g) with mean curvature H, satisfying a certain non-degeneracy condition.

This non-degeneracy condition is usually taken to be either that (M, γ) contains no closed minimal surfaces (except possibly the boundary), or that the boundary is outer minimising. Without such a condition the Bartnik mass would always trivially be zero.

By Huisken and Ilmanen's proof of the Riemannian Penrose inequality, we have that the Hawking mass bounds the Bartnik mass by below:

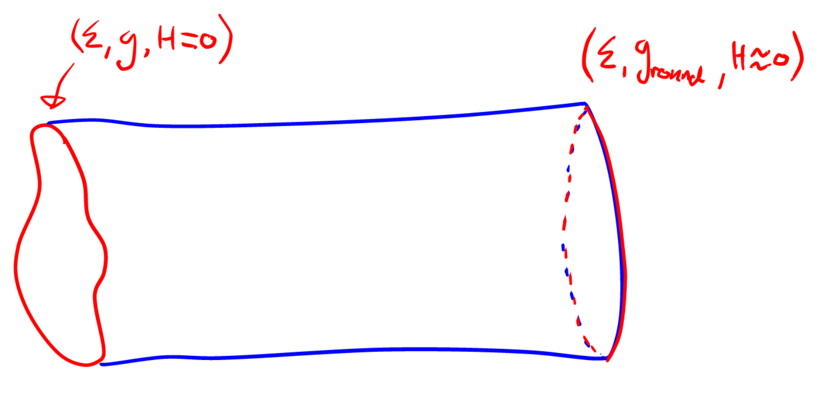
$$\mathfrak{m}_H(\Sigma, g, H) \leq \mathfrak{m}_B(\Sigma, g, H),$$

provided we take the degeneracy condition to be such that the boundary of an admissible extension is outer minimising.
If g is a round metric and H is constant, then an exterior portion of the Schwarzschild manifold with mass parameter $m = \mathfrak{m}_H(\Sigma, g, H)$ provides an admissible extension of (Σ, g, H) and therefore we have equality,

$$\mathfrak{m}_H(\Sigma, g, H) = \mathfrak{m}_B(\Sigma, g, H).$$

However, in general equality should not hold.

In the case where the data (Σ, g, H) corresponds to a stable minimal surface in an initial data set ($H = 0$ and g satisfies a stability condition), we also have equality between these two masses (Mantoulidis–Schoen, 2015).



Following similar ideas as Mantoulidis and Schoen, similar estimates can be obtained for H being some positive constant (Cabrera Pacheco, Cederbaum, M^{c}C., Miao).

Since we can't expect equality between the Hawking mass and the Bartnik mass in general, we obtain an upper bound for the Bartnik mass that is larger than the Hawking mass in general.

Note: Qualitatively similar, but slightly different, estimates for the Bartnik mass were obtained also by Lin and Sormani using different ideas.

To define any mass, an appropriate reference should be fixed. In the case of the usual Bartnik mass, the reference space is Euclidean space – we consider asymptotically flat extensions.

In the presence of a negative cosmological constant, it is natural to consider asymptotically hyperbolic extensions in the definition of the Bartnik mass.

We take the 'hyperbolic Bartnik mass' to be the infimum of the total hyperbolic mass* over an analogous space of admissible extensions (with scalar curvature $R \geq -6$).

Theorem (Cabrera Pacheco, Cederbaum, M^{c}C.)
Let (Σ, g, H) be Bartnik data satisfying $K(g) > -3$ and H is a non-negative constant. Then the asymptotically hyperbolic Bartnik mass satisfies

$$\mathfrak{m}_B^{AH}(\Sigma, g, H) \leq \mathfrak{m}_H^{AH} + Err(H\sqrt{\alpha}), \tag{3}$$

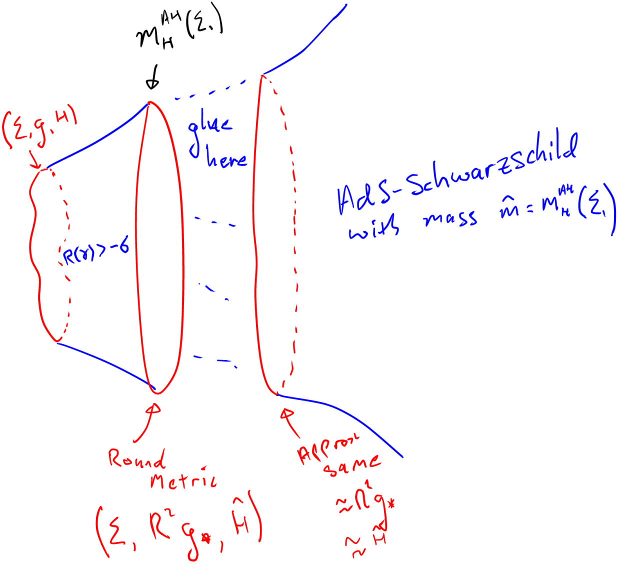
where $Err(y)$ is an error term that goes to zero as y goes to zero, and α is a parameter measures how much g deviates from being round.

In particular, for Bartnik data close to being round or a minimal surface, then the AH Bartnik mass is close to the AH Hawking mass.

Recall, the AH Hawking mass is given by

$$\mathfrak{m}_H^{AH} := \left(\frac{|\Sigma|_g}{16\pi}\right)^{1/2}\left(1 - \frac{1}{16\pi}\int_\Sigma (H^2 - 4)\, dS\right). \tag{4}$$

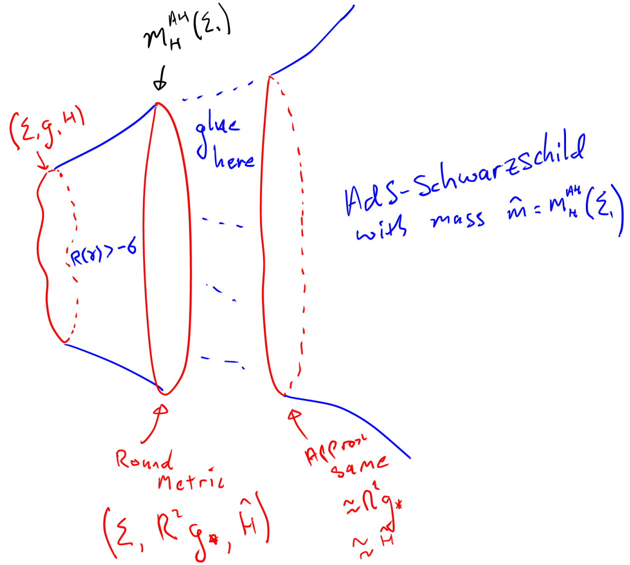
OVERVIEW OF PROOF


m_H^AH(Σ_1)
(Σ, g, H)
glue here
R(γ) > -6
AdS-Schwarzschild with mass m̂ = m_H^AH(Σ_1)
Round Metric
(Σ, R² g_*, Ĥ)
Approx same
≈ R² g_*
≈ Ĥ

1. Construct a collar manifold $([0,1]\times\Sigma,\gamma_c)$ such that $\Sigma_0=\{0\}\times\Sigma$ realises the given Bartnik data (Σ,g,H) and $\Sigma_1=\{1\}\times\Sigma$ is round with AH Hawking mass close to that of the given Bartnik data. We also must ensure that the collar has scalar curvature bound below by -6 and satisfies the non-degeneracy condition.

2. Smoothly glue the collar to an AdS–Schwarzschild manifold with mass parameter m close to the AH Hawking mass at the end of the collar.

"SKETCH"



$m_H^{AH}(\Sigma_1)$
(Σ, g, H)
glue here
$R(\gamma) > -6$
AdS-Schwarzschild with mass $\hat{m} = m_H^{AH}(\Sigma_1)$
Round Metric
$(\Sigma, R^2 g_*, \hat{H})$
Approx same
$\approx R^2 g_*$
$\approx \hat{H}$

COLLARS AND PATHS OF METRICS

Consider the manifold $[0,1] \times \Sigma$ equipped with a metric γ of the form

$$\gamma = A^2 ds^2 + E(s)^2 g(s), \tag{5}$$

where $g(s)$ is a one-parameter family of metrics on Σ.

Under the assumption that $\mathrm{tr}_{g(s)}(\dot{g}(s)) = 0$, the mean curvature of $s =$constant slices is easily computed as

$$H(s) = \frac{2E'(s)}{AE(s)} \tag{6}$$

and the scalar curvature $R(\gamma)$ is

$$R(\gamma) = 2K(g(s)) + A^{-2}\,[\ldots] \tag{7}$$

$$\gamma = A^2 ds^2 + E(s)^2 g(s), \tag{8}$$

where $g(s)$ is a one-parameter family of metrics on Σ.

$$H(s) = \frac{2E'(s)}{AE(s)} \tag{9}$$

$$R(\gamma) = 2K(g(s)) + A^{-2}\,[\ldots] \tag{10}$$

In the case where $H = 0$, we can construct a collar with scalar curvature $R(\gamma) \geq -6$, mean curvature of the cross-sections small, and with the area of $\Sigma_1 = \{1\} \times \Sigma$ close to that of Σ_0. This is by choosing A large, E close to constant, and $K(g(s)) > -3$.

By choosing $g(1)$ to be round, we can glue on a Schwarzschild-AdS manifold near the horizon with mass $m \approx \mathfrak{m}_H^{AH}(\Sigma, g, 0)$.

The key to constructing such a collar (and therefore also extensions) is the path $g(s)$ of metrics on Σ. The path must satisfy

(i) $\mathrm{tr}_{g(s)}\dot{g}(s) = 0$ (area form preserved),

(ii) the Gauss curvature satisfies $K(g(t)) > -3$,

(iii) $\dot{g}(s) = 0$ in a neighbourhood of $s = 1$,

(iv) and $g(1)$ is a round metric.

In the paper of Mantoulidis and Schoen, it was shown that given a path of metrics with constant total area, it is possible to modify the path to ensure that in fact (i) is satisfied.

It is also straightforward to approximate any given path with a path satisfying (iii), by making a suitable modification near $s = 1$.

So we seek to find a path of metrics $g(s)$, connecting a given metric g_o to a round metric, while preserving the area and maintaining a (negative) Gauss curvature lower bound.

We can do this with normalised Ricci flow:

$$\partial_t g(t) = 2\left(r_o^{-1} - K(g(t))\right) g(t), \tag{11}$$

where $r_o = \sqrt{\frac{|\Sigma|_{g_o}}{4\pi}}$ is the area radius of g_o. Now well-known results of Hamilton and Chow imply that the solution exists for all time and converges exponentially fast to a sphere.

The Gauss curvature satisfies a parabolic equation, and by a maximum principle argument, we know that if $K(g_o) > -3$ then $K(g(t)) > -3$ for all time.

Reparametrising gives the desired path.

CONSTRUCTING THE COLLARS

$$\gamma = A^2 ds^2 + E(s)^2 g(s), \tag{12}$$

where $g(s)$ is a one-parameter family of metrics on Σ.

$$H(s) = \frac{2E'(s)}{AE(s)} \tag{13}$$

$$R(\gamma) = 2K(g(s)) + A^{-2}[...] \tag{14}$$

In the case where $H = 0$, we can construct a collar with scalar curvature $R(\gamma) \geq -6$, mean curvature of the cross-sections small, and with the area of $\Sigma_1 = \{1\} \times \Sigma$ close to that of Σ_0. This is by choosing A large, E close to constant, and $K(g(s) > -3$.

By choosing $g(1)$ to be round, we can glue on a Schwarzschild-AdS manifold near the horizon with mass $m \approx \mathfrak{m}_H^{AH}(\Sigma, g, 0)$.

THE CASE $H_o \neq 0$

We saw that if the Bartnik data (Σ, g_o, H_o) satisfies $H_o = 0$ then we can construct a cylinder connecting the data to a round metric with nearly the same area, and with small mean curvature. Ie. the end of the collar has close to the same AH Hawking mass as that of the given data.

In the case $H_o > 0$, the area increases along the collar so we cannot stretch it out too much if we hope to control the AH Hawking mass along it.

Remember: we would like to glue on a Schwarzschild-AdS exterior with the smallest possible mass. So we need to keep the AH Hawking mass at the end of our collars, as small as possible.

The standard AdS-Schwarzschild manifold of mass m can be expressed as

$$ds^2 + u_{m,1}(s)^2 g_* \tag{15}$$

where g_* is the standard round metric of area 4π and $u_{m,1}$ is a positive function satisfying

$$u'_{m,1}(s) = \sqrt{1 + u_{m,1}(s)^2 - \frac{2m}{u_{m,1}(s)}}. \tag{16}$$

Notably, the AH Hawking mass is constant, equalling m on each $s =$constant slice.

This motivates us to consider the collar

$$\gamma = A^2 ds^2 + u_{m,1}(Aks)^2 r_o^{-2} g(s) \tag{17}$$

We also introduce an additional parameter b as follows. The metric

$$ds^2 + u_{m,b}(s)^2 g_* \tag{18}$$

where $u_{m,b}$ is a positive function satisfying

$$u'_{m,b}(s) = \sqrt{1 + b u_{m,b}(s)^2 - \frac{2m}{u_{m,b}(s)}}. \tag{19}$$

can be viewed as the AdS-Schwarzschild manifold in the case of a cosmological constant $\Lambda = -3b$ and has scalar curvature equal to $-6b$.

This motivates us to instead consider the collar

$$\gamma_{m,b} = A^2 ds^2 + u_{m,b}(Aks)^2 r_o^{-2} g(s) \tag{20}$$

$H > 0$ collars

By choosing k in terms of the parameters m and b, the mean curvature at $s = 0$ can be prescribed to be H_o. We then have freedom in m and b to ensure the scalar curvature satisfies $R(\gamma_{m,b}) > -6$, without increasing the AH hawking mass of the end of the collar too much.

Note that $R(\gamma_{m,b})$ depends on the path $g(s)$, and therefore so must the parameters, and also the Hawking mass at the end of the collar, Σ_1. In particular, it depends on the Gauss curvature along the path and the parameter

$$\alpha := \frac{1}{4} \max_{\Sigma, s\in[0,1]} |\dot{g}(s)|_{g(s)}. \tag{21}$$

$H > 0$ collars

There is no obvious choice for free parameters, but qualitatively many choices are similar and give a Hawking mass of Σ_1 as

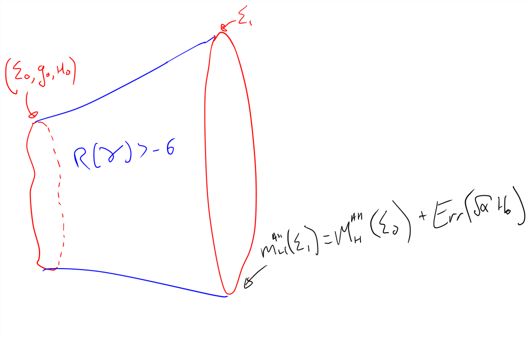
$$\mathfrak{m}_H^{AH}(\Sigma_0, g_o, H_o) = \mathfrak{m}_H^{AH}(\Sigma_1) + \mathrm{Err}(\sqrt{\alpha}H_o). \tag{22}$$

In particular, the Hawking mass at the end of the collar becomes close to the Hawking mass of the given data if the data is close to round or H is small.

$H > 0$ collars

That is, for given Bartnik data (Σ, g_o, H_o) with $H_o \geq 0$ and $K(g_o) > -3$, we construct collar metrics, γ on $[0,1] \times \Sigma$ satisfying:

- $\gamma_{s=0} = g_o, \qquad H_{s=0} = H_o, \qquad R(\gamma) > -6$
- Σ_0 minimises area among homologous competitors
- $\mathfrak{m}_H^{AH}(\Sigma_1) = \mathfrak{m}_H(\Sigma_0) + \mathrm{Err}(\sqrt{\alpha} H_o)$



GLUING COLLARS TO ADS-SCHWARZSCHILD

Lemma

Let $f_i : [a_i, b_i] \to \mathbb{R}^+$, $i = 1, 2$, be two smooth positive functions, g_* be the standard metric on S^n, and $\tau \in (-\infty, 0]$. Suppose

(i) *the metric $\gamma_i := ds^2 + f_i(s)^2 g_*$ has scalar curvature $R(\gamma_i) > \tau$,*

(ii) $f_1(b_1) < f_2(a_2)$,

(iii) $0 < f_1'(b_1) < \sqrt{1 - \frac{\tau}{n(n-1)} f_1(b_1)^2}$, *and*

(iv) $-\sqrt{1 - \frac{\tau}{n(n-1)} f_2(a_2)^2} < f_2'(a_2) \leq f_1'(b_1)$.

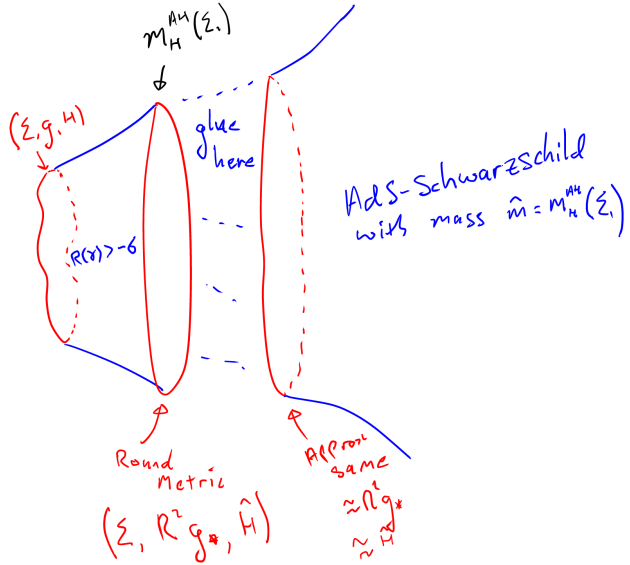
Then, after translating $[a_i, b_i]$ appropriately, there is f such that

(I) $f \equiv f_1$ *on* $[a_1, \frac{a_1+b_1}{2}]$ *and* $f \equiv f_2$ *on* $[\frac{a_2+b_2}{2}, b_2]$ *and*

(II) *the metric $\gamma := dt^2 + f(s)^2 g_*$ has scalar curvature $R(\gamma_b) > \tau$ on $[a_1, b_2] \times S^n$.*

In addition, if $f_i' > 0$ on $[a_i, b_i]$ then $([a_1, b_2] \times S^n, \gamma)$ is foliated by mean convex CMC spheres.

The gluing lemma can be easily proved following the AF case (Mantoulidis–Schoen). As can a lemma for beinging the AdS-Schwarzschild manifold where we would like to glue it to the collars, to bump up the scalar curvature to strictly > -6 near where we would like to glue



$m_H^{AH}(\Sigma_1)$
(Σ, g, H)
glue here
$R(\gamma) > -6$
AdS-Schwarzschild with mass $\hat{m} = m_H^{AH}(\Sigma_1)$
Round Metric
$(\Sigma, R^2 g_*, \hat{H})$
Approx same
$\approx R^2 g_*$
$\approx \hat{H}$

Theorem (Cabrera Pacheco, Cederbaum, M^cC.)
Let (Σ, g, H) *be Bartnik data satisfying* $K(g) > -3$ *and* H *is a non-negative constant. Then the asymptotically hyperbolic Bartnik mass satisfies*

$$\mathfrak{m}_B^{AH}(\Sigma, g, H) \leq \mathfrak{m}_H^{AH} + Err(H\sqrt{\alpha}), \tag{23}$$

where $Err(y)$ *is an error term that goes to zero as* y *goes to zero, and* α *is a parameter measures how much* g *deviates from being round.*

In particular, for Bartnik data close to being round or a minimal surface, then the AH Bartnik mass is close to the AH Hawking mass.

The estimates depend on a quantity α associated to paths of metrics on Σ, which relates to the 'roundness' of the given g_o.

Miao and Xie used a similar path construction and were able to make this statement more precise. They proved that the path can be chosen so that if g is sufficiently close to the round metric g_* in $C^{2,\tau}$ then α for this path satisfies $\alpha \leq C\|g - g_o\|^2_{C^{0,\tau}}$.

Thanks for listening