“Dice”-sion Making under Uncertainty: When Can a Random Decision Reduce Risk?

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Imperial College Business School

joint work with Erick Delage and Daniel Kuhn

Tilburg University, Dec 2017

Facility Location under Uncertainty

Facility Location under Uncertainty

0.45 vs. 0.59

μ = 0.52

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0.04 vs. 1.79
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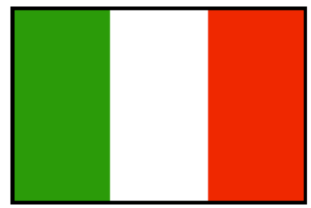

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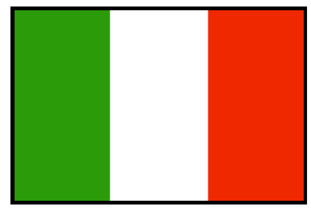

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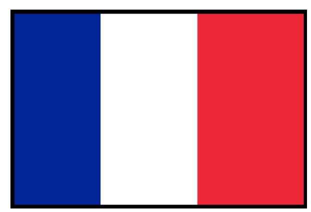

-0.10 vs. 5.39
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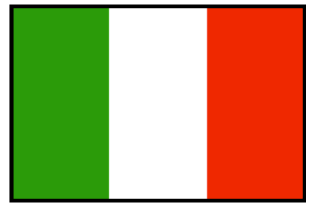

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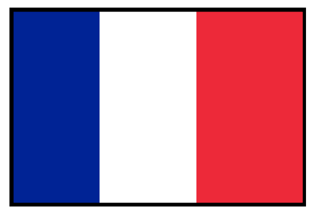

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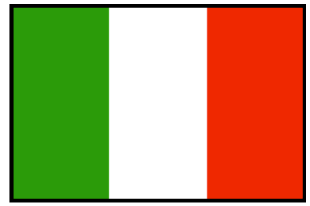

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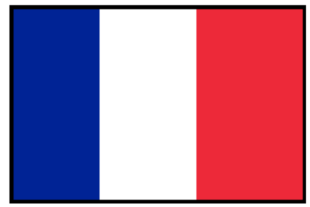

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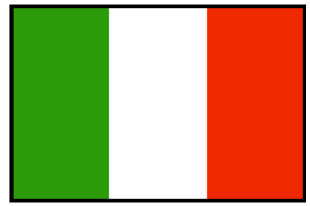

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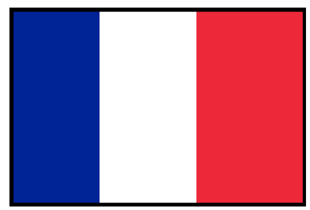

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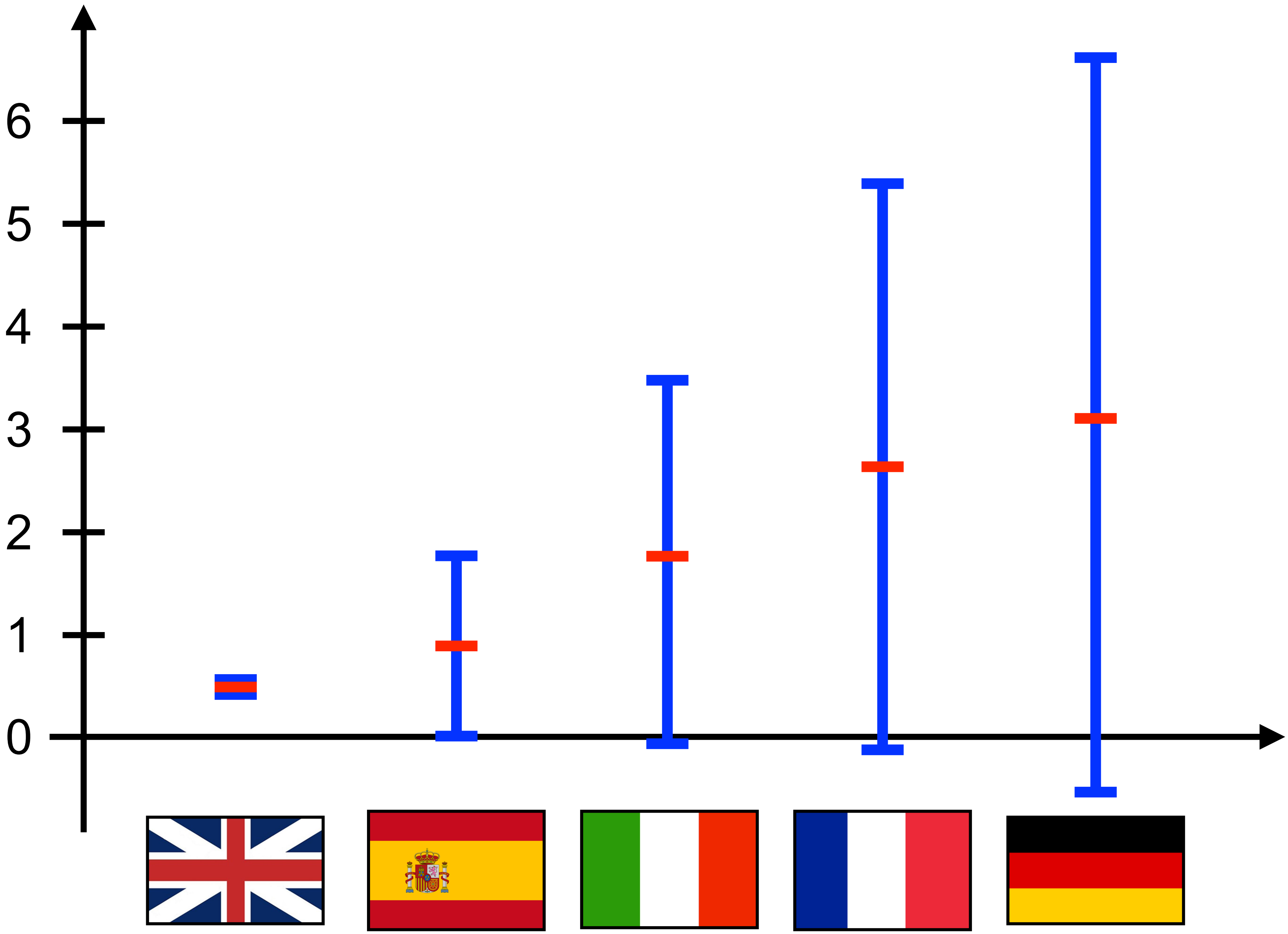
Risk Averse Decision-Making



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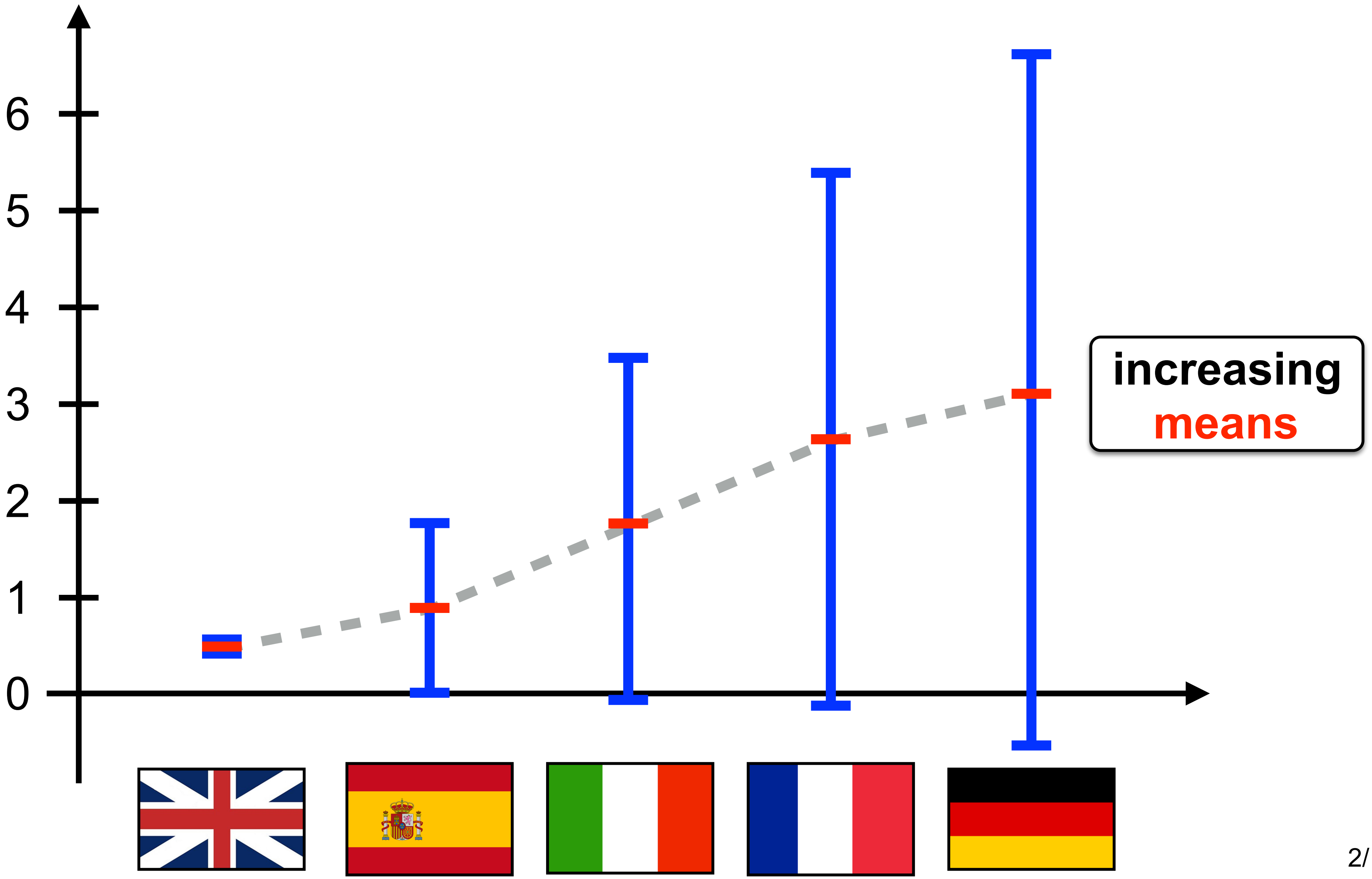

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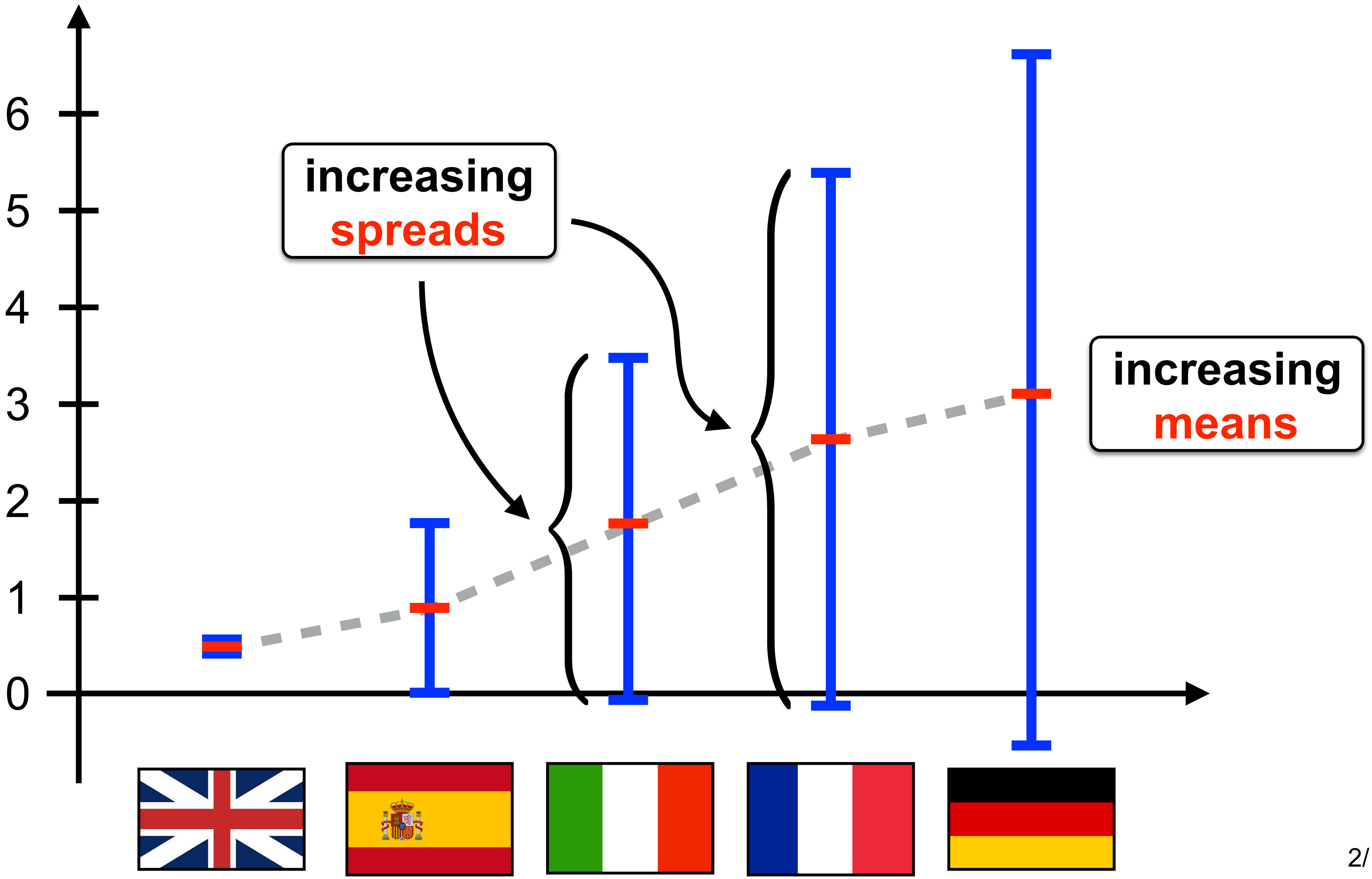
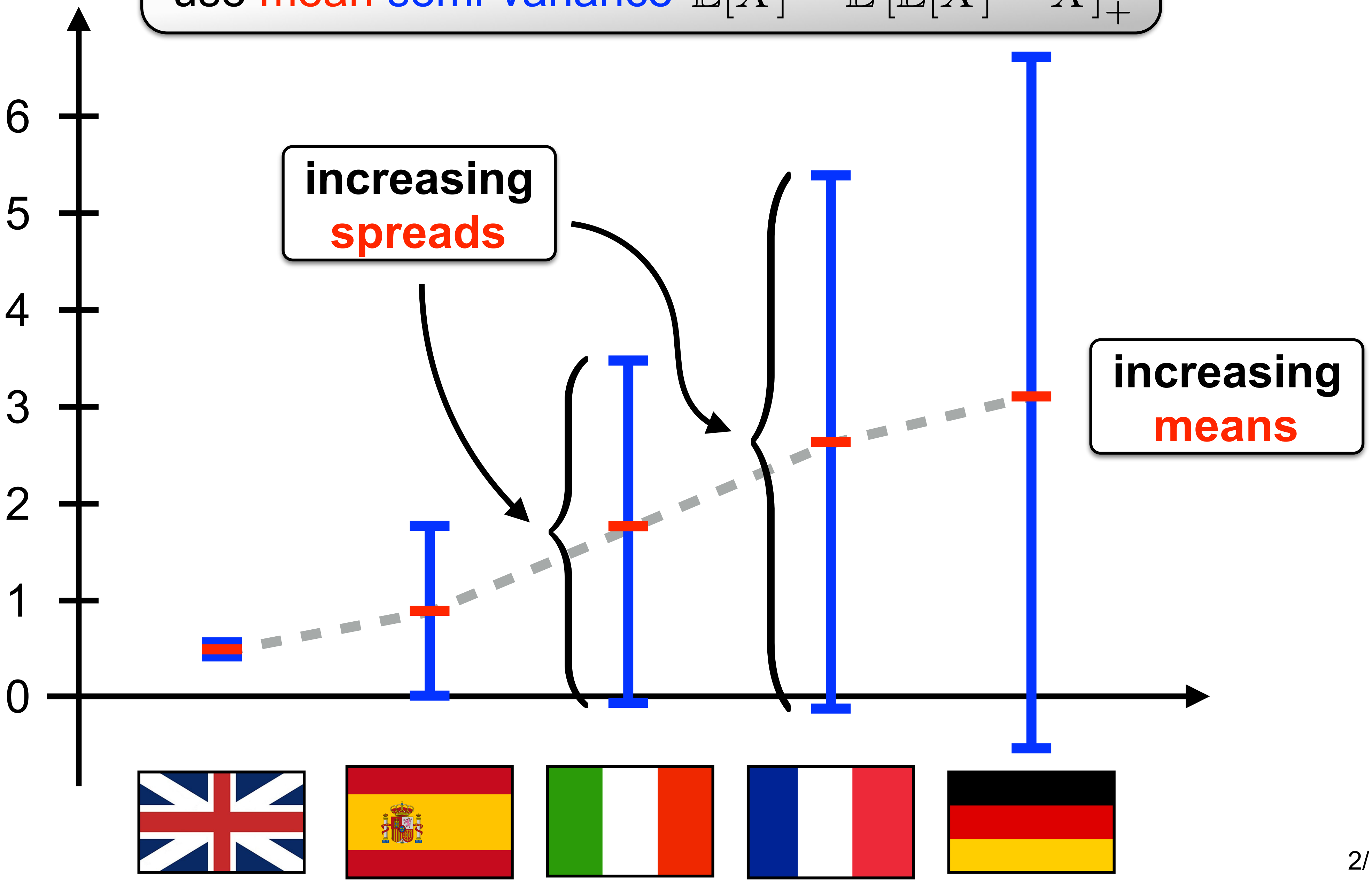


Risk Averse Decision-Making

To account for the profit variation:

use mean semi-variance $\mathbb{E}[X] - \mathbb{E}\left[\mathbb{E}[X] - X\right]_+^2$



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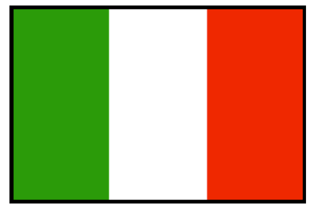

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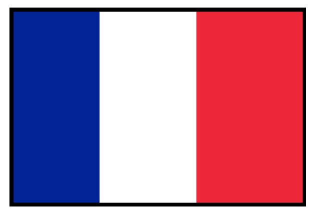

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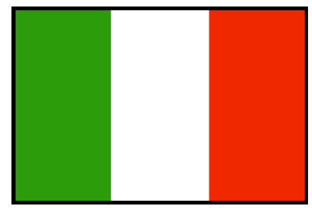

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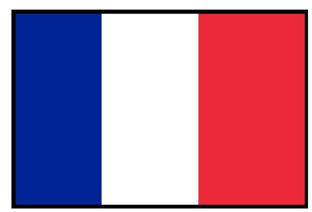

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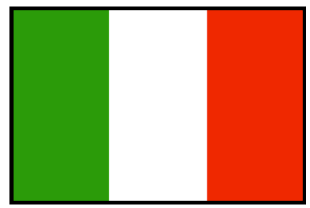

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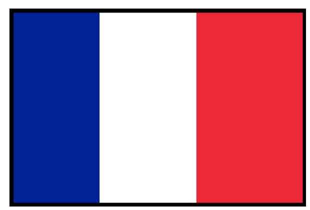

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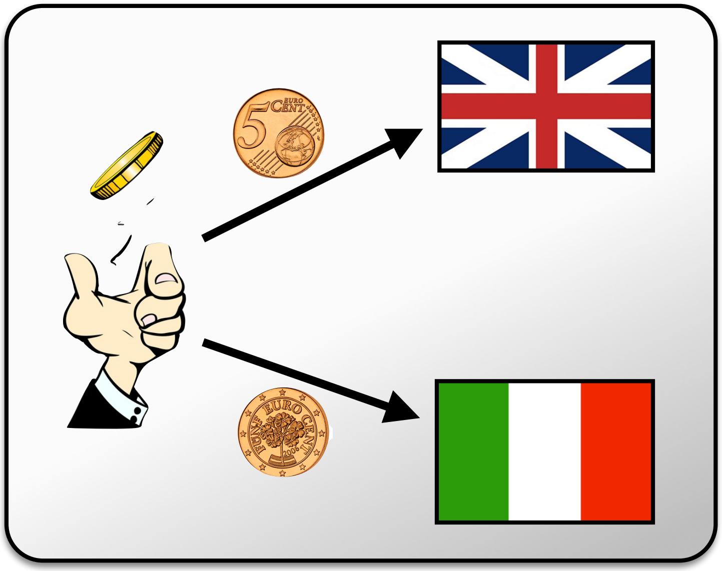

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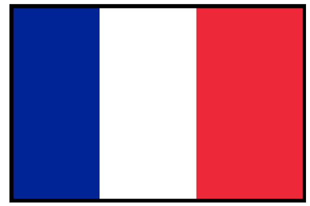

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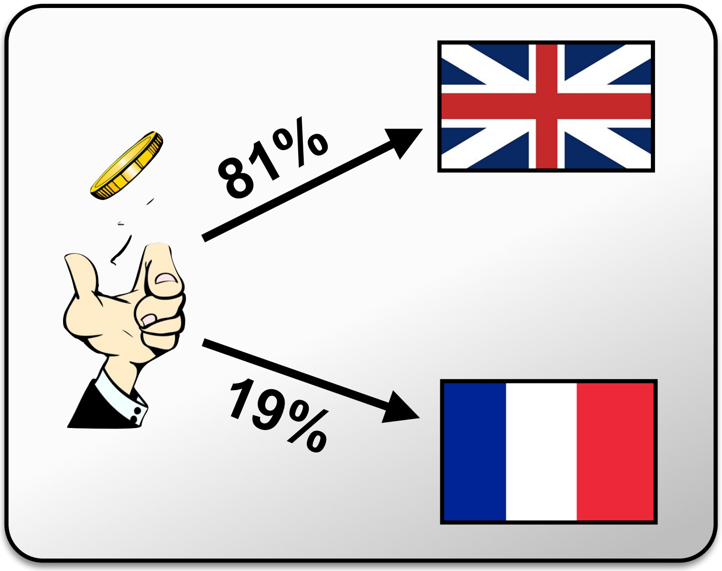

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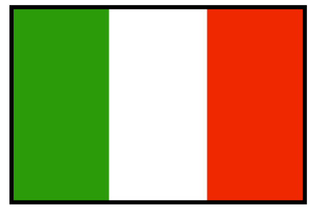

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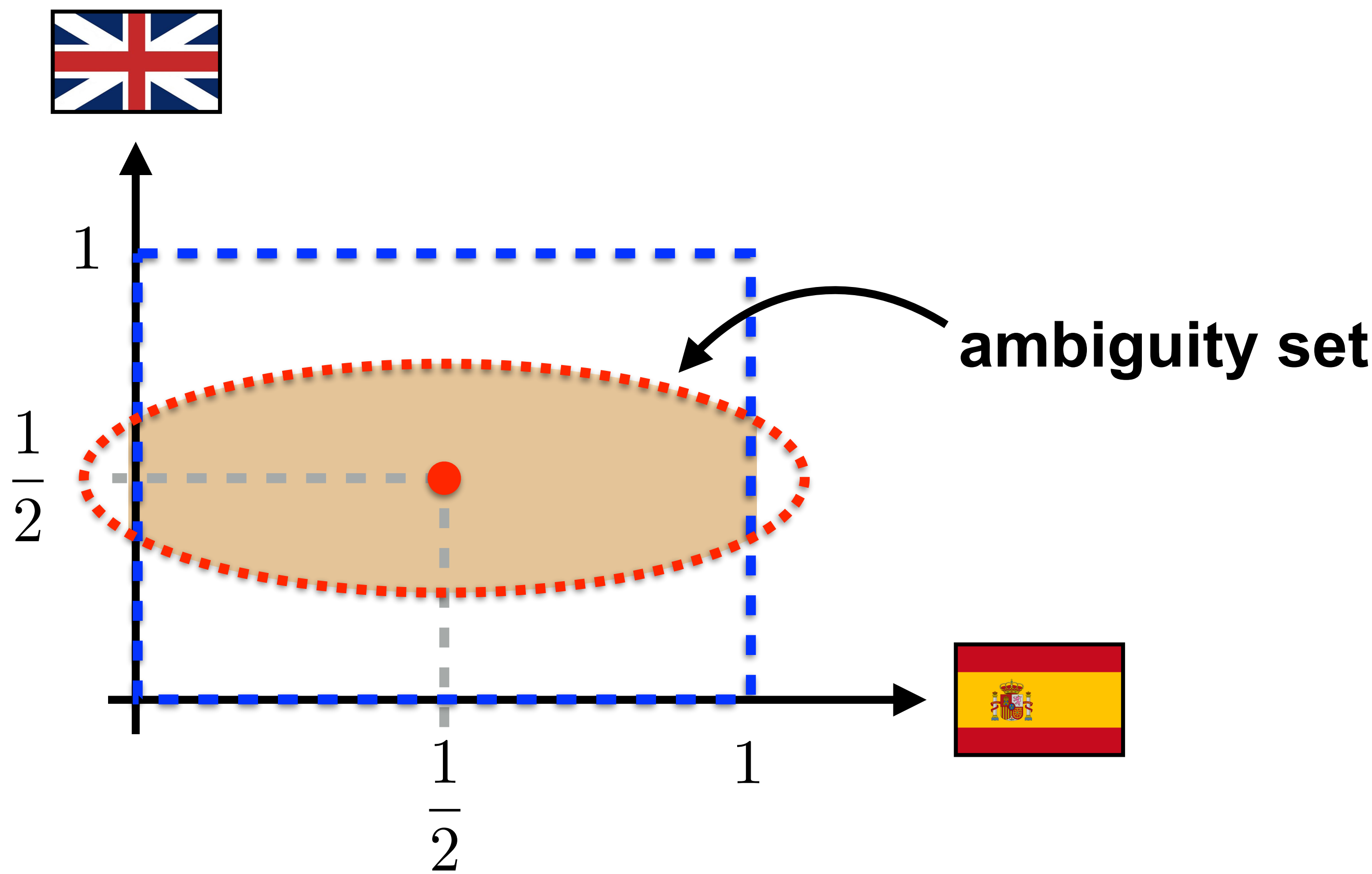
Ambiguity Averse Decision-Making

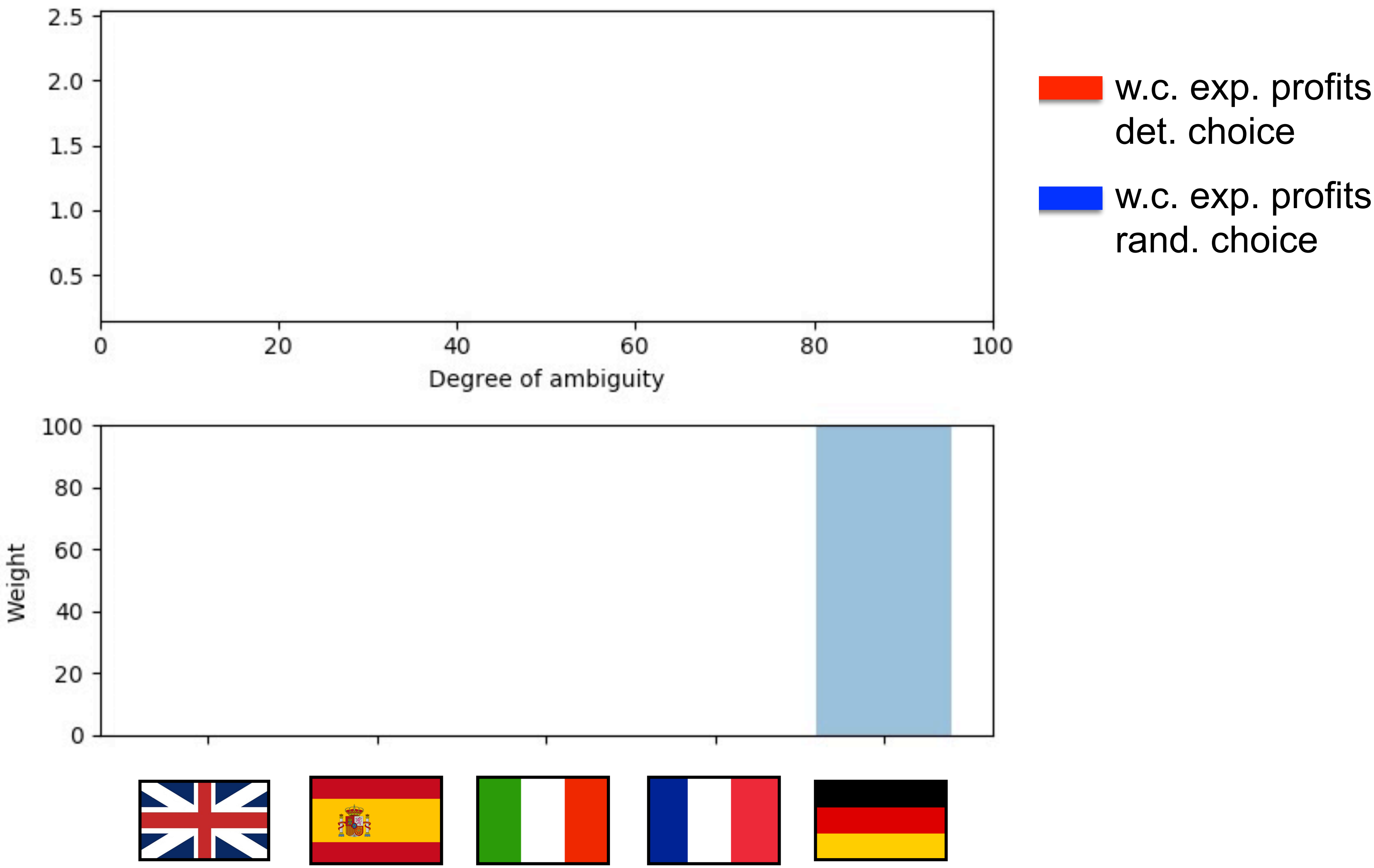
In practice, the probabilities for the profit scenarios may only be partially known:



Distributionally robust optimization: Optimize a risk measure over worst distribution in ambiguity set

Ambiguity Averse Decision-Making

Assume we want to maximize expected profits under the worst probability distribution in the ambiguity set.



Agenda

1. ~~Motivation~~

2. **Randomization under Distributional Ambiguity**

   - ☑ **Background**
   - ☑ Problem Setup
   - ☑ The Power of Randomization

3. Discussion

Ambiguous Probability Spaces

We model uncertainty via an *ambiguous* probability space:

$$(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$$

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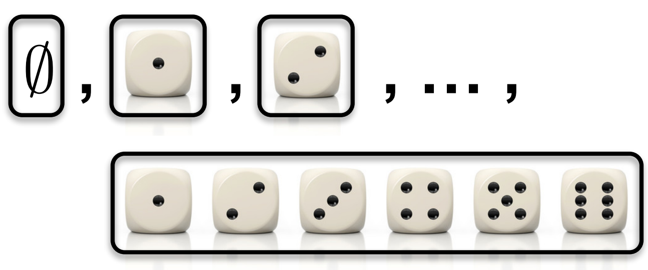
$\mathcal{F}_0$ is the σ-algebra of events:



(2⁶ = 64 sets)

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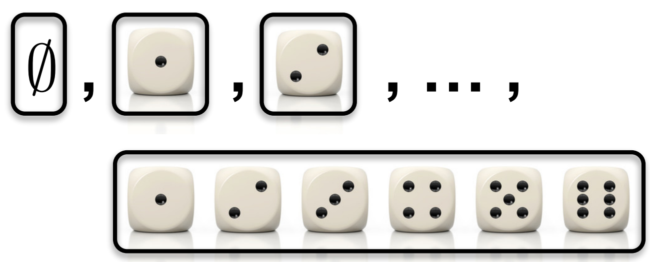

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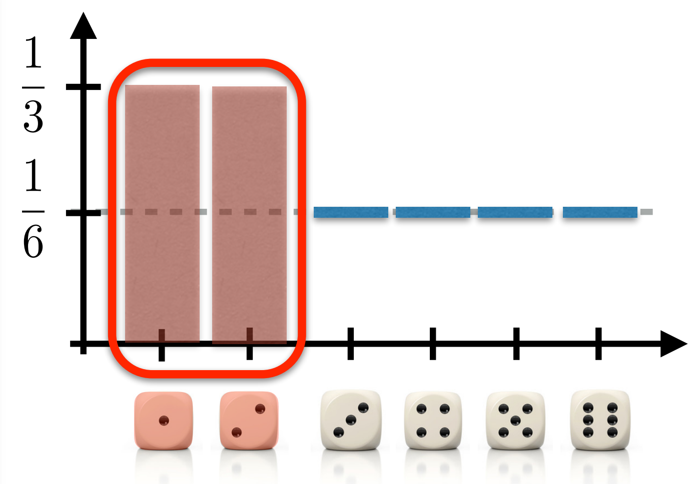
$\mathcal{P}_0$ is the ambiguity set:



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We denote by $\mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ the real-valued random variables that are essentially bounded w.r.t. *all* $\mathbb{P} \in \mathcal{P}_0$:

$$\mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \;=\; \bigcap_{\mathbb{P} \in \mathcal{P}_0} \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathbb{P})$$

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 $\rightarrow -10$  $\rightarrow 10$

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we think of X as revenues

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$$F_X^{\mathbb{P}}(x) = \mathbb{P}(X \leq x) \quad \forall x \in \mathbb{R}$$

Let $\mathcal{D}$ be the set of all distribution functions

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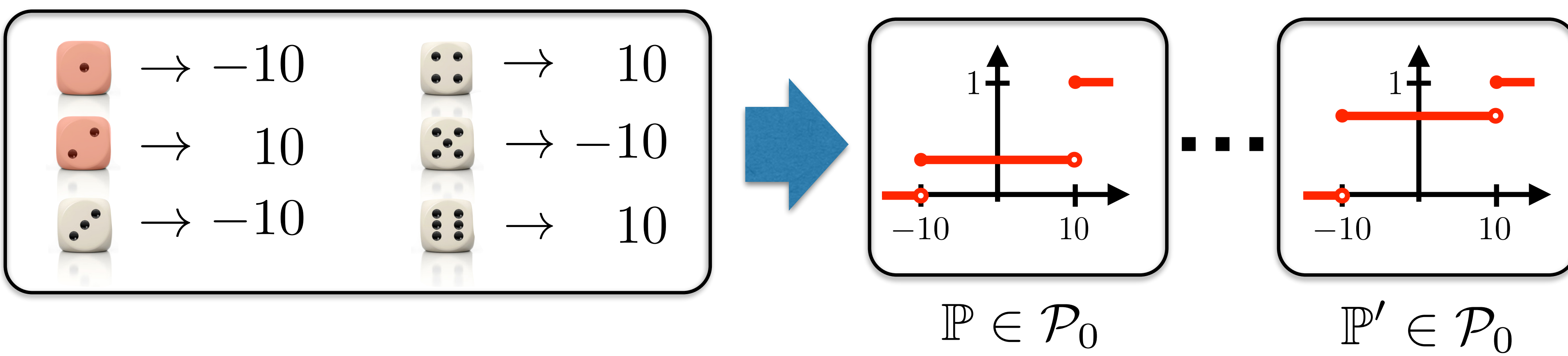

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An ambiguous probability space $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is non-atomic if:

$\exists U_0 \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ that follows a uniform distribution on $[0, 1]$ under *every* probability measure $\mathbb{P} \in \mathcal{P}_0$.

Risk Measures

A risk measure assigns each random variable a risk index:

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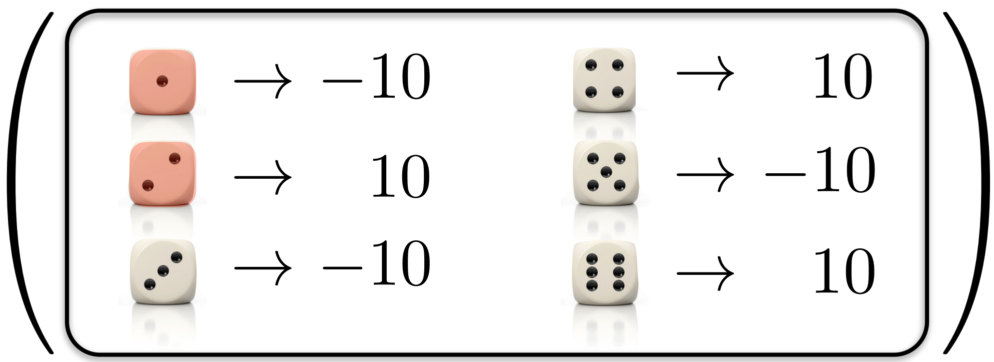

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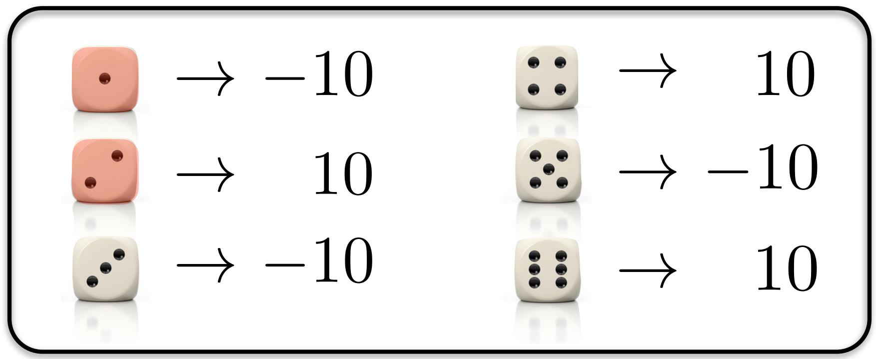
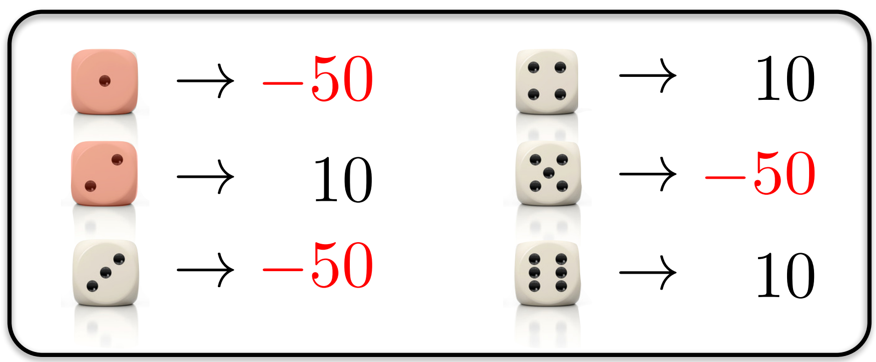


$\Big) = 1$

$\rho_0 \Big($



$\Big) = 2$

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A risk measure ρ_0 is law invariant if it satisfies:

$$\left\{F_X^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}_0\right\} = \left\{F_Y^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}_0\right\} \Rightarrow \rho_0(X) = \rho_0(Y) \quad \forall X, Y \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$$

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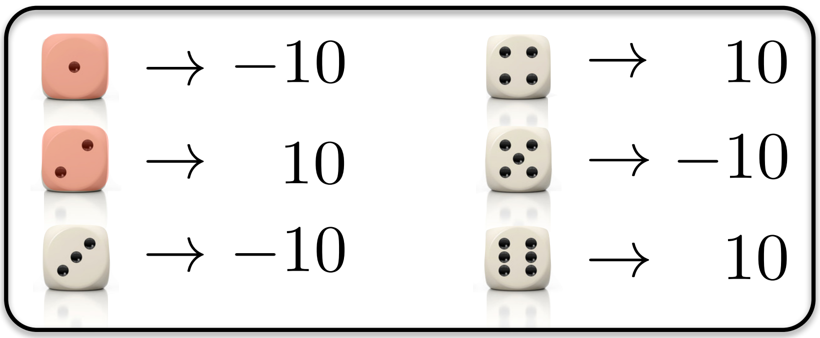

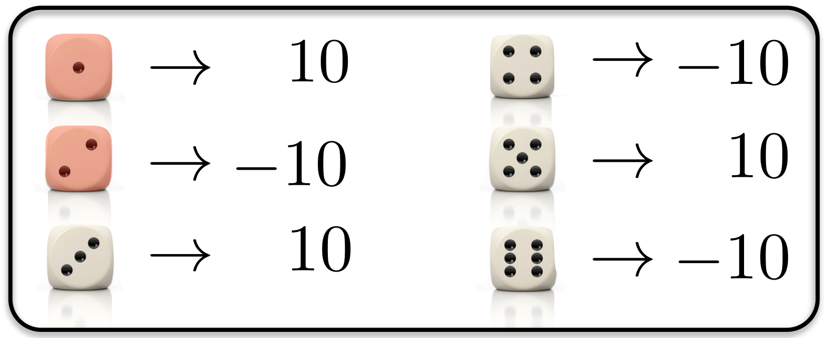

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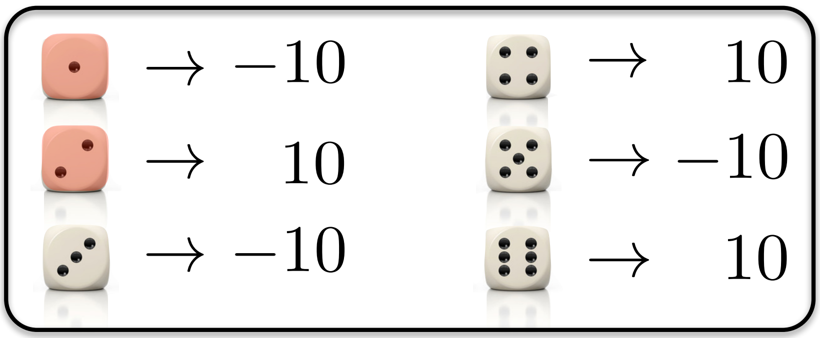

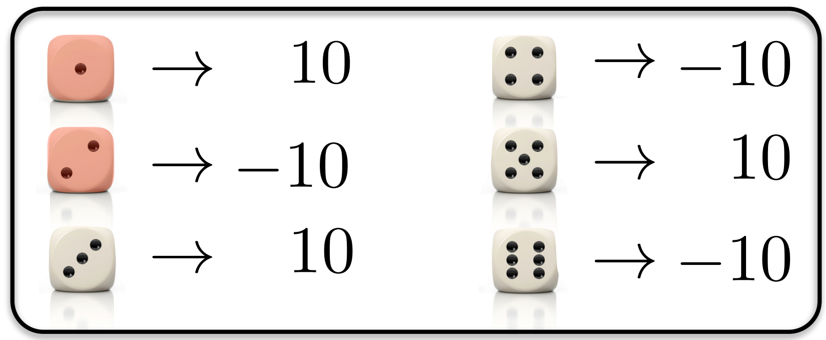

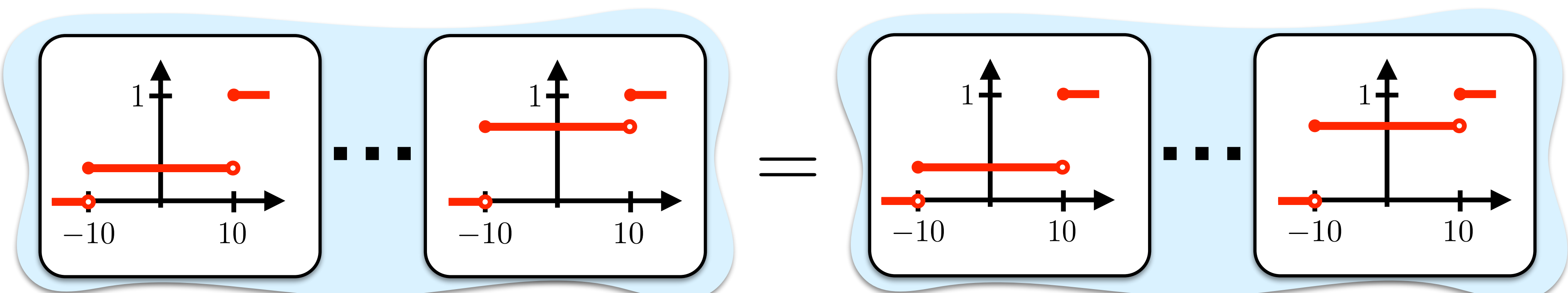

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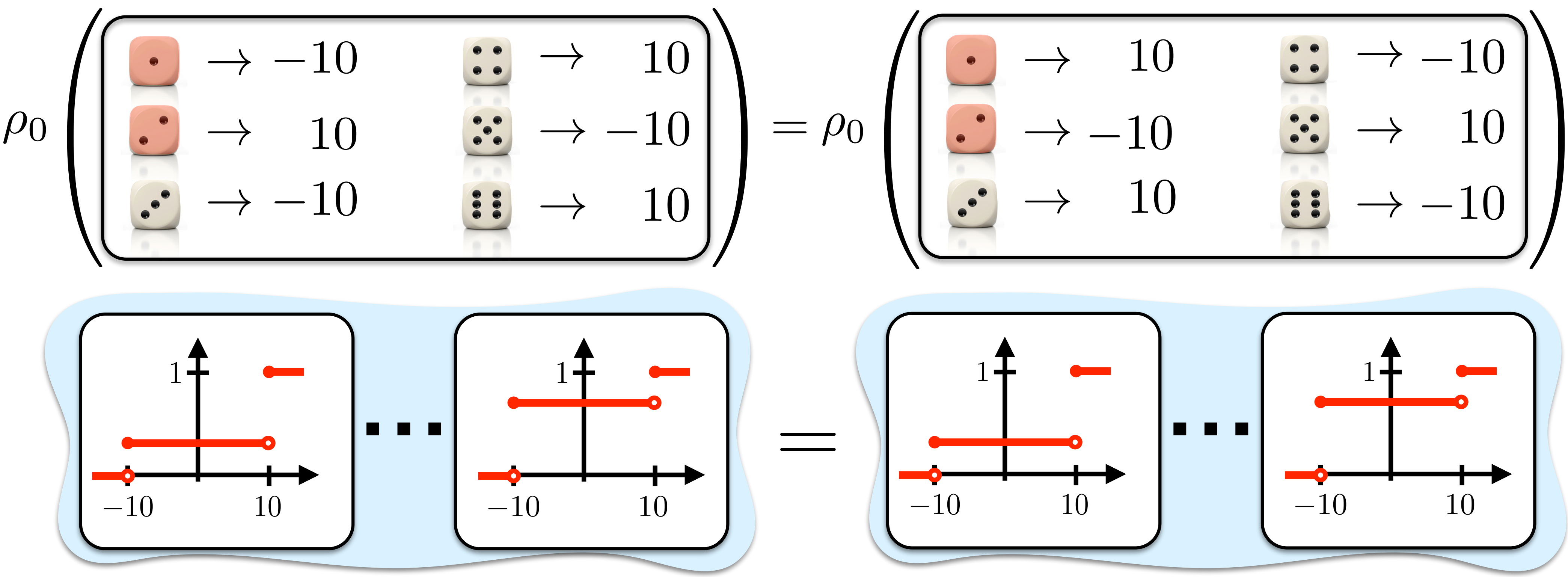

Representation of Risk Measures

Proposition: Assume that $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is non-atomic and that ρ_0 is law invariant.

- ☑ For all $F \in \mathcal{D}$ there is $X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ with $F_X^{\mathbb{P}} = F$ for all $\mathbb{P} \in \mathcal{P}_0$.
- ☑ There exists a unique $\varrho_0 : \mathcal{D} \to \mathbb{R}$ satisfying

$$\rho_0(X) = \varrho_0(F_X) \quad \forall X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) : F_X^{\mathbb{P}} = F_X \;\; \forall \mathbb{P} \in \mathcal{P}_0.$$

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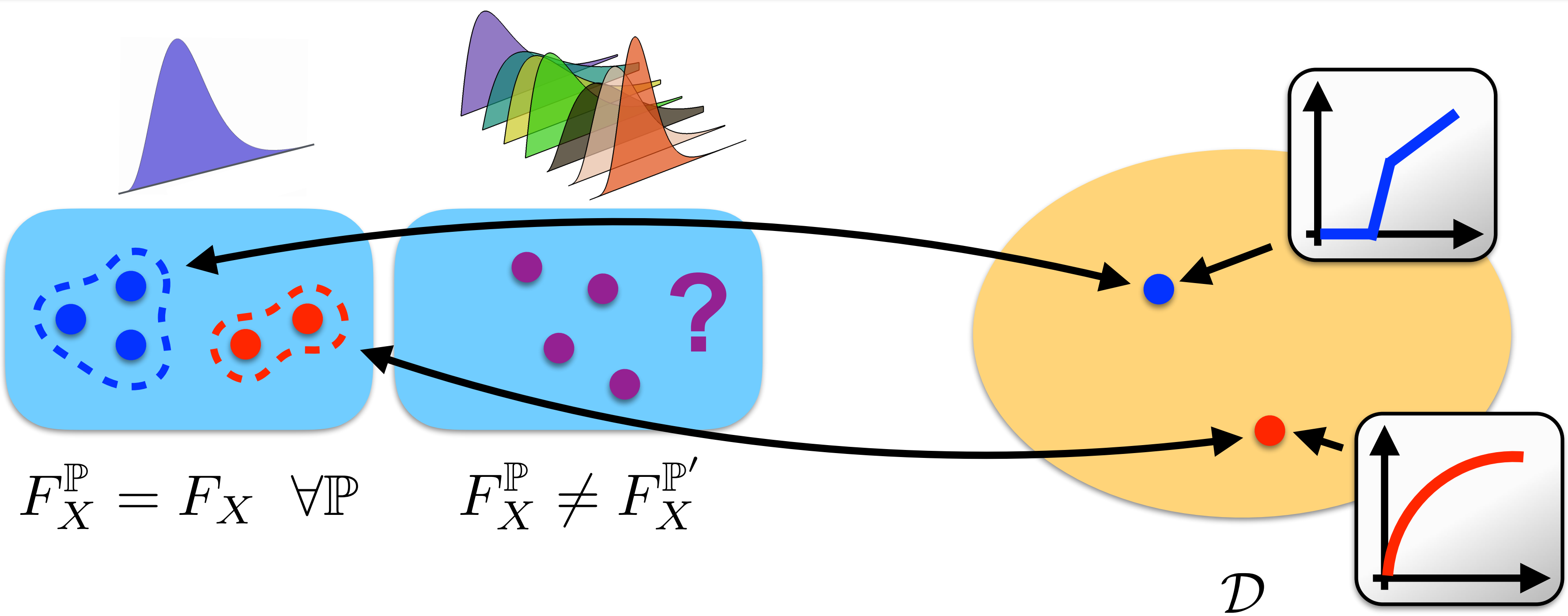
$$\rho_0(X) = \varrho_0(F_X) \quad \forall X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) : F_X^{\mathbb{P}} = F_X \;\; \forall \mathbb{P} \in \mathcal{P}_0.$$



Ambiguity Averse Risk Measures

Definition: A risk measure ρ_0 is called ambiguity averse if it satisfies for all $X, Y \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$:

- **Ambiguity aversion:** If $\{F_X^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}_0\} \subseteq \{F_Y^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}_0\}$, then $\rho_0(X) \leq \rho_0(Y)$.
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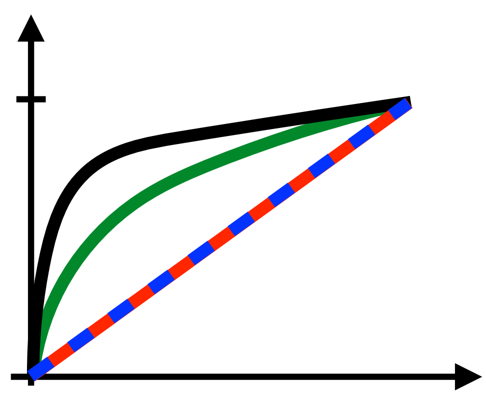
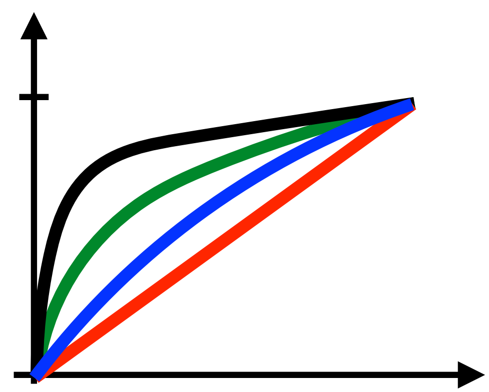


$\{F_X^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}_0\}$ $\subseteq$



$\{F_Y^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}_0\}$

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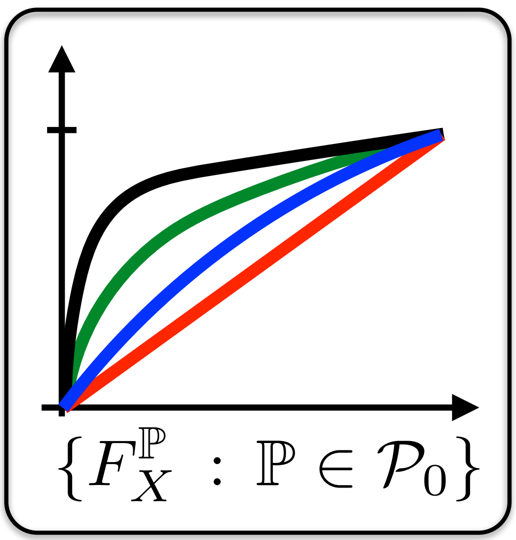

$$\varrho_0(F_X^{\mathbb{P}_1}) \leq \varrho_0(F_Y^{\mathbb{P}_1})$$
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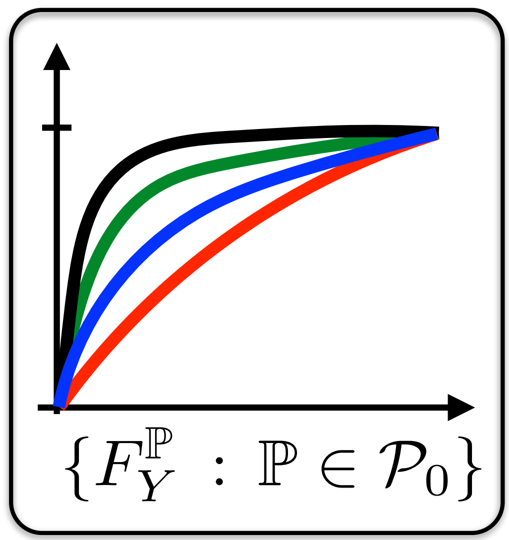

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Representation of Risk Measures

Proposition: Assume that $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is non-atomic and that ρ_0 is ambiguity averse and translation invariant.

Then the risk measure satisfies

$$\rho_0(X) = \sup_{\mathbb{P} \in \mathcal{P}_0} \varrho_0(F_X^{\mathbb{P}}) \quad \forall X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0).$$

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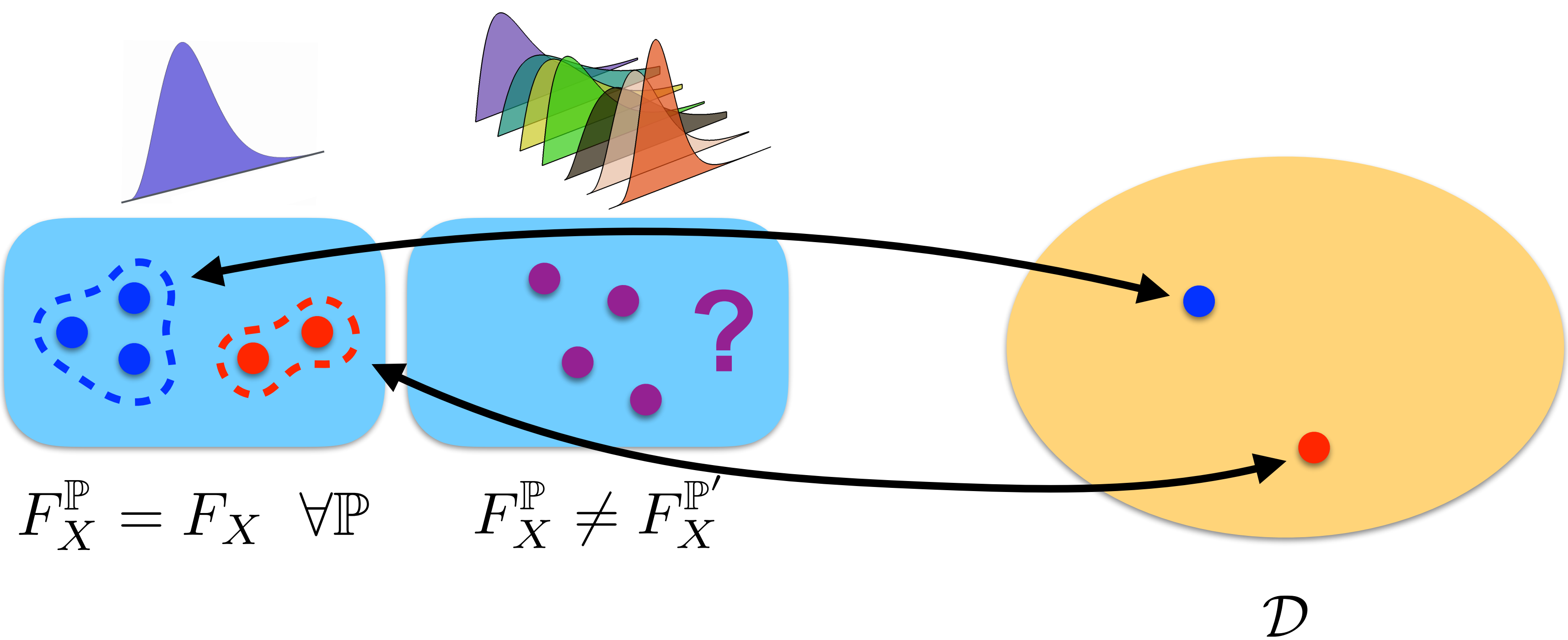

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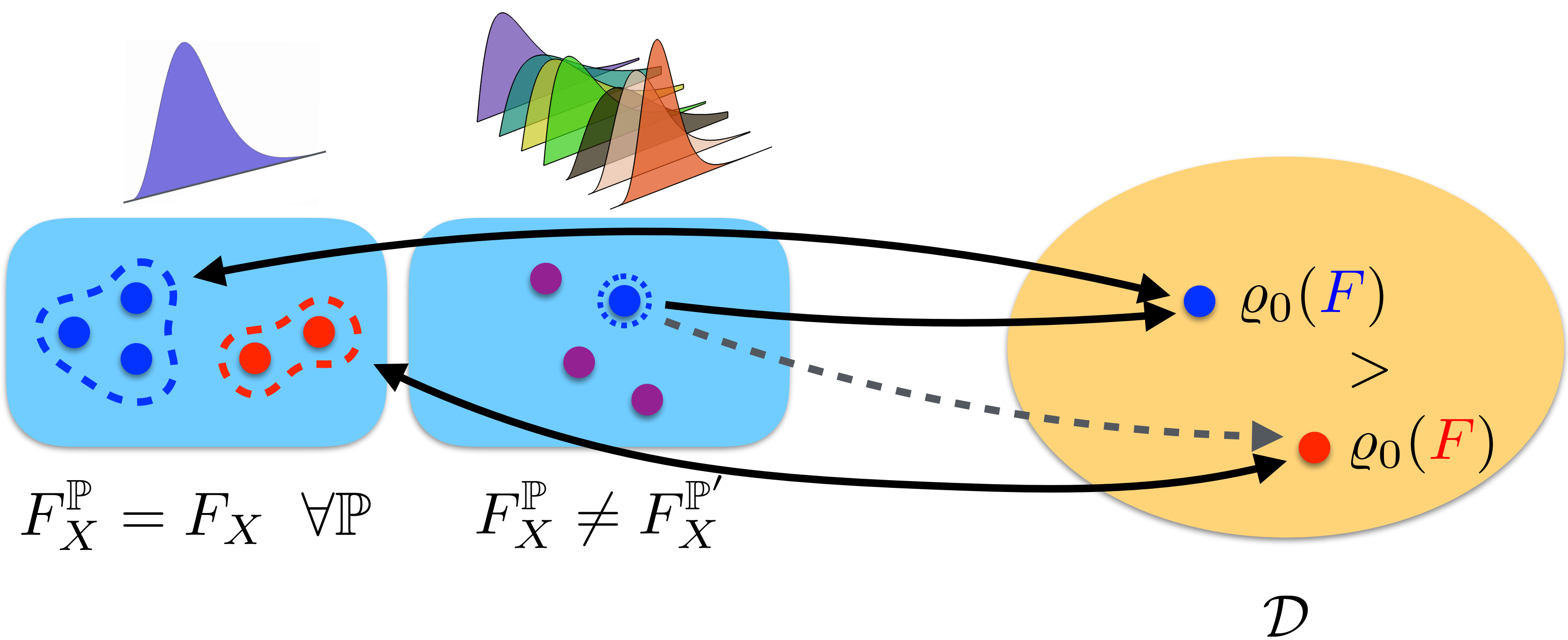

Agenda

1. ~~Motivation~~
2. **Randomization under Distributional Ambiguity**
   - ☑ ~~Background~~
   - ☑ **Problem Setup**
   - ☑ The Power of Randomization
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Pure Strategy Problem

We consider the abstract optimization problem

$$\underset{X \in \mathcal{X}_0}{\text{minimize}} \;\; \rho_0(X) \qquad \text{(PSP)}$$

where $\mathcal{X}_0 \subseteq \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ denotes the feasible region.

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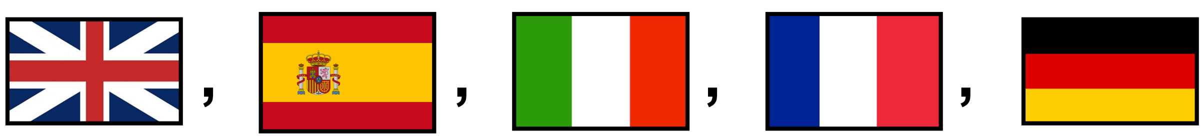
Example: Facility location

$\mathcal{X}_0 = \{$  $\}$

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<u>Example:</u> Portfolio optimization

$$\mathcal{X}_0 = \{\boldsymbol{r}^\top \boldsymbol{w} \; : \; \boldsymbol{w} \geq \mathbf{0}, \mathrm{e}^\top \boldsymbol{w} = 1\} \quad \text{with} \quad \boldsymbol{r} = \begin{pmatrix} r_{\text{IBM}} \\ r_{\text{Walmart}} \\ r_{\text{pepsi}} \\ r_{\text{P\&G}} \end{pmatrix}$$

From Deterministic to Random Decisions

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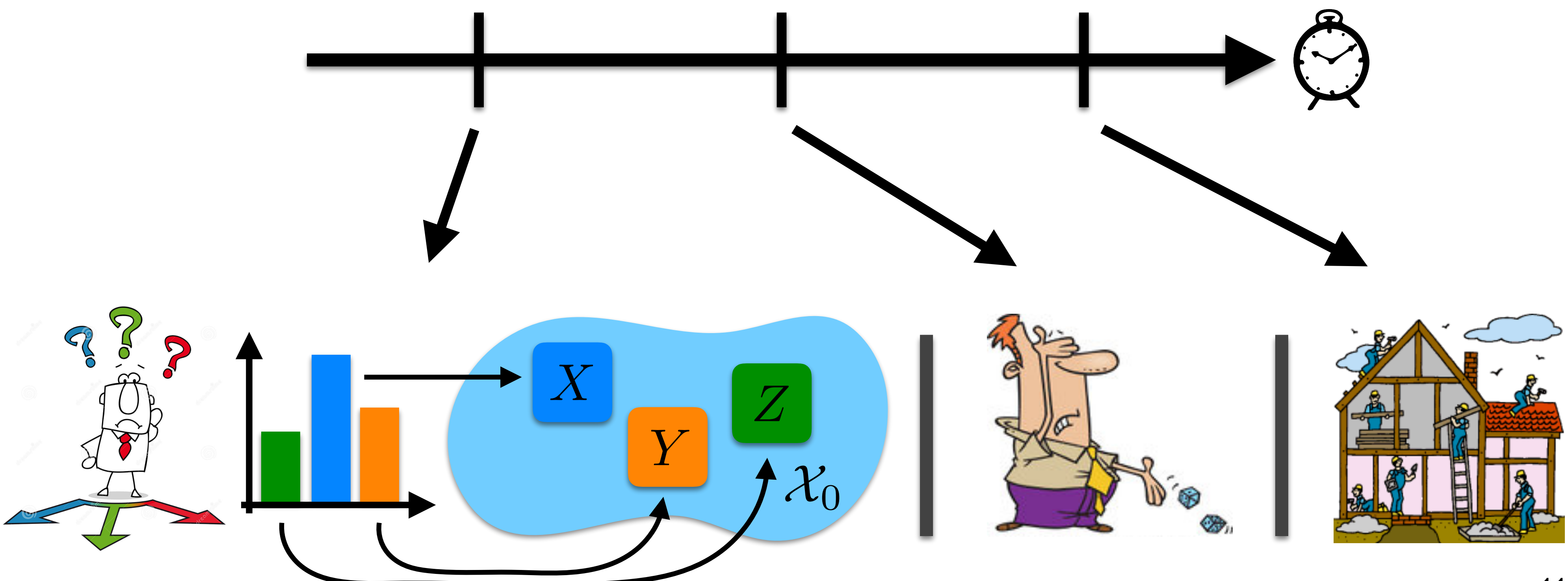

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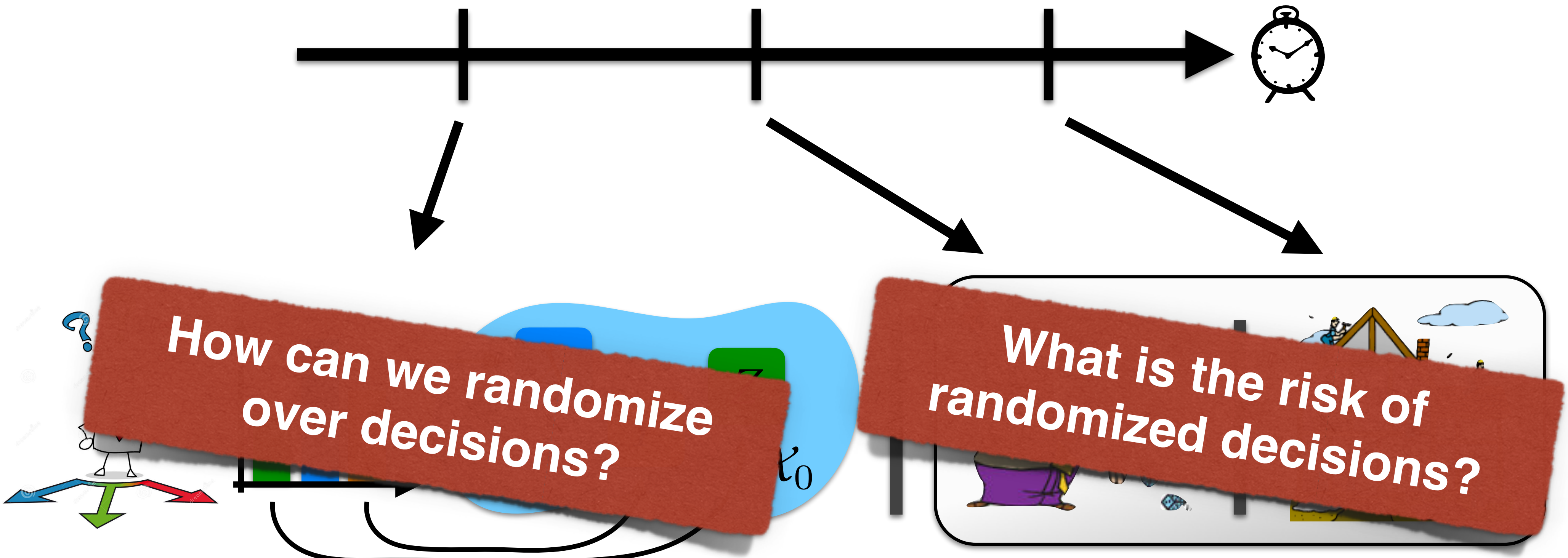
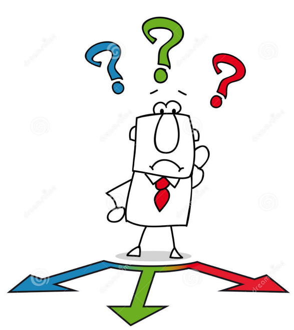
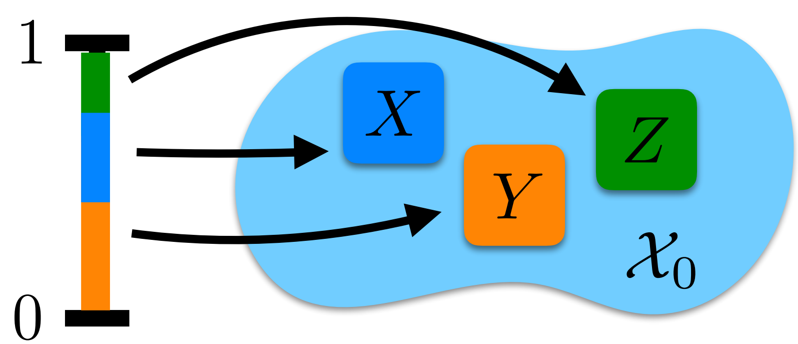


Randomization Devices

We assume we have a randomization device **that generates uniform samples from $[0,1]$:**

pure strategy problem	**randomized strategy problem**
$(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$	$(\Omega, \mathcal{F}, \mathcal{P})$ with $\Omega = \Omega_0 \times [0,1]$; $\mathcal{F} = \mathcal{F}_0 \otimes \mathcal{B}_{[0,1]}$; $\mathcal{P} = \{\mathbb{P} \times \mathbb{U} : \mathbb{P} \in \mathcal{P}\}$
$\mathcal{X}_0 \subseteq \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$	$\mathcal{X} = \{X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}) : X(\cdot, u) \in \mathcal{X}_0 \ \forall u \in [0,1]\}$





Risk of Randomized Decisions

Proposition: Assume that $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is non-atomic and that ρ_0 is ambiguity averse and translation invariant.

The unique extension of ρ_0 to an ambiguity averse risk measure ρ on $\mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P})$ is given by

$$\rho(X) = \sup_{\mathbb{P} \in \mathcal{P}} \varrho_0(F_X^{\mathbb{P}}) \quad \forall X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}).$$

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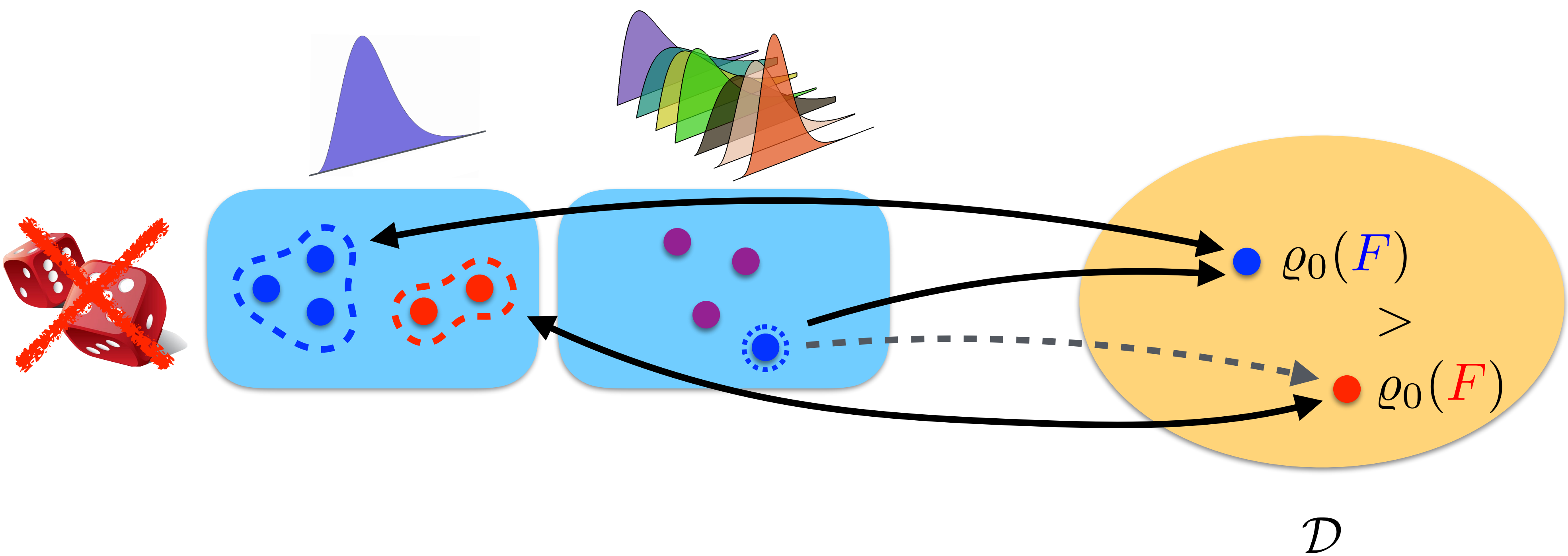

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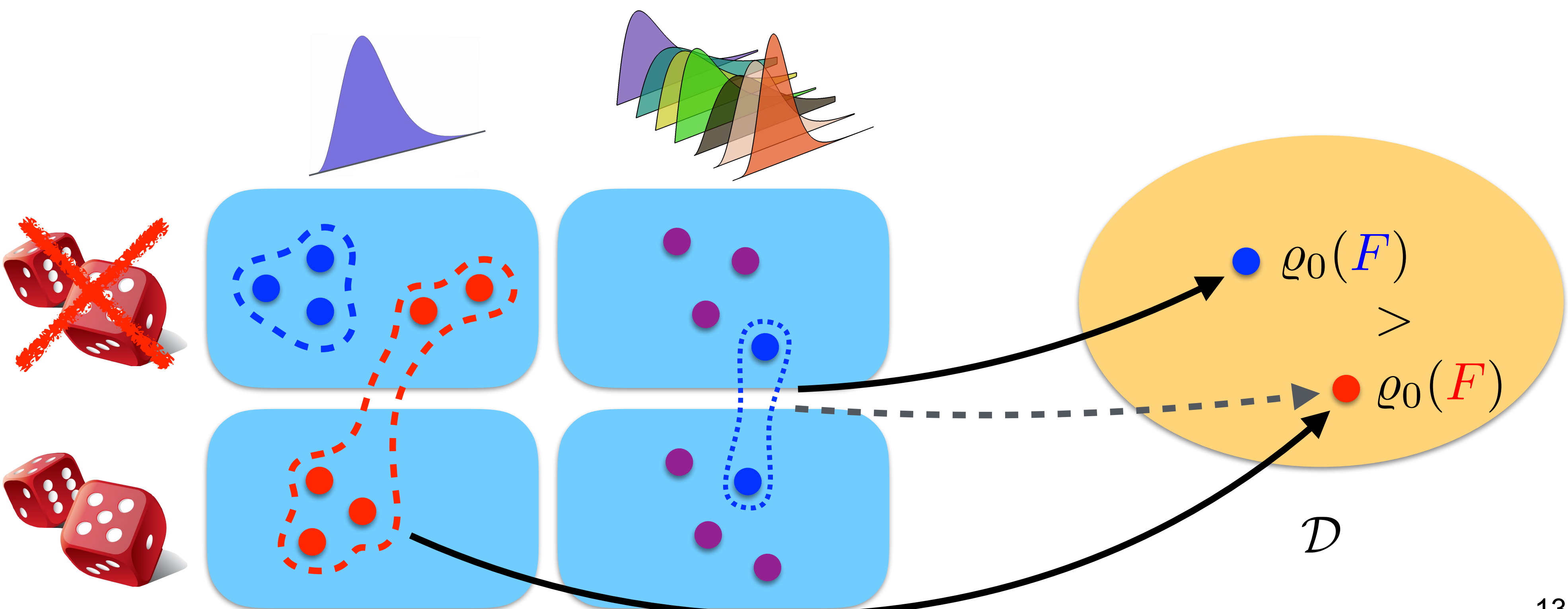

Randomized Strategy Problem

We define the randomized strategy problem

$$\underset{X \in \mathcal{X}}{\text{minimize}} \;\; \rho(X) \qquad \text{(RSP)}$$

where the extended risk measure ρ is defined via

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and $\mathcal{X}$ denotes the enlarged feasible region:

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The feasible region contains all pure strategies:

$X_0 \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ ⇒ $X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P})$ with $X(\omega, u) = X_0(\omega) \;\; \forall u \in [0,1]$

Agenda

1. ~~Motivation~~
2. **Randomization under Distributional Ambiguity**
   - ☑ ~~Background~~
   - ☑ ~~Problem Setup~~
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The Power of Randomization

Definition: The ambiguity averse risk measure ρ_0 has the Lebesgue property if

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Theorem: Assume that

- ☑ $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is non-atomic and has a maximally ambiguous random variable,
- ☑ ρ_0 is ambiguity averse and satisfies the Lebesgue property.

Then there is $\mathcal{X}_0$ such that (PSP) > (RSP).

The Rainbow Urn Game

Consider an urn with balls of K different colors where:

- ☑ the number of balls is unknown
- ☑ the proportions of colors are unknown

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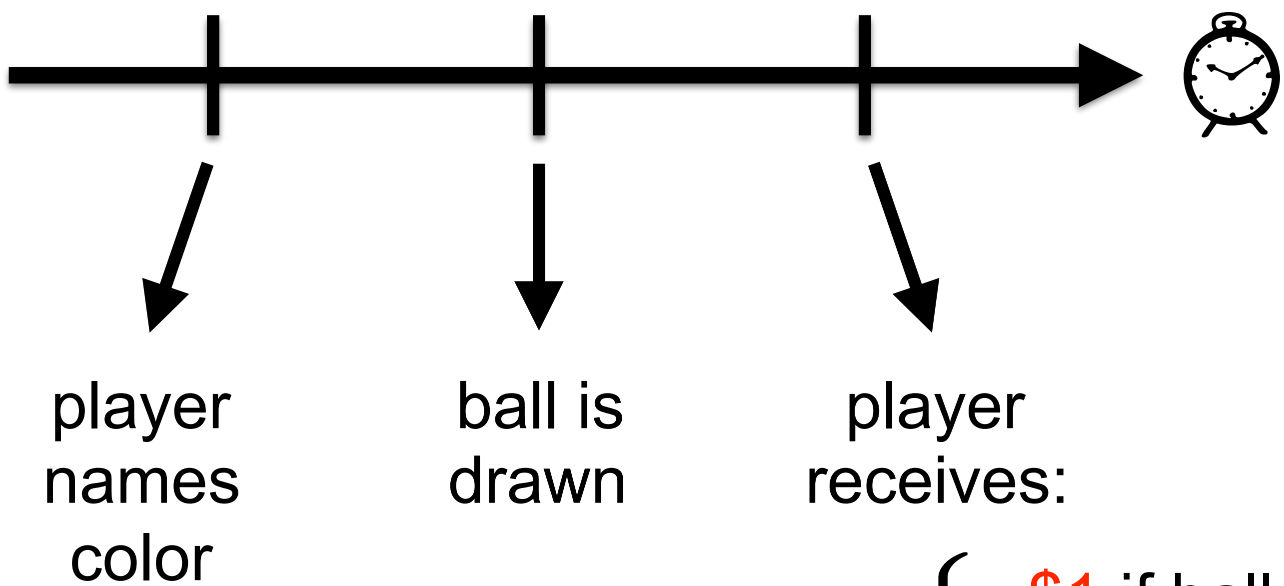
A player is offered the following game:



- -\$1 if ball is of stated color
- +\$1 if ball is *not* of stated color

The Rainbow Urn Game

Assume the player uses an ambiguity averse risk measure ρ_0:

The Rainbow Urn Game

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Strategy

Worst-case outcome

Any *pure strategy* is as good as losing $1 with certainty

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Assume the player uses an ambiguity averse risk measure ρ_0:

Strategy		**Worst-case outcome**
Any *pure strategy* is as good as losing $1 with certainty	⇒	All balls are of stated color
The *randomized strategy* that names each color with probability $1/K$ suppresses the ambiguity	⇒	-$1 with probability $\frac{1}{K}$ +$1 with probability $\frac{K-1}{K}$

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Strategy	**Worst-case outcome**
Any *pure strategy* is as good as losing $1 with certainty	All balls are of stated color
The *randomized strategy* that names each color with probability $1/K$ suppresses the ambiguity	-$1 with probability $\frac{1}{K}$; +$1 with probability $\frac{K-1}{K}$

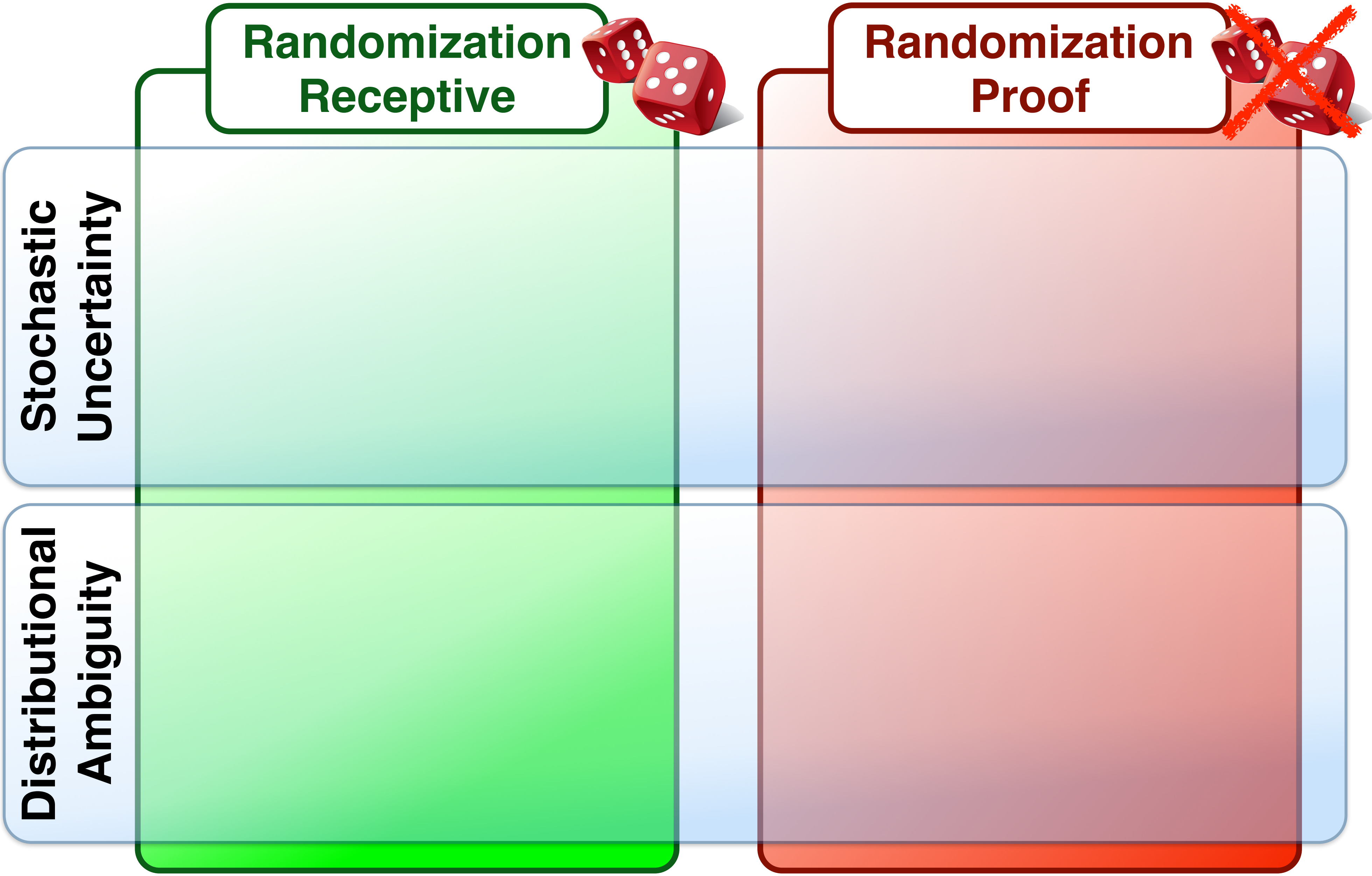
If ρ_0 has the Lebesgue property, then this is as attractive as receiving +$1 for sure as $K \to \infty$!

Randomization can serve as a cure for ambiguity

Agenda

1. ~~Motivation~~
2. ~~Randomization under Stochastic Uncertainty~~
3. **Discussion**

Summary



Randomization Receptive
Randomization Proof
Stochastic Uncertainty
Distributional Ambiguity

Summary

Randomization Receptive 

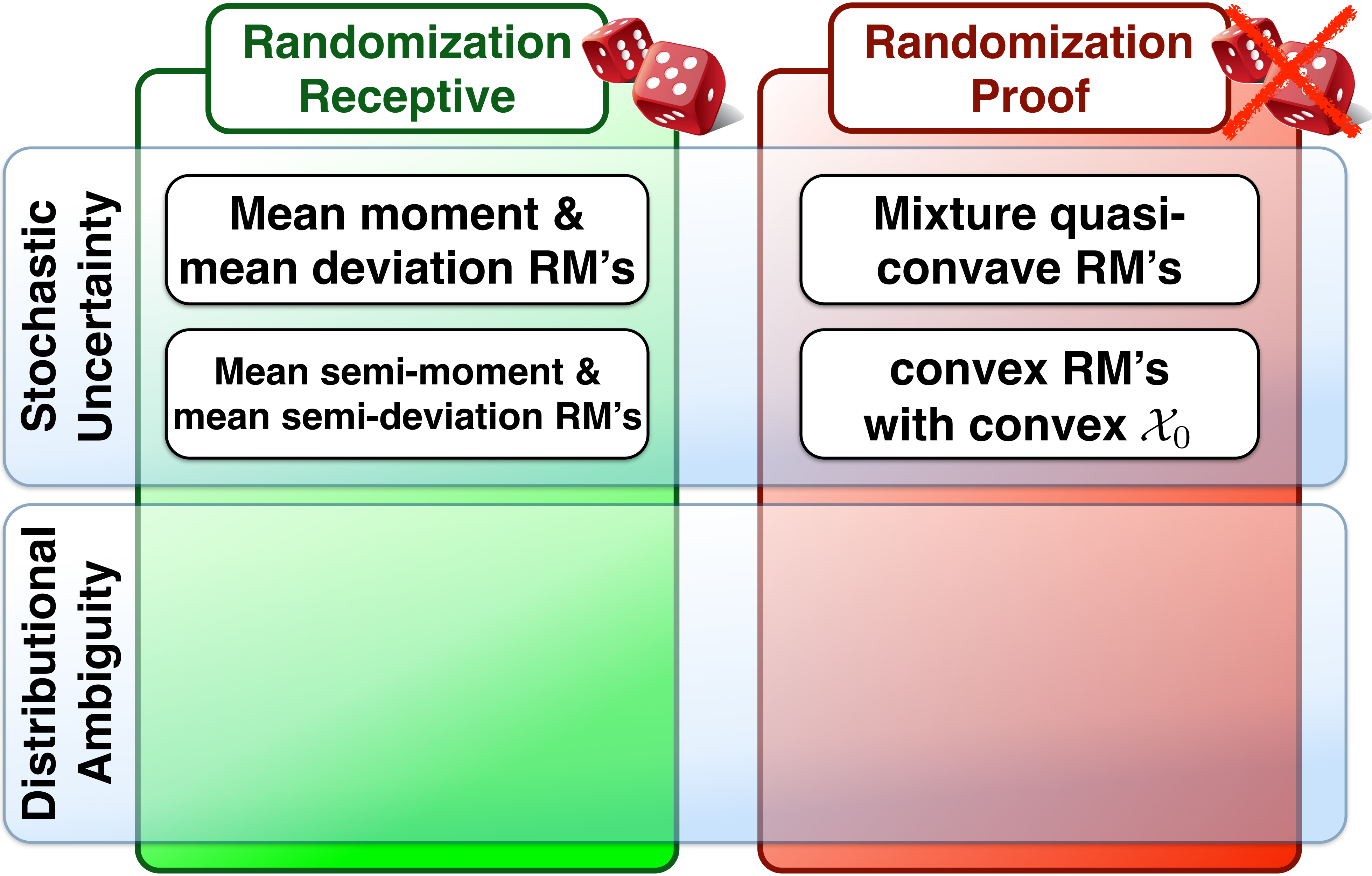
Randomization Proof 

Stochastic Uncertainty

- Mixture quasi-convave RM's
- convex RM's with convex $\mathcal{X}_0$

Distributional Ambiguity

Summary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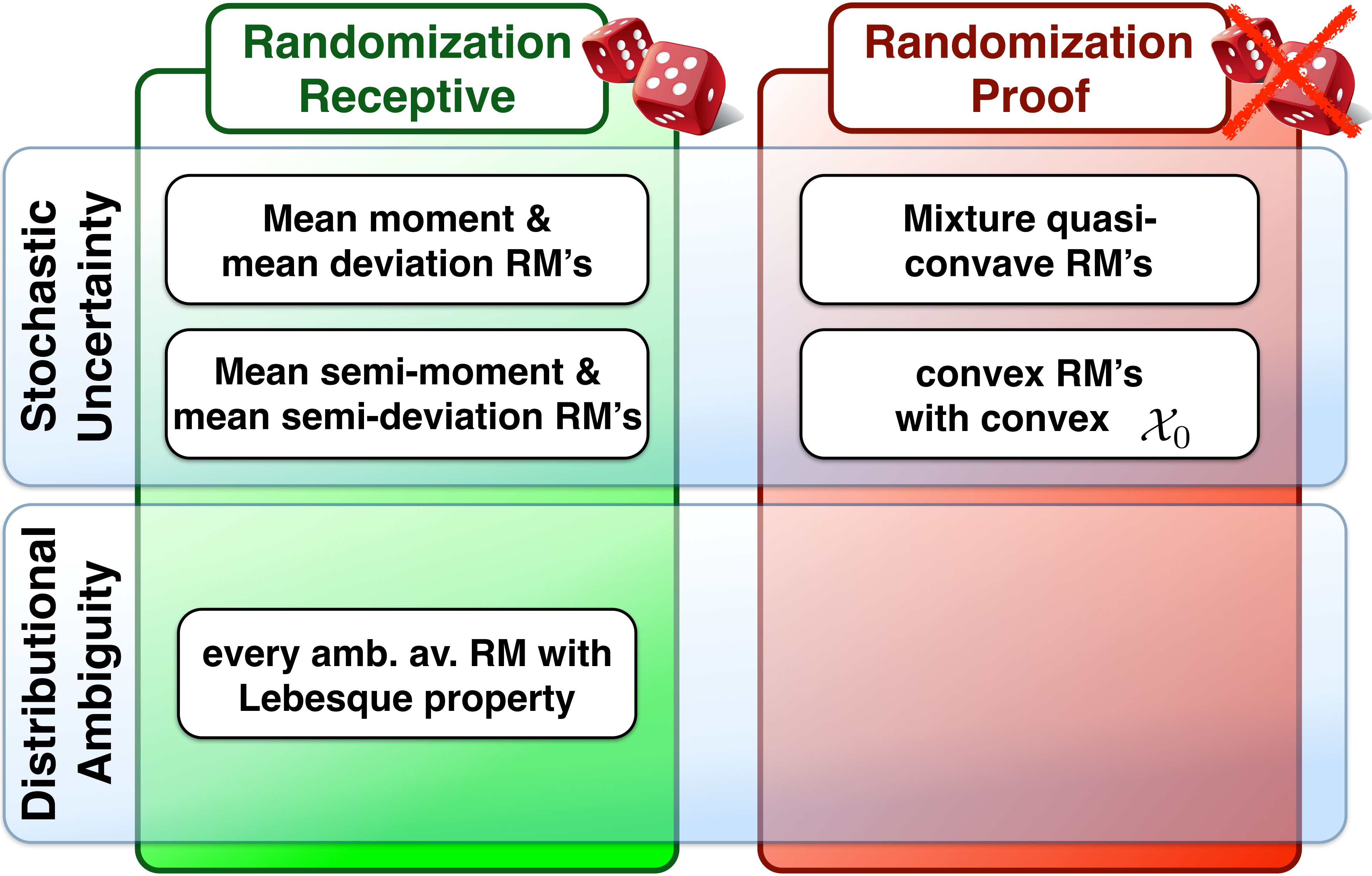
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The Issue of Time Consistency

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Remember the randomized strategy problem:

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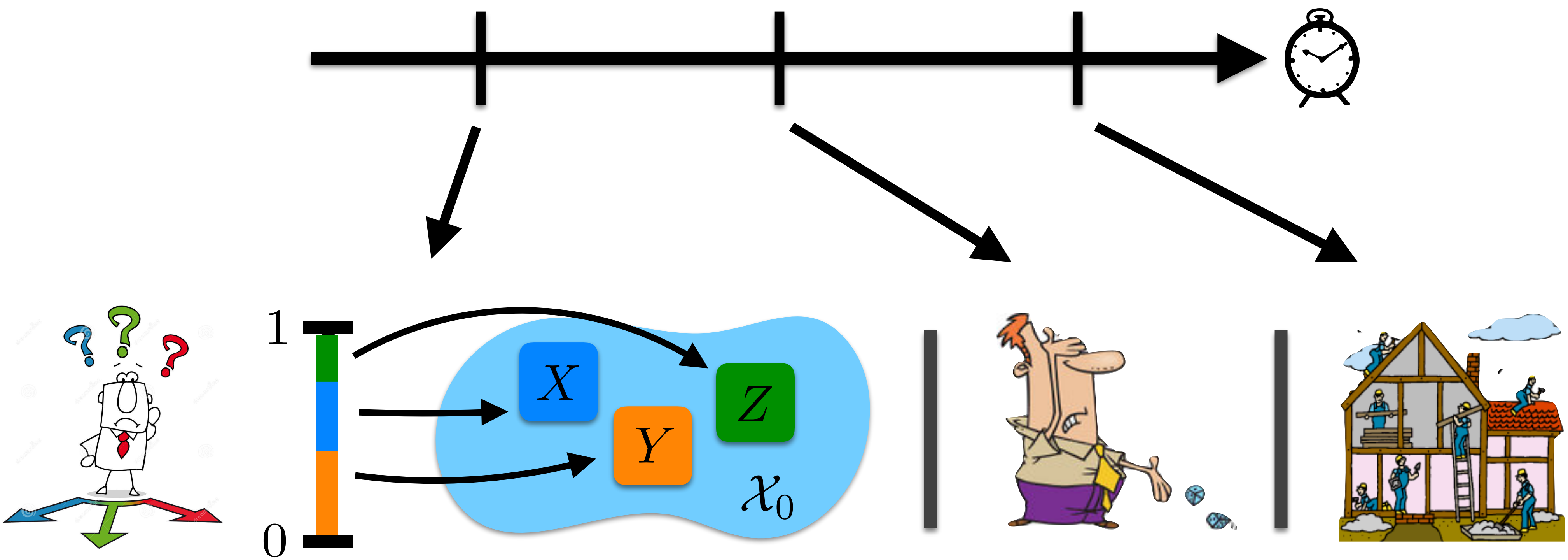

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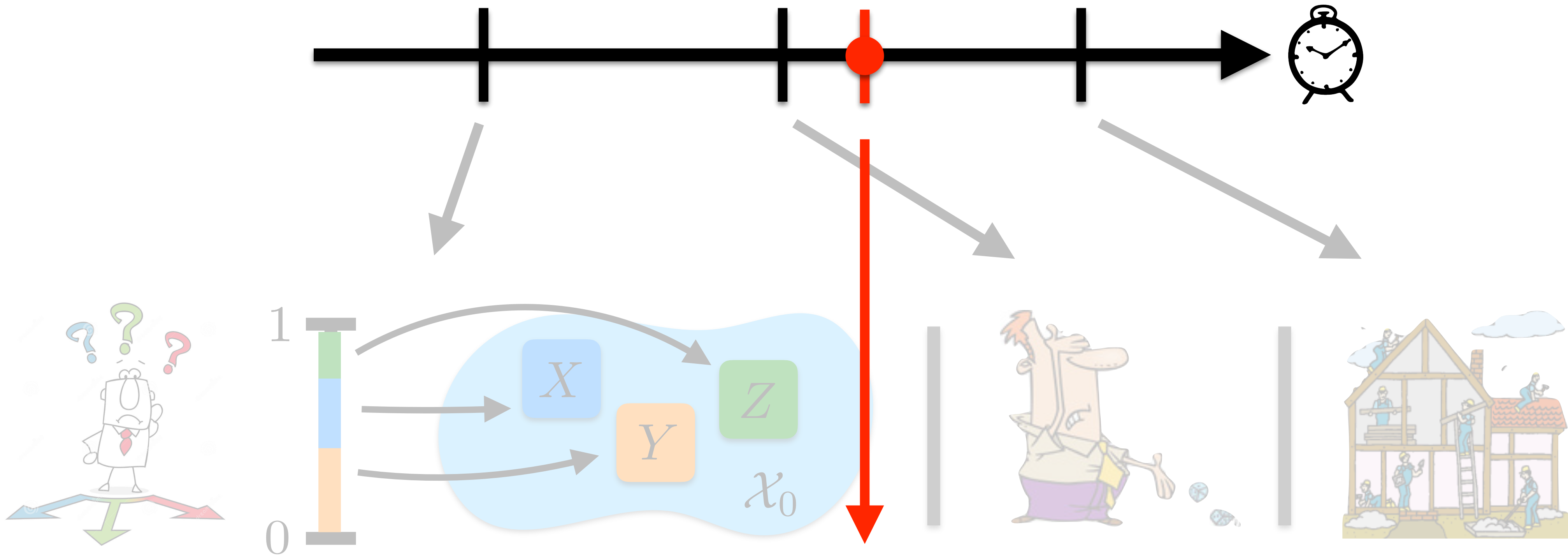

Once we observe the outcome of the randomization, we have an incentive to deviate in favour of the optimal pure choice!

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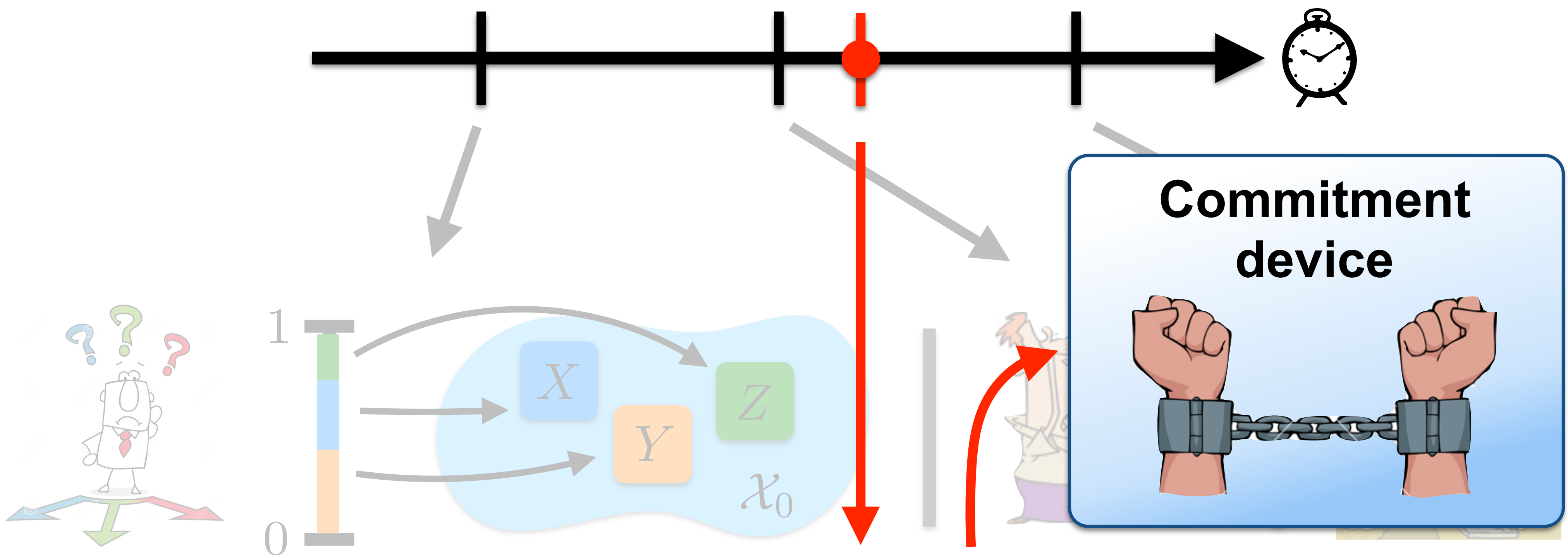

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